On Wohlfahrt series and wreath products

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On Wohlfahrt series and wreath products

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Abstract. Suppose that a group \( A \) contains only a finite number of subgroups of index \( d \) for each positive integer \( d \). Let \( G \wr S_n \) be the wreath product of a finite group \( G \) with the symmetric group \( S_n \) on \( \{1, \ldots, n\} \). For each positive integer \( n \), let \( K_n \) be a subgroup of \( G \wr S_n \) containing the commutator subgroup of \( G \wr S_n \). If the sequence \( \{K_n\}_{n=0}^{\infty} \) satisfies a certain compatible condition, then the exponential generating function \( \sum_{n=0}^{\infty} \frac{|\text{Hom}(A, K_n)|}{n!} X^n / |G|^n \) of the sequence \( \{|\text{Hom}(A, K_n)|\}_{n=0}^{\infty} \) takes the form of a sum of exponential functions.

1. Introduction

Let \( A \) be a group and \( \mathcal{F}_A \) the set of subgroups \( B \) of \( A \) of finite index \(|A : B|\). Suppose that \( A \) contains only a finite number of subgroups of index \( d \) for each positive integer \( d \). Then for any finite group \( K \), the set \( \text{Hom}(A, K) \) of homomorphisms from \( A \) to \( K \) is a finite set. We denote by \(|\text{Hom}(A, K)|\) the number of homomorphisms from \( A \) to a finite group \( K \). Let \( S_n \) be the symmetric group on \( [n] = \{1, \ldots, n\} \) and \( S_0 \) the group consisting of only the identity. In [17] Wohlfahrt proves that

\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(A, S_n)|}{n!} X^n = \exp \left( \sum_{B \in \mathcal{F}_A} \frac{1}{|A : B|} X^{|A : B|} \right). \tag{WF}
\]

This formula interests us in various exponential formulas.

Given a sequence \( \{K_n\}_{n=0}^{\infty} \) of finite groups, the Wohlfahrt series \( E_A(X : \{K_n\}_{0}^{\infty}) \) is the exponential generating function

\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(A, K_n)|}{n!} X^n.
\]

Previous studies of Wohlfahrt series have given some exponential formulas, each of which is a sum of exponential functions. In this paper we extend the approach to the exponential formulas. The approach is based on character theory of finite groups.

Keyword and phrases: generating function, symmetric group, linear character, wreath product, reflection group, finite abelian group.
Let $G$ be a finite group and $G^{(n)}$ the direct product of $n$ copies of $G$. If $H$ is a subgroup of $S_n$, then the wreath product

$$G \wr H = \{(g_1, \ldots, g_n)h \mid (g_1, \ldots, g_n) \in G^{(n)}, h \in H\}$$

is the semidirect product $G^{(n)} \rtimes H$, in which each $h \in H$ acts as an inner automorphism on $G^{(n)}$:

$$h(g_1, \ldots, g_n)h^{-1} = (g_{h^{-1}(1)}, \ldots, g_{h^{-1}(n)}).$$

We consider $G \wr S_0 = S_0$. In [10, 11, 15, 16] the Wohlfahrt formula (WF) is extended to formulas for $E_A(X : \{G \wr S_n\}_0^\infty)$ and $E_A(X/|G| : \{G \wr S_n\}_0^\infty)$ (cf. Corollary 2.7).

Let $1_{S_n}$ be the trivial $C$-character of $S_n$ and $\delta_n$ the linear $C$-character of $S_n$ such that $\delta_n(h)$ is the sign of $h$ for all $h \in S_n$, where $C$ is the complex numbers. We denote by $\mathbf{e}$ the sequence $\{1_{S_n}\}_0^\infty$ and denote by $\text{sgn}$ the sequence $\{\delta_n\}_0^\infty$. Let $\chi$ be a linear $C$-character of $G$, and let $\zeta(\chi, \mathbf{e}, n)$ and $\zeta(\chi, \text{sgn}, n)$ be linear $C$-characters of $G \wr S_n$ defined by

$$\zeta(\chi, \mathbf{e}, n)((g_1, \ldots, g_n)h) = \chi(g_1 \cdots g_n)1_{S_n}(h)$$

and

$$\zeta(\chi, \text{sgn}, n)((g_1, \ldots, g_n)h) = \chi(g_1 \cdots g_n)\delta_n(h)$$

for all $(g_1, \ldots, g_n) \in G^{(n)}$ and $h \in S_n$. Given a linear $C$-character $\zeta$ of $G \wr S_n$, there exists a linear $C$-character $\chi_0$ of $G$ such that $\zeta = \zeta(\chi_0, \mathbf{e}, n)$ or $\zeta = \zeta(\chi_0, \text{sgn}, n)$.

Let $z \in \{\mathbf{e}, \text{sgn}\}$. We define $K(\chi, z, n)$ to be the kernel of $\zeta(\chi, z, n)$, and consider $K(\chi, \mathbf{e}, 0) = K(\chi, \text{sgn}, 0) = S_0$. Let $1_G$ be the trivial $C$-character of $G$, and let $A_n$ be the alternating group on $[n]$. Then $G \wr S_n = K(1_G, \mathbf{e}, n)$ and $G \wr A_n = K(1_G, \text{sgn}, n)$.

The Wohlfahrt series $E_A(X : \{K(1_G, z, n) \cap K(\chi, z, n)\}_0^\infty)$ with $|G/\text{Ker} \chi| \leq 2$ is described as a sum of exponential functions by Müller and Shareshian [11]. The form of $E_A(X/|G| : \{G \wr A_n\}_0^\infty)$ is also studied in [16] (cf. Corollary 2.8). Moreover, $E_A(X/|G| : \{K(\chi, \mathbf{e}, n)\}_0^\infty)$ with $|G/\text{Ker} \chi| = p$, where $p$ is a prime, takes the form of a sum of exponential functions, and so does $E_A(X/|G| : \{K(\chi, \text{sgn}, n)\}_0^\infty)$ with $|G/\text{Ker} \chi| = 2$ [16, Theorem 1].

Given linear $C$-characters $\chi_1, \ldots, \chi_s$ of $G$ and an element $(z_1, \ldots, z_s)$ of the Cartesian product $\{\mathbf{e}, \text{sgn}\}^s$ of $s$ copies of $\{\mathbf{e}, \text{sgn}\}$, we define

$$K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n) = \bigcap_{i \in \{1, \ldots, s\}} K(\chi_i, z_i, n).$$

Every subgroup of $G \wr S_n$ containing the commutator subgroup of $G \wr S_n$ is considered as such a subgroup, because any subgroup of a finite abelian group is expressed as the intersection of kernels of linear $C$-characters. In Section 2 we study the form of

$$\sum_{n=0}^\infty \frac{|\text{Hom}(A, K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n))|}{|G|^n n!} X^n,$$
which is described as a sum of exponential functions (cf. Theorem 2.1).

Let \( m \) be a positive integer, and let \( \omega \) be a primitive \( m \)th root of unity in \( \mathbb{C} \). If \( G \) is the cyclic group \( \langle \omega \rangle \) generated by \( \omega \) and if \( \chi(\omega) = \omega^{m/r} \), where \( r \) is a divisor of \( m \), then we identify \( K(\chi, e, n) \) with the imprimitive complex pseudo-reflection group \( G(m, r, n) \) [8], and define

\[
H(m, r, n) = K(\chi, e, n) \cap (G \wr A_n) = K(\chi, 1_G, e, \text{sgn}, n)
\]

and

\[
L(m, r, n) = K(\chi, \text{sgn}, n).
\]

The form of \( E_A(X/p : \{G(p, p, n)\}_0^\infty) \) and the form of \( E_A(X/2 : \{L(2, 2, n)\}_0^\infty) \) are studied in [16]. In Section 3 we study the form of \( E_A(X/m : \{K_n\}_0^\infty) \) where \( K_n \) is \( G(m, r, n) \), \( H(m, r, n) \) or \( L(m, r, n) \) (cf. Theorem 3.2).

The Weyl group \( W(D_n) \) of type \( D_n \) is isomorphic to \( G(2, 2, n) \). When \( A \) is a finite abelian group, the explicit forms of \( E_A(X : \{G \wr A_n\}_0^\infty) \) and \( E_A(X : \{W(D_n)\}_0^\infty) \) are given in [11]. In Section 4 we study the form of \( E_P(X/p : \{G(p, p, n)\}_0^\infty) \) where \( P \) is a finite abelian \( p \)-group, together with that of \( E_P(X/2 : \{L(2, 2, n)\}_0^\infty) \) and that of \( E_P(X : \{A_n\}_0^\infty) \) where \( P \) is a finite abelian 2-group (cf. Theorems 4.8 and 4.12).

The argument about the descriptions of these Wohlfahrt series is essentially due to Müller and Shareshian (see [11, Section 4]).

In Sections 5 and 6 we present some examples.

2. The form of Wohlfahrt series

Let \( \chi_1, \ldots, \chi_s \) be linear \( \mathbb{C} \)-characters of \( G \), and let \( (z_1, \ldots, z_s) \in \{e, \text{sgn}\}^{(s)} \). In this section we study the form of \( E_A(X/G : \{K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n)\}_0^\infty) \).

Let \( i \in \{1, \ldots, s\} \). Suppose that the factor group \( G/\ker \chi_i \) is of order \( r_i' \). Put \( r_i = r_i' \) if \( r_i' \) is even or \( z_i = e \), and \( r_i = 2r_i' \) otherwise. Then the linear \( \mathbb{C} \)-character \( \zeta(\chi_i, z_i, n) \) is a homomorphism from \( G \wr S_n \) to the cyclic group \( \langle \omega_{r_i} \rangle \) generated by a primitive \( r_i \)th root \( \omega_{r_i} \) of unity in \( \mathbb{C} \). Define

\[
\Phi_{r_i}(A) = \bigcap_{\alpha \in \text{Hom}(A, \langle \omega_{r_i} \rangle)} \ker \alpha.
\]

Then \( \Phi_{r_i}(A) \) is a normal subgroup of \( A \) and the factor group \( A/\Phi_{r_i}(A) \) is a finite abelian group. Write \( R_i = A/\Phi_{r_i}(A) \), and let \( \bar{a} \) denote the coset \( a\Phi_{r_i}(A) \) of \( \Phi_{r_i}(A) \) in \( A \) containing \( a \in A \). Given \( \varphi \in \text{Hom}(A, G \wr S_n) \) and \( \bar{c} \in R_i \), it is clear that \( \zeta(\chi_i, z_i, n)(\varphi(c)) \) with \( c \in \bar{c} \) is independent of the choice of \( c \) in \( \bar{c} \).

Let \( B \in \mathcal{F}_A \). We define a homomorphism \( \text{sgn}_{A/B} \) from \( A \) to \( \mathbb{C} \) by

\[
\text{sgn}_{A/B}(a) = \begin{cases} 
1 & \text{if } a \in A \text{ is an even permutation on } A/B, \\
-1 & \text{if } a \in A \text{ is an odd permutation on } A/B,
\end{cases}
\]
where $A/B$ is the left $A$-set consisting of all left cosets of $B$ in $A$ with the action given by $a.cB = acB$ for all $a, c \in A$.

Suppose that $|A : B| = d$ and $T_B^d = \{a_1, \ldots, a_d\}$ is a left transversal of $B$ in $A$. For each normal subgroup $N$ of $B$ containing the commutator subgroup $B'$, let $V_{A-B/N}$ be the transfer from $A$ to the factor group $B/N$ defined by

$$V_{A-B/N}(a) = \prod_{j=1}^{d} a_j^{-1}aa_jN \quad \text{with} \quad aa_j \in a_j'B$$

for all $a \in A$, which is independent of the choice of $T_B^d$, and is a homomorphism.

Let $\alpha \in \text{Hom}(B, \mathbb{C}^\times)$, $\mathbb{C}^\times$ the multiplicative group of $\mathbb{C}$. Then $B' \leq \text{Ker} \alpha$. Let $\alpha_0$ be the homomorphism from $B'/B''$ to $\mathbb{C}^\times$ defined by $\alpha_0(b'B') = \alpha(b)$ for all $b \in B$. Let $\alpha^{\otimes A}$ be the homomorphism from $A$ to $\mathbb{C}^\times$ given by

$$\alpha^{\otimes A}(a) = \alpha_0(V_{A-B/B'}(a))$$

for all $a \in A$, which is the representation afforded by a tensor induced $CA$-module (see [4, (13.12) Proposition]). Let $\kappa \in \text{Hom}(B, G)$. Given $\overline{a} \in R_i$, it is clear that $(\chi_i \circ \kappa)^{\otimes A}(c)$ with $c \in \pi$ is independent of the choice of $c$ in $\pi$.

Set $I = \{i \mid z_i = \text{sgn}\}$. Given $\overline{a} \in R_i$ with $i \in I$, $\text{sgn}_{A/B}(c)$ with $c \in \overline{a}$ is independent of the choice of $c$ in $\overline{a}$.

Put $R = R_1 \times \cdots \times R_s$. Given $(\overline{a_1}, \ldots, \overline{a_s}) \in R$, we define

$$\rho_B(\overline{a_1}, \ldots, \overline{a_s}) = \text{sgn}_{A/B} \left( \prod_{i \in I} c_i \right) \sum_{\kappa \in \text{Hom}(B, G)} \prod_{i=1}^{s} (\chi_i \circ \kappa)^{\otimes A}(c_i).$$

We are successful in finding the following formula.

**Theorem 2.1**

$$\sum_{n=0}^{\infty} \frac{\left| \text{Hom}(A, K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n) \right|}{|G|^n n!} X^n = \frac{1}{|R|} \sum_{(\overline{a_1}, \ldots, \overline{a_s}) \in R} \exp \left( \sum_{B \in F_A} \frac{\rho_B(\overline{a_1}, \ldots, \overline{a_s})}{|G| |A : B|} X^{[A : B]} \right).$$

Let us prove this theorem. We start with the following lemma, which plays a crucial role in this description of $E_A(X/G) : \{K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n)\}^\infty$).

**Lemma 2.2** Let $\varphi \in \text{Hom}(A, G \wr S_n)$. Then for each integer $i$ with $1 \leq i \leq s$,

$$\frac{1}{|R_i|} \sum_{\overline{a} \in R_i} \zeta(\chi_i, z_i, n)(\varphi(a)) = \begin{cases} 1 & \text{if } \text{Im} \varphi \leq K(\chi_i, z_i, n), \\ 0 & \text{otherwise,} \end{cases}$$

where the sum $\sum_{\overline{a} \in R_i} \zeta(\chi_i, z_i, n)(\varphi(a))$ is over all left cosets $\overline{a} \in R_i$ with $a \in A$. 
Proof. Define a $\mathbb{C}$-character $\alpha_i$ of $R_i$ by setting

$$\alpha_i(\sigma) = \zeta(\chi_i, z_i, n)(\varphi(a))$$

for all $\sigma \in R_i$ with $a \in A$. Then $\text{Im} \varphi \leq K(\chi_i, z_i, n)$ if and only if $\alpha_i$ is the trivial $\mathbb{C}$-character of $R_i$. Hence it follows from the first orthogonality relation [4, (9.21) Proposition] that

$$\frac{1}{|R_i|} \sum_{\sigma \in R_i} \alpha_i(\sigma) = \begin{cases} 1 & \text{if } \text{Im} \varphi \leq K(\chi_i, z_i, n), \\ 0 & \text{otherwise}, \end{cases}$$

which proves the lemma. \qed

This lemma enables us to get the following proposition.

**Proposition 2.3**

$$\sum_{n=0}^{\infty} \frac{|\text{Hom}(A, K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n))|}{n!} X^n = \frac{1}{|R|} \sum_{(c_1, \ldots, c_s) \in R} \left( \prod_{i=1}^{s} \zeta(\chi_i, z_i, n)(\varphi(c_i)) \right) X^n.$$

**Proof.** If $\varphi \in \text{Hom}(A, G \wr S_n)$, then by Lemma 2.2, we have

$$\prod_{i=1}^{s} \left( \frac{1}{|R_i|} \sum_{\sigma \in R_i} \zeta(\chi_i, z_i, n)(\varphi(c_i)) \right) = \begin{cases} 1 & \text{if } \text{Im} \varphi \leq \bigcap_{i \in \{1, \ldots, s\}} K(\chi_i, z_i, n), \\ 0 & \text{otherwise}. \end{cases}$$

Hence it turns out that

$$|\text{Hom}(A, K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n))|$$

$$= \sum_{\varphi \in \text{Hom}(A, G \wr S_n)} \left( \frac{1}{|R_i|} \sum_{\sigma \in R_i} \zeta(\chi_i, z_i, n)(\varphi(c_i)) \right)$$

$$= \frac{1}{|R|} \sum_{(c_1, \ldots, c_s) \in R} \left( \prod_{i=1}^{s} \zeta(\chi_i, z_i, n)(\varphi(c_i)) \right),$$

completing the proof of the proposition. \qed

We consider the Cartesian product $G \times [n]$ of $G$ and $[n]$ to be the left $G \wr S_n$-set with the left action of $G \wr S_n$ given by

$$(g_1, \ldots, g_n)h.(g, i) = (gh(i)g, h(i)).$$
for all \((g_1, \ldots, g_n) \in G^{(n)}\), \(h \in S_n\), and \((g, i) \in G \times [n]\) [9, 2.11], so that \(G \wr S_n\) is isomorphic to the automorphism group of the free right \(G\)-set \(G \times [n]\) with the right action of \(G\) given by \((g, i).y = (gy, i)\) for all \((g, i) \in G \times [n]\) and \(y \in G\) (see [1, Proposition 6.11], [16, Proposition 1]).

Let \(\nu_n\) be the homomorphism from \(G \wr S_n\) to \(S_n\) defined by

\[
\nu_n((g_1, \ldots, g_n)h) = h
\]

for all \((g_1, \ldots, g_n) \in G^{(n)}\) and \(h \in S_n\).

Set \(\mathcal{F}_A(n) = \{B \in \mathcal{F}_A \mid \lvert A : B \rvert \leq n\}\). We now show a recurrence formula like Dey’s theorem [5, (6.10)], namely,

**Proposition 2.4** If \(n\) is a positive integer, then

\[
\sum_{\varphi \in \text{Hom}(A, G \wr S_n)} \prod_{i=1}^{s} \zeta(\chi_i, z_i, n)(\varphi(c_i)) \cdot \frac{|G|^n(n-1)!}{|G|^n(n-1)!}
\]

\[
= \sum_{B \in \mathcal{F}_A(n)} \frac{\rho_B(\tau_1, \ldots, \tau_s)}{|G|} \sum_{\psi \in \text{Hom}(A, G \wr S_{n-\lvert A : B \rvert})} \prod_{i=1}^{s} \zeta(\chi_i, z_i, n-\lvert A : B \rvert)(\psi(c_i)) \cdot \frac{|G|^{n-\lvert A : B \rvert}(n-\lvert A : B \rvert)!}{|G|^{n-\lvert A : B \rvert}(n-\lvert A : B \rvert)!}
\]

with \(c_1, \ldots, c_s \in A\).

The proof is analogous to that of [15, Theorem 3.1].

**Proof of Proposition 2.4.** If \(B \in \mathcal{F}_A\), then we fix a left transversal \(T^A_B\) containing the identity \(\epsilon_A\) of \(A\). We denote by \(\epsilon\) the identity of \(G\).

Let \(\varphi \in \text{Hom}(A, G \wr S_n)\). Define a subgroup \(B\) of \(A\) by

\[
B = \{a \in A \mid \nu_n(\varphi(a))(1) = 1\},
\]

and define a homomorphism \(\kappa\) from \(B\) to \(G\) by

\[
\varphi(b). (\epsilon, 1) = (\kappa(b), 1)
\]

for all \(b \in B\). We then have \(\lvert A : B \rvert \leq n\). Suppose that \(T^A_B = \{a_1, \ldots, a_d\}\) with \(a_1 = \epsilon_A\) and \(d = \lvert A : B \rvert\). Define an injection \(\iota\) from \([d]\) into \([n]\) with \(\iota(1) = 1\) by

\[
\iota(j) = \nu_n(\varphi(a_j))(1)
\]

for all \(j \in [d]\), and define an element \((y_1, \ldots, y_d)\) of the Cartesian product \(G^{(d)}\) of \(d\) copies of \(G\) with \(y_1 = \epsilon\) by

\[
\varphi(a_j). (\epsilon, 1) = (y_j, \iota(j))
\]
for all \( j \in [d] \). If \( a \in A \) and if \( j \in [d] \), then we have
\[
\varphi(a).(\epsilon, \iota(j)) = (y_j.a (a_j^{-1} a a_j^{-1}) y_j^{-1}, \iota(j)) \quad \text{with} \quad a a_j \in a_j B.
\] (I)

Suppose that \( \{\iota(1), \ldots, \iota(d)\} \cup \{k_1, \ldots, k_{n-d}\} = [n] \) and \( k_1 < \cdots < k_{n-d} \). If \( h \in \operatorname{Im}(\nu_n \circ \varphi) \), then we define a permutation \( \hat{h} \) on \([n-d]\) by \( h(k_t) = k_{h(t)} \) for all \( t \in [n-d] \). Let \( \psi \) be the mapping from \( A \) to \( G \wr S_{n-d} \) defined by
\[
\psi(a) = (g_{k_1}, \ldots, g_{k_{n-d}}) \hat{h} \quad \text{with} \quad h = \nu_n(\varphi(a)), \quad \varphi(a) = (g_1, \ldots, g_n) h
\] (II)

for all \( a \in A \). Then it is easily checked that \( \psi \) is a homomorphism.

We have got a quintet \( (B, \kappa, \iota, (y_1, \ldots, y_d), \psi) \) satisfying the condition
\[
\begin{cases}
B \in F_A \text{ with } d = |A : B| \leq n, \\
\kappa \in \text{Hom}(B, G), \\
\iota \text{ is an injection from } [d] \text{ to } [n] \text{ with } \iota(1) = 1 \\
(y_1, \ldots, y_d) \in G^{(d)} \text{ with } y_1 = \epsilon, \\
\psi \in \text{Hom}(A, G \wr S_{n-d})
\end{cases}
\] (III)

and by (I) and (II), we obtain
\[
\prod_{i=1}^s \zeta(\chi_i, z_i, n)(\varphi(c_i)) = \operatorname{sgn}_{A/B} \left( \prod_{i \in I} c_i \right) \cdot \prod_{i=1}^s (\chi_i \circ \kappa)^{\otimes_A}(c_i) \cdot \zeta(\chi_i, z_i, n - d)(\psi(c_i)).
\] (IV)

The preceding map
\[
\Gamma : \varphi \rightarrow (B, \kappa, \iota, (y_1, \ldots, y_d), \psi)
\]
from \( \text{Hom}(A, G \wr S_n) \) to the set of quintets \( (B, \kappa, \iota, (y_1, \ldots, y_d), \psi) \) satisfying (III) is clearly injective. Moreover, it is easily verified that \( \Gamma \) is surjective (see the proof of [15, Theorem 3.1]). Combining this fact with (IV), we have
\[
\sum_{\varphi \in \text{Hom}(A, G \wr S_n)} \prod_{i=1}^s \zeta(\chi_i, z_i, n)(\varphi(c_i))
\]
\[
= \sum_{B \in F_A(n)} \left\{ \rho_B(\overline{t_1}, \ldots, \overline{t_s}) \frac{(n-1)!}{(n-|A : B|)!} |G|^{|A:B|-1} \right\} \times \sum_{\psi \in \text{Hom}(A, G \wr S_{n-d} : A, B)} \prod_{i=1}^s \zeta(\chi_i, z_i, n - |A : B|)(\psi(c_i))
\]
This completes the proof of the proposition. □

If \( \chi_1 = \cdots = \chi_s = 1_G \) and if \( \mathbf{z}_1 = \cdots = \mathbf{z}_s = \mathbf{e} \), then this proposition is the recurrence formula [15, Theorem 3.1] of \( |\text{Hom}(A, G \wr S_n)| \), which is a generalization of the recurrence formula [17, Satz] of \( |\text{Hom}(A, S_n)| \).

As a result of Proposition 2.4, we obtain the following proposition.

**Proposition 2.5** Suppose that \( c_1, \ldots, c_s \in A \). Then

\[
\sum_{n=0}^{\infty} \frac{1}{G^n n!} \left\{ \sum_{\varphi \in \text{Hom}(A, G \wr S_n)} \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \right\} X^n
\]

\[= \exp \left( \sum_{B \in \mathcal{F}_A} \frac{\rho_B(\mathbf{c}_1, \ldots, \mathbf{c}_s)}{|G| |A : B|} X^{|A:B|} \right).\]

**Proof.** Put \( \gamma(n) = \prod_{i=1}^{s} \zeta(\chi_i, \mathbf{z}_i, n)(\varphi(c_i)) \) with \( \varphi \in \text{Hom}(A, G \wr S_n) \), and put \( \beta(B) = \rho_B(\mathbf{c}_1, \ldots, \mathbf{c}_s) \) with \( B \in \mathcal{F}_A \) for convenience. We denote by \( \Xi(n) \) the set of sequences \( \{n_B\} \in \mathcal{F}_A \) of nonnegative integers corresponding to \( B \in \mathcal{F}_A \) such that \( \sum_{B \in \mathcal{F}_A} n_B |A : B| = n \), and abbreviate \( \{n_B\} \in \mathcal{F}_A \) to \( (n_B) \). It suffices to show that for each nonnegative integer \( n \),

\[
\sum_{\varphi \in \text{Hom}(A, G \wr S_n)} \frac{\gamma(n)}{G^n n!} = \sum_{(n_B) \in \Xi(n)} \prod_{B \in \mathcal{F}_A} \frac{\beta(B)^{n_B}}{|G|^n |A : B|^n n_B!}.\]

We use induction on \( n \). Evidently, this formula is true if \( n = 0 \). Suppose that \( n \geq 1 \). Then Proposition 2.4 yields

\[
\sum_{\varphi \in \text{Hom}(A, G \wr S_n)} \frac{\gamma(n)}{G^n n!} = \sum_{B \in \mathcal{F}_A(n)} \frac{\beta(B)}{|G|} \sum_{\psi \in \text{Hom}(A, G \wr S_{n-|A:B|})} \frac{\gamma(n-|A : B|)}{|G|^{n-|A : B|} |n-|A : B|! |A : B|^n n_B!}.\]

Moreover, given \( B \in \mathcal{F}_A(n) \), the inductive assumption means that

\[
\sum_{\psi \in \text{Hom}(A, G \wr S_{n-|A:B|})} \frac{\gamma(n-|A : B|)}{|G|^{n-|A:B|} (n-|A : B|)! |A : B|^n n_B!} = \sum_{(n_K) \in \Xi(n-|A:B|)} \prod_{K \in \mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K} |A : K|^n n_K!}.\]
Hence we obtain
\[
\sum_{\varphi \in \text{Hom}(A, G \wr S_n)} \frac{\gamma_{\varphi}(n)}{|G|^n n!} = \frac{1}{n} \sum_{B \in \mathcal{F}_A(n)} \frac{\beta(B)}{|G|} \sum_{(n_K) \in \Xi(n-|A:B|)} \prod_{K \in \mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K} |A : K|^{n_K n_K!}}
\]
\[
= \frac{1}{n} \sum_{B \in \mathcal{F}_A(n)} \left( \sum_{B \in \mathcal{F}_A(n)} n_B |A : B| \prod_{K \in \mathcal{F}_A} \frac{\beta(K)^{n_K}}{|G|^{n_K} |A : K|^{n_K n_K!}} \right)
\]
\[
= \prod_{(n_K) \in \Xi(n)} \frac{\beta(K)^{n_K}}{|G|^{n_K} |A : K|^{n_K n_K!}}
\]
as required. \[\square\]

**Remark 2.6** Proposition 2.5 is also a consequence of a categorical fact, namely, [16, Proposition 5] (see the second half of the proof of [16, Theorem 1]). It should be stated in this connection that the categorical proof of the Wohlfahrt formula (WF) was given by Yoshida (see [18, 6.4]).

By virtue of Propositions 2.3 and 2.5, we have established Theorem 2.1.

Recall that \(G \wr S_n = K(1_G, e, n)\) and \(G \wr A_n = K(1_G, sgn, n)\). The next results are corollaries to Theorem 2.1.

**Corollary 2.7 ([10, 11, 15, 16])** We have
\[
\sum_{n=0}^{\infty} \frac{\text{Hom}(A, G \wr S_n)}{|G|^{n} n!} X^n = \exp \left( \sum_{B \in \mathcal{F}_A} \frac{\text{Hom}(B, G)}{|G| |A : B|} X^{|A:B|} \right).
\]

**Corollary 2.8 ([16])** We have
\[
\sum_{n=0}^{\infty} \frac{\text{Hom}(A, G \wr A_n)}{|G|^{n} n!} X^n = \frac{1}{|A : \Phi_2(A)|} \sum_{c \in A/\Phi_2(A)} \exp \left( \sum_{B \in \mathcal{F}_A} \frac{\text{sgn}_{A/B}(c) \cdot \text{Hom}(B, G)}{|G| |A : B|} X^{|A:B|} \right).
\]

**Remark 2.9** When \(A\) is a finite cyclic group, Corollary 2.7 is shown in [3, 12] and Corollary 2.8 is shown in [3].
3. Imprimitive complex pseudo reflection groups and related groups

Keep the notation of Section 2, and suppose that $G = \langle \omega \rangle$ with $\omega$ a primitive $m$th root of unity in $\mathbb{C}$. Assume that for any integer $i$ with $1 \leq i \leq s$, $\chi_i(\omega) = \omega^r$, where $q_i$ is a positive integer. Let $B \in \mathcal{F}_A$, and define

$$\Phi_m(B) = \bigcap_{\alpha \in \text{Hom}(B, \omega)} \text{Ker} \alpha.$$ 

Let $i \in \{1, \ldots, s\}$. Since the order of $\langle \omega \rangle / \text{Ker} \chi_i$ divides $r_i$, it follows that $q_i r_i$ is a multiple of $m$. Then the order of $V_{A-B}/\Phi_m(B)(e^\theta)$ with $c \in A$ divides $r_i$. Hence, given $\pi \in R_i$, $V_{A-B}/\Phi_m(B)(e^\theta)$ with $c \in \pi$ is independent of the choice of $c$ in $\pi$.

Now define a homomorphism $F_{A-B}/\Phi_m(B)$ from $R$ to $B/\Phi_m(B)$ by

$$F_{A-B}/\Phi_m(B)(\overline{c_1}, \ldots, \overline{c_s}) = V_{A-B}/\Phi_m(B) \left( \prod_{i=1}^{s} e^{\overline{c_i}} \right)$$

for all $(\overline{c_1}, \ldots, \overline{c_s}) \in R$. Let $c_1, \ldots, c_s \in A$. We can identify $\text{Hom}(B, \langle \omega \rangle)$ with $\text{Hom}(B/\Phi_m(B), \langle \omega \rangle)$. Hence it turns out that

$$\sum_{\kappa \in \text{Hom}(B, \langle \omega \rangle)} \prod_{i=1}^{s} (\chi_i \circ \kappa)^{\otimes A}(c_i) = \sum_{\kappa \in \text{Hom}(B/\Phi_m(B), \langle \omega \rangle)} \prod_{i=1}^{s} \kappa(V_{A-B}/\Phi_m(B)(c_i))^{q_i}$$

$$= \sum_{\kappa \in \text{Hom}(B/\Phi_m(B), \langle \omega \rangle)} \kappa \left( F_{A-B}/\Phi_m(B)(\overline{c_1}, \ldots, \overline{c_s}) \right).$$

Moreover, the $\mathbb{C}$-character

$$\sum_{\kappa \in \text{Hom}(B/\Phi_m(B), \langle \omega \rangle)} \kappa$$

of $B/\Phi_m(B)$ is afforded by the left regular module $\mathbb{C}(B/\Phi_m(B))$. Thus

$$\sum_{\kappa \in \text{Hom}(B, \langle \omega \rangle)} \prod_{i=1}^{s} (\chi_i \circ \kappa)^{\otimes A}(c_i) = \begin{cases} |B : \Phi_m(B)| & \text{if } (\overline{c_1}, \ldots, \overline{c_s}) \in \text{Ker } F_{A-B}/\Phi_m(B), \\ 0 & \text{otherwise.} \end{cases}$$

Combining the preceding fact with Theorem 2.1, we conclude that

$$\sum_{n=0}^{\infty} \frac{|\text{Hom}(A, K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n))|}{m^n n!} X^n$$

$$= \frac{1}{|R|} \sum_{(\overline{c_1}, \ldots, \overline{c_s}) \in R} \exp \left( \sum_{B \in \Omega_A(\overline{c_1}, \ldots, \overline{c_s})} \text{sgn}_{A/B} \left( \prod_{i \in I} c_i \right) \frac{|B : \Phi_m(B)|}{m |A : B|} X^{[A:B]} \right),$$

(V)

where

$$\Omega_A(\overline{c_1}, \ldots, \overline{c_s}) = \left\{ B \in \mathcal{F}_A \mid (\overline{c_1}, \ldots, \overline{c_s}) \in \text{Ker } F_{A-B}/\Phi_m(B) \right\}.$$
Remark 3.1 There exists a divisor $r$ of $m$ such that $K(\chi_1, \ldots, \chi_s, z_1, \ldots, z_s, n)$ is $G(m, r, n)$, $H(m, r, n)$, or $L(m, r, n)$.

The following theorem is an immediate consequence of the formula (V).

Theorem 3.2 Let $r$ be a divisor of $m$. Given $c \in A$, set

$$\Omega_{A}(\bar{c}) = \{B \in \mathcal{F}_A \mid c^{m/r} \in \operatorname{Ker} V_{A \rightarrow B}/\Phi_{m(B)}\}.$$ 

Put $r_0 = r$ if $r$ is even, and $r_0 = 2r$ if $r$ is odd. Then

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, G(m, r, n))|}{m^n n!} X^n = \frac{1}{|A : \Phi_r(A)|} \sum_{\bar{c} \in A/\Phi_r(A)} \exp \left( \sum_{B \in \Omega_{A}(\bar{c})} \frac{|B : \Phi_{m(B)}|}{m|A : B|} X^{[A : B]} \right),$$

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, H(m, r, n))|}{m^n n!} X^n = \frac{1}{|A : \Phi_r(A)|} \frac{|A : \Phi_2(A)|}{|A : \Phi_2(A)|} \times \sum_{(\bar{c}_1, \bar{c}_2) \in (A/\Phi_r(A)) \times (A/\Phi_2(A))} \exp \left( \sum_{B \in \Omega_{A}(\bar{c}_1)} \frac{\operatorname{sgn}_{A/B}(c_2)|B : \Phi_{m(B)}|}{m|A : B|} X^{[A : B]} \right),$$

and

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, L(m, r, n))|}{m^n n!} X^n = \frac{1}{|A : \Phi_{r_0}(A)|} \sum_{\bar{c} \in A/\Phi_{r_0}(A)} \exp \left( \sum_{B \in \Omega_{A}(\bar{c})} \frac{\operatorname{sgn}_{A/B}(c)|B : \Phi_{m(B)}|}{m|A : B|} X^{[A : B]} \right).$$

Corollary 3.3 ([16]) Keep the notation of Theorem 3.2, and assume further that $m = r = 2$. Then

$$\sum_{n=0}^{\infty} \frac{|\operatorname{Hom}(A, W(D_n))|}{2^n n!} X^n = \frac{1}{|A : \Phi_2(A)|} \sum_{\bar{c} \in A/\Phi_2(A)} \exp \left( \sum_{B \in \Omega_{A}(\bar{c})} \frac{|B : \Phi_2(B)|}{2|A : B|} X^{[A : B]} \right).$$
Example 3.4 Suppose that $A$ is a finite cyclic group of order $\ell$ and is generated by an element $c$. Let $p$ be a prime. For a subgroup $B$ of $A$, we have

$$\text{sgn}_{A/B}(c) = \begin{cases} 1 & \text{if } |A : B| \text{ is odd,} \\ -1 & \text{if } |A : B| \text{ is even,} \end{cases}$$

and $V_{A \to B/\Phi_p(B)}(c) = e^{[A:B] \Phi_p(B)}$. Considering $A$ as $\mathbb{Z}/\ell \mathbb{Z}$, we obtain the following.

1. $$\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/\ell \mathbb{Z}, S_n)|}{n!} X^n = \exp \left( \sum_{d|\ell} \frac{1}{d} X^d \right).$$
2. $$\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/\ell \mathbb{Z}, A_n)|}{n!} X^n = \frac{1}{2} \exp \left( \sum_{d|\ell} \frac{1}{pd} X^d \right) + \frac{1}{2} \exp \left( \sum_{d|\ell} \frac{(-1)^{d-1}}{d} X^d \right).$$
3. $$\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/\ell \mathbb{Z}, G(p, p, n))|}{p^n n!} X^n = \frac{1}{p} \exp \left( \sum_{d|\ell, p|(\ell/d)} \frac{1}{pd} X^d \right) \left\{ \exp \left( \sum_{d|\ell, p|(\ell/d)} \frac{1}{d} X^d \right) + p - 1 \right\}.$$
Remark 3.5 The formula (1) is given in [2] and (2) is given in [13, Chapter 4, Problem 22] and [3]. When \( p = 2 \), the formula (3) is shown in [3].

4. Finite abelian \( p \)-groups

Suppose that \( A \) is a finite abelian group. Let \( \hat{A} \) be the set of irreducible \( \mathbb{C} \)-characters of \( A \), and define a multiplication in \( \hat{A} \) by \( \alpha_1 \alpha_2(a) = \alpha_1(a)\alpha_2(a) \) for all \( \alpha_1, \alpha_2 \in \hat{A} \) and \( a \in A \). Then \( \hat{A} \) becomes a group, and the groups \( A \) and \( \hat{A} \) are isomorphic [7, 5.1]. If \( B \) is a subgroup of \( A \), we put

\[
B^\perp = \{ \alpha \in \hat{A} \mid \alpha(b) = 1 \text{ for all } b \in B \}.
\]

If \( U \) is a subgroup of \( \hat{A} \), then we put

\[
U^\perp = \{ a \in A \mid \alpha(a) = 1 \text{ for all } \alpha \in U \}.
\]

We use the following lemmas, which are parts of [7, 5.5, 5.6].

Lemma 4.1 ([7]) Let \( B \) be a subgroup of \( A \). Then

\[
\hat{A}/B \cong B^\perp \quad \text{and} \quad \hat{A}/B^\perp \cong \hat{B}.
\]

Lemma 4.2 ([7]) Let \( B \) be a subgroup of \( A \), and let \( U \) be a subgroup of \( \hat{A} \). Then

\[
B^{\perp \perp} = B \quad \text{and} \quad U^{\perp \perp} = U.
\]

Lemma 4.3 ([7]) Let \( B_1, B_2 \) be subgroups of \( A \). Then

\[
(B_1 \cap B_2)^\perp = B_1^\perp B_2^\perp \quad \text{and} \quad (B_1B_2)^\perp = B_1^\perp \cap B_2^\perp.
\]

Lemma 4.4 ([7]) Let \( U_1, U_2 \) be subgroups of \( \hat{A} \). Then

\[
(U_1 \cap U_2)^\perp = U_1^\perp U_2^\perp \quad \text{and} \quad (U_1U_2)^\perp = U_1^\perp \cap U_2^\perp.
\]

Let \( \epsilon_A \) be the identity of \( A \). For each positive integer \( k \), we define

\[
\Omega_k(A) = \{ a \in A \mid a^k = \epsilon_A \} \quad \text{and} \quad \emptyset_k(A) = \{ a^k \mid a \in A \}.
\]

We provide a part of [7, 5.8], namely,

Lemma 4.5 ([7]) \( \Omega_k(A) = \emptyset_k(\hat{A}) \), and equivalently, \( \Omega_k(A) = \emptyset_k(\hat{A})^\perp \).
A partition is a sequence $\lambda = (\lambda_1, \ldots, \lambda_\ell, \ldots)$ of nonnegative integers containing only finitely many non-zero terms where $\lambda_1 \geq \cdots \geq \lambda_\ell \geq \cdots$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell, \ldots)$, we define

$$m_i(\lambda) = \# \{ t \mid \lambda_t = i \}$$

and

$$\lambda'_i = \# \{ t \mid \lambda_t \geq i \}.$$

Then $\lambda' = (\lambda'_1, \ldots, \lambda'_i, \ldots)$ is a partition, and is called the conjugate of $\lambda$.

Let $p$ be a prime. If $P$ is a finite abelian $p$-group, then there is a unique partition $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0, \ldots)$ such that $P$ is isomorphic to the direct product

$$\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_\ell}\mathbb{Z}$$

of cyclic $p$-groups $\mathbb{Z}/p^{\lambda_1}\mathbb{Z}$, $\ldots$, $\mathbb{Z}/p^{\lambda_\ell}\mathbb{Z}$, and we call $\lambda$ the type of $P$.

Now let $P$ be a finite abelian $p$-group, and let $\epsilon_P$ be the identity of $P$. We have

$$\Phi_p(P) = \mathcal{U}_p(P) \quad \text{and} \quad P/\Phi_p(P) \cong \Omega_p(P).$$

In order to describe the Wohlfahrt series $E_P(X/p : \{G(p, p, n)\}^{\infty}_0)$, we must show the following.

**Lemma 4.6** Let $P_0$ be a subgroup of $P$. Suppose that $c \in P$ and $c \notin \Phi_p(P)$. Then $c \notin \text{Ker} V_{P \rightarrow P_0/\Phi_p(P_0)}$ if and only if $P = \langle c \rangle P_0$ and $P_0$ contains $\Omega_p(P_0)$.

**Proof.** We have $\Phi_p(P_0) = \mathcal{U}_p(P_0)$ and $V_{P \rightarrow P_0/\Phi_p(P_0)}(c) = c^{[P:P_0]}\Phi_p(P_0)$. Assume that $c \notin \text{Ker} V_{P \rightarrow P_0/\Phi_p(P_0)}$. Then $c^{[P:P_0]} \notin \mathcal{U}_p(P_0)$, and thereby $P = \langle c \rangle P_0$. Moreover, if $P_0$ does not contain $\Omega_p(P)$, then $c^{[P:P_0]} = \epsilon_P$, contrary to the assumption. Hence $P_0$ contains $\Omega_p(P)$. Conversely, assume that $P = \langle c \rangle P_0$ and $P_0$ contains $\Omega_p(P)$. Since $c \notin \Phi_p(P)$, it follows that $c \notin \text{Ker} V_{P \rightarrow P_0/\Phi_p(P_0)}$. Hence we assume that $P \neq P_0$. Clearly, $c^{[P:P_0]} \notin P_0$. Now suppose that $c^{[P:P_0]} \in \mathcal{U}_p(P_0)$ and $a$ is an element of $P_0$ such that $a^p = c^{[P:P_0]}$. Then $a^{-1}c^{[P:P_0]}a/p$ is not contained in $P_0$ and is of order $p$. But every element of order $p$ is contained in $P_0$. This is a contradiction. Thus $c^{[P:P_0]} \notin \mathcal{U}_p(P_0)$, and hence $c \notin \text{Ker} V_{P \rightarrow P_0/\Phi_p(P_0)}$, which proves the lemma. \square

Suppose that $P$ is of type $\lambda = (\lambda_1, \ldots, \lambda_\ell, 0, \ldots)$ and $P = \langle a_1 \rangle \times \cdots \times \langle a_\ell \rangle$, where $\langle a_i \rangle$ is a cyclic group generated by $a_i$ and is of order $p^{\lambda_i}$. We assume that $\lambda_\ell > 0$, and set

$$T(P) = \{ a_1^{e_1} \cdots a_\ell^{e_\ell} \mid 0 \leq e_1, \ldots, e_\ell \leq p-1 \},$$

which is a left transversal of $\Phi_p(P)$ in $P$. Given a positive integer $j$, we define $T_j(P)$ to be the set of all elements of order $p^j$ in $T(P)$. Then

$$T_j(P) = p^{\lambda_1 - \lambda_{j+1}} - p^{\lambda_1 - \lambda'_j}.$$

We have the following.
Lemma 4.7 Suppose that \( c \in T_j(P) \). Let \( k \) be a nonnegative integer, and let \( \mathcal{M}(\langle c \rangle; k) \) be the set of all subgroups \( P_0 \) of \( P \) containing \( \Omega_p(P) \) such that \( P = \langle c \rangle P_0 \) and \( |P : P_0| = p^k \). Then

\[
\sharp \mathcal{M}(\langle c \rangle; k) = \begin{cases} 0 & \text{if } k < j, \\ p^{w_\lambda(k)} & \text{if } k \geq j, \end{cases}
\]

where

\[
w_\lambda(k) = \left\{ k \sum_{i=k+1}^{\lambda_1} m_i(\lambda) + \sum_{i=1}^{k} (i-1) m_i(\lambda) \right\} - k.
\]

Proof. Suppose that \( P_0 \in \mathcal{M}(\langle c \rangle; k) \). Then by Lemma 4.5, \( P_0^\perp \) is contained in \( \widehat{\Omega}_p(P) \). Since \( P^\perp = \{1_P\} \), it follows from Lemma 4.3 that \( \langle c \rangle^\perp \cap P_0^\perp = \{1_P\} \), where \( 1_P \) is the trivial character of \( P \). Moreover, by Lemma 4.1 we have

\[
P/P_0 \cong \widehat{P}/P_0 \cong P_0^\perp.
\]

Thus \( P_0^\perp \) is a cyclic group of \( \widehat{\Omega}_p(P) \) such that \( \langle c \rangle^\perp \cap P_0^\perp = \{1_P\} \) and \( |P_0^\perp| = p^k \).

Now let \( \mathcal{N}(\langle c \rangle^\perp; k) \) be the set of all cyclic subgroups \( U \) of \( \widehat{\Omega}_p(P) \) such that \( \langle c \rangle^\perp \cap U = \{1_P\} \) and \( |U| = p^k \). If \( P_0 \in \mathcal{M}(\langle c \rangle; k) \), then by the preceding argument, \( P_0^\perp \in \mathcal{N}(\langle c \rangle^\perp; k) \). Define a map \( f \) from \( \mathcal{M}(\langle c \rangle; k) \) to \( \mathcal{N}(\langle c \rangle^\perp; k) \) by \( f(P_0) = P_0^\perp \) for all \( P_0 \in \mathcal{M}(\langle c \rangle; k) \). Then Lemma 4.2 implies that \( f \) is injective.

Suppose that \( U \in \mathcal{N}(\langle c \rangle^\perp; k) \). Then by Lemma 4.5, \( U^\perp \) contains \( \Omega_p(P) \). Since \( \{1_P\}^\perp = P \), it follows from Lemmas 4.2 and 4.4 that \( P = \langle c \rangle U^\perp \). Moreover, by Lemmas 4.1 and 4.2, we have

\[
P/U^\perp \cong \widehat{P}/U^\perp \cong U,
\]

whence \( |P : U^\perp| = |U| = p^k \). Thus we obtain \( U^\perp \in \mathcal{M}(\langle c \rangle; k) \). This fact, together with Lemma 4.2, means that \( f \) is surjective. Consequently, \( f \) is bijective.

In order to prove the statement, it suffices to verify that

\[
\sharp \mathcal{N}(\langle c \rangle^\perp; k) = \begin{cases} 0 & \text{if } k < j, \\ p^{w_\lambda(k)} & \text{if } k \geq j. \end{cases}
\]

Suppose that \( c = a_1^{e_1} \cdots a_\ell^{e_\ell} \), where \( e_1, \ldots, e_\ell \) are nonnegative integers less than \( p \). Since \( c \not\in \epsilon_P \), we assume that \( e_i = 0 \) with \( i < t_0 \) and \( e_{t_0} \neq 0 \), where \( 1 \leq t_0 \leq \ell \). Put

\[
D = \langle a_1 \rangle \times \cdots \times \langle a_{t_0-1} \rangle \times \langle a_{t_0+1} \rangle \times \cdots \times \langle a_\ell \rangle.
\]

Then \( P = \langle c \rangle \times D \), and hence \( \widehat{P} = \langle c \rangle^\perp \times D^\perp \) by Lemma 4.3. Moreover, it follows from Lemma 4.1 that

\[
D^\perp \cong \widehat{P}/\langle c \rangle^\perp \cong \langle c \rangle^\perp \cong \langle c \rangle \text{ and } \langle c \rangle^\perp \cong \widehat{P}/D^\perp \cong \widehat{D} \cong D.
\]
Thus there exists a bijection from \( N((c)^{\perp}; k) \) to the set \( \mathcal{W}(D; k) \) of all cyclic subgroups \( Y \) of \( \tilde{\mathcal{O}}_d(P) \) such that \( D \cap Y = \{ \epsilon_P \} \) and \( |Y| = p^k \). If \( k \geq j \), then clearly \( \mathcal{W}(D; k) = \emptyset \), and hence \( \sharp N((c)^{\perp}; k) = \sharp \mathcal{W}(D; k) = 0 \). Suppose that \( k < j \).

We set \( I_1 = \{ t \mid \lambda_t > k, t \neq t_0 \} \) and \( I_2 = \{ t \mid \lambda_t \leq k \} \). For each sequence \((n_1, \ldots, n_{t_0-1}, n_{t_0+1}, \ldots, n_{t})\) of positive integers, put

\[
y_{(n_1,\ldots,n_{t_0-1},n_{t_0+1},\ldots,n_{t})} = e^{p^j-k} \left( \prod_{t \in I_1} a_t^{p^{\lambda_t-k} n_t} \right) \left( \prod_{t \in I_2} a_t^{p n_t} \right).
\]

Then

\[
\mathcal{W}(D; k) = \left\{ \left( y_{(n_1,\ldots,n_{t_0-1},n_{t_0+1},\ldots,n_{t})} \right) \mid \begin{array}{ll}
1 \leq n_t \leq p^k & \text{if } t \in I_1, \\
1 \leq n_t \leq p^{\lambda_t-1} & \text{if } t \in I_2,
\end{array} \right\},
\]

and \( \sharp \mathcal{W}(D; k) = p^{w_X(k)} \). Thus we conclude that \( \sharp N((c)^{\perp}; k) = p^{w_X(k)} \), and the proof is completed. \( \square \)

Theorem 3.2, together with Lemmas 4.6 and 4.7, enables us to get the following.

**Theorem 4.8** Keep the notation of Lemma 4.7. We have

\[
\sum_{n=0}^{\infty} |\text{Hom}(P, G(p, p, n))| \frac{X^n}{p^n n!} = \frac{1}{p^k} \left\{ \sum_{n=0}^{\infty} \frac{|\text{Hom}(P, (\mathbb{Z}/p\mathbb{Z}) \wr S_n)|}{p^n n!} X^n \right\} \times \left\{ 1 + \sum_{j \geq 1} (p^{\lambda_j-\lambda_{j+1}} - p^{\lambda_j-\lambda_j}) \exp \left( -p^{j-1} \sum_{k=0}^{j-1} p^{w_X(k)-k} X^p^k \right) \right\}.
\]

We now turn to the forms of \( E_P(X/2 : \{ L(2,2,n) \}_{0}^{\infty}) \) and \( E_P(X : \{ A_n \}_{0}^{\infty}) \).

First, we need a consequence of [15, Lemma 2.1], namely,

**Lemma 4.9** Let \( P_0 \) be a subgroup of \( P \), and let \( c \in P \). Then \( \text{sgn}_{P/P_0}(c) = -1 \) if and only if \( P \neq P_0 \) and \( P = \langle c \rangle P_0 \).

The proof of the next lemma is straightforward.

**Lemma 4.10** Let \( P_0 \) be a subgroup of \( P \), and let \( c \in P - \{ \epsilon_P \} \). Then \( P = \langle c \rangle \times P_0 \) if and only if \( P = \langle c \rangle P_0 \) and \( P_0 \) does not contain \( \Omega_p(P) \).

By an argument similar to that in the proof of Lemma 4.7, we get the following.
Lemma 4.11 Suppose that \( c \in T_j(P) \). Let \( k \) be a nonnegative integer. Then the number of all subgroups \( P_0 \) of \( P \) such that \( P = \langle c \rangle P_0 \) and \( |P : P_0| = p^k \) is 0 if \( k > j \), and is \( p^s_{\lambda}(k) \) if \( k \leq j \), where

\[
s_{\lambda}(k) = \left\{ k \left( \sum_{i=k+1}^{\lambda_1} m_i(\lambda) + \sum_{i=1}^{k} im_i(\lambda) \right) - k. \right.\]

Combining Theorem 3.2 with Lemmas 4.6, 4.9, 4.10, and 4.11, we can now state the following.

Theorem 4.12 Keep the notation of Lemma 4.11, and assume further that \( p = 2 \). Then

\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(P, L(2, 2, n))|}{2^n n!} X^n = \frac{1}{2^\ell} \left\{ \sum_{n=0}^{\infty} \frac{|\text{Hom}(P, (\mathbb{Z}/2\mathbb{Z}) \wr S_n)|}{2^n n!} X^n \right\} \times \left\{ 1 + \sum_{j \geq 1} (2^{\lambda'_1-\lambda'_j+1} - 2^{\lambda'_1-\lambda'_j}) \exp \left( -2^{\ell-1} \sum_{k=0}^{j} 2^{s_{\lambda}(k)-k} X^{2^k} \right) \right\},
\]

and

\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(P, A_n)|}{n!} X^n = \frac{1}{2^\ell} \left\{ \sum_{n=0}^{\infty} \frac{|\text{Hom}(P, S_n)|}{n!} X^n \right\} \times \left\{ 1 + \sum_{j \geq 1} (2^{\lambda'_1-\lambda'_j+1} - 2^{\lambda'_1-\lambda'_j}) \exp \left( -2^{\sum_{k=1}^{j} 2^{s_{\lambda}(k)-k} X^{2^k}} \right) \right\}.
\]

Remark 4.13 The form of \( E_P(X : \{ A_n \}_{0}^{\infty}) \) in the theorem above is also a consequence of Lemma 4.11 and [15, Theorem 1.1].

5. Explicit formulas

Keep the notation of Section 4, and further assume that \( \lambda_1 = \cdots = \lambda_{\ell-1} = u \) and \( \lambda_\ell = v \), where \( \ell \geq 1 \) and \( u \geq v > 0 \). Then \( P \simeq (\mathbb{Z}/p^u \mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/p^v \mathbb{Z} \), whence \( \sharp T_u(P) = p^\ell - p \) and \( \sharp T_v(P) = p - 1 \).
Example 5.1 By Theorem 4.8, we have
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}((\mathbb{Z}/p^n\mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/p^n\mathbb{Z}, G(p, p, n))|}{p^n n!} X^n
\]
\[
= \frac{1}{p^\ell} \left\{ \sum_{n=0}^{\infty} \frac{|\text{Hom}((\mathbb{Z}/p^n\mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/p^n\mathbb{Z}, (\mathbb{Z}/p\mathbb{Z}) * S_n)|}{p^n n!} X^n \right\}
\times \left\{ 1 + (p-1) \exp \left( -p^{\ell-1} \sum_{k=0}^{v-1} p^{(\ell-2)k} X^k \right) \right.
\left. + (p^\ell - p) \exp \left( -p^{\ell-1} \sum_{k=0}^{w-1} p^{(\ell-2)k} X^k - p^{\ell-1} \sum_{k=v}^{w-1} p^{(\ell-3)k+v-1} X^k \right) \right\}.
\]

By Theorem 4.12,
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}((\mathbb{Z}/2^n\mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/2^n\mathbb{Z}, L(2, 2, n))|}{2^n n!} X^n
\]
\[
= \frac{1}{2^\ell} \left\{ \sum_{n=0}^{\infty} \frac{|\text{Hom}((\mathbb{Z}/2^n\mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/2^n\mathbb{Z}, (\mathbb{Z}/2\mathbb{Z}) * S_n)|}{2^n n!} X^n \right\}
\times \left\{ 1 + \exp \left( -2^{\ell-1} \sum_{k=1}^{v} 2^{(\ell-2)k} X^k \right) \right.
\left. + (2^\ell - 2) \exp \left( -2^{\ell-1} \sum_{k=0}^{v} 2^{(\ell-2)k} X^k + 2^{\ell-1} \sum_{k=v+1}^{w} 2^{(\ell-3)k+v} X^k \right) \right\},
\]
and
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}((\mathbb{Z}/2^n\mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/2^n\mathbb{Z}, A_n)|}{n!} X^n
\]
\[
= \frac{1}{2^\ell} \left\{ \sum_{n=0}^{\infty} \frac{|\text{Hom}((\mathbb{Z}/2^n\mathbb{Z})^{(\ell-1)} \times \mathbb{Z}/2^n\mathbb{Z}, S_n)|}{n!} X^n \right\}
\times \left\{ 1 + \exp \left( -2 \sum_{k=1}^{v} 2^{(\ell-2)k} X^k \right) \right.
\left. + (2^\ell - 2) \exp \left( -2 \sum_{k=1}^{v} 2^{(\ell-2)k} X^k - 2 \sum_{k=v+1}^{w} 2^{(\ell-3)k+v} X^k \right) \right\}.
\]

Remark 5.2 The formulas of $E_P(X : \{W(D_n)\})_0^\infty$ and $E_P(X : \{A_n\})_0^\infty$ where $P = (\mathbb{Z}/2^n\mathbb{Z})^{(\ell)}$ are due to Müller and Shareshian [11].
We next suppose that $P \cong \mathbb{Z}/p^u\mathbb{Z} \times \mathbb{Z}/p^v\mathbb{Z}$, where $u \geq v > 0$. Given a nonnegative integer $k$, let $N_P(k)$ be the number of subgroups of order $p^k$ in $P$.

**Proposition 5.3** Let $k$ be a nonnegative integer. Then

$$N_P(k) = \begin{cases} 
1 + p + \cdots + p^k & \text{if } 0 \leq k < v, \\
1 + p + \cdots + p^v & \text{if } v \leq k \leq u, \\
1 + p + \cdots + p^{u+v-k} & \text{if } u < k \leq u + v.
\end{cases}$$

**Proof.** We proceed by induction on $u+v$. Obviously, the assertion is true if $u+v = 0$. Assume that $u+v > 0$ and $P = \langle a \rangle \times \langle b \rangle$, where $a$ has order $p^u$ and $b$ order $p^v$. Put $M = \langle a^p \rangle \times \langle b \rangle$. If $k < v$, then $N_P(k) = N_M(k)$ because every subgroup of order less than $p^v$ is contained in $M$, and hence by the inductive assumption,

$$N_P(k) = 1 + p + \cdots + p^k.$$

Case (1) Assume that $u = v$. Then by [14, Corollary], we obtain

$$N_P(v) = N_M(v-1) + p^v.$$

Hence by the inductive assumption,

$$N_P(v) = 1 + p + \cdots + p^v.$$

Case (2) Assume that $u > v$. If $v \leq k < u$, then clearly $N_P(k) = N_M(k)$. Moreover, it follows from [14, Corollary] that

$$N_P(u) = N_M(u-1).$$

Hence if $v \leq k \leq u$, then by the inductive assumption,

$$N_P(k) = 1 + p + \cdots + p^v.$$

Since $N_P(k) = N_P(u + v - k)$, the assertion of the proposition follows. □

**Lemma 5.4** Let $k$ be a positive integer. Then the number of cyclic subgroups of order $p^k$ in $P$ is $p^{k-1} + p^k$ if $0 < k \leq v$, and is $p^v$ if $v < k \leq u$.

The next result is a consequence of Proposition 5.3 and Lemma 5.4.
Proposition 5.5 We have
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/p^n \mathbb{Z} \times \mathbb{Z}/p^n \mathbb{Z}, S_n)|}{p^n n!} X^n
= \exp \left( \sum_{k=0}^{u-1} \frac{1 + \cdots + p^k}{p^k} X^{p^k} + \sum_{k=v}^{u} \frac{1 + \cdots + p^v}{p^k} X^{p^k} + \sum_{k=u+1}^{u+v} \frac{1 + \cdots + p^{u+v-k}}{p^k} X^{p^k} \right)
\]
and
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/p^n \mathbb{Z} \times \mathbb{Z}/p^n \mathbb{Z}, (\mathbb{Z}/p\mathbb{Z}) \wr S_n)|}{p^n n!} X^n
= \exp \left( \sum_{k=0}^{u-1} \frac{p + \cdots + p^{k+1}}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p + \cdots + p^{v}}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p^v}{p^k} X^{p^k} + \sum_{k=u}^{u+v-1} \frac{p + \cdots + p^{u+v-k}}{p^k} X^{p^k} + \sum_{k=u}^{u+v-1} \frac{p^{u+v-k-1}}{p^k} X^{p^k} \right)
\]

We are now in position to determine the form of \(E_P(X/p : \{G(p,p,n)\}_0^\infty)\), \(E_P(X/2 : \{L(2,2,n)\}_0^\infty\), and \(E_P(X : \{A_n\}_0^\infty\).

Theorem 5.6 We have
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/p^n \mathbb{Z} \times \mathbb{Z}/p^n \mathbb{Z}, G(p,p,n))|}{p^n n!} X^n
= \frac{1}{p^2} \exp \left( \sum_{k=1}^{u-1} \frac{p + \cdots + p^k}{p^k} X^{p^k} + \sum_{k=v}^{u-1} \frac{p + \cdots + p^{v}}{p^k} X^{p^k} \right.
+ \left. \sum_{k=u}^{u+v-1} \frac{p + \cdots + p^{u+v-k}}{p^k} X^{p^k} + \sum_{k=u}^{u+v-1} \frac{p^{u+v-k-1}}{p^k} X^{p^k} \right)
\times \left\{ \exp \left( \sum_{k=0}^{u-1} p X^{p^k} + \sum_{k=v}^{u-1} \frac{p^v}{p^k} X^{p^k} \right) + (p-1) \exp \left( \sum_{k=u}^{u+v-1} \frac{p^{u+v-k}}{p^k} X^{p^k} \right) + p(p-1) \right\},
\]
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/2^n \mathbb{Z} \times \mathbb{Z}/2^n \mathbb{Z}, L(2, 2, n))|}{2^n n!} X^n
\]
\[
= \frac{1}{2^2} \exp \left( \sum_{k=1}^{u+v-1} \frac{2 + \cdots + 2^k}{2k} X^{2k} + \sum_{k=1}^{u+v} \frac{2 + \cdots + 2^{u+v-k}}{2k} X^{2k} - \sum_{k=1}^{u-1} \frac{2^v}{2k} X^{2k} \right)
\times \left\{ \exp \left( \sum_{k=0}^{v} 2X^{2k} + \sum_{k=v+1}^{u} \frac{2^{v+1}}{2k} X^{2k} \right) + \exp \left( \sum_{k=v+1}^{u} \frac{2^{v+1}}{2k} X^{2k} \right) + 2 \right\},
\]
and
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}/2^n \mathbb{Z} \times \mathbb{Z}/2^n \mathbb{Z}, A_n)|}{n!} X^n
\]
\[
= \frac{1}{2^2} \exp \left( X - \sum_{k=1}^{u} \frac{1}{2k} X^{2k} + \sum_{k=u+1}^{u+v} \frac{1 + \cdots + 2^{u+v-k}}{2k} X^{2k} \right)
\times \left\{ \exp \left( \sum_{k=1}^{v} 2X^{2k} + \sum_{k=v+1}^{u} \frac{2^{v+1}}{2k} X^{2k} \right) + \exp \left( \sum_{k=v+1}^{u} \frac{2^{v+1}}{2k} X^{2k} \right) + 2 \right\}.
\]

**Remark 5.7** In [15, Example 6.2], the formula of \( E_P(X/2 : \{W(D_n)\}_0) \), where \( P = \mathbb{Z}/2^n \mathbb{Z} \times \mathbb{Z}/2^n \mathbb{Z} \), is not correct, and neither is the formula of \( E_P(X : \{A_n\}_0) \); either of them has a wrong term.

6. The additive group of \( p \)-adic integers

Let \( \mathbb{Z}_p \) be the additive group of \( p \)-adic integers. The subgroups of finite index in \( \mathbb{Z}_p \) are \( p^k \mathbb{Z}_p, \) \( k = 0, 1, 2, \ldots \). Moreover, \( \mathbb{Z}_p/p^k \mathbb{Z}_p \cong \mathbb{Z}/p^k \mathbb{Z} \) for each nonnegative integer \( k \). In [6] Dress and Yoshida pointed out that
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}_p, S_n)|}{n!} X^n = \exp \left( \sum_{k=0}^{\infty} \frac{1}{p^k} X^{p^k} \right);
\]
this is called the Artin-Hasse exponential. We conclude this paper with a presentation of the following consequences of Theorem 3.2:
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}_p, A_n)|}{n!} X^n = \frac{1}{2} \exp \left( \sum_{k=0}^{\infty} \frac{1}{2^k} X^{2^k} \right) + \frac{1}{2} \exp \left( X - \sum_{k=1}^{\infty} \frac{1}{2^k} X^{2^k} \right);
\]
\[
\sum_{n=0}^{\infty} \frac{|\text{Hom}(\mathbb{Z}_p, G(p, p, n))|}{p^n n!} X^n = \frac{1}{p} \exp \left( \sum_{k=0}^{\infty} \frac{1}{p^k} X^{p^k} \right) + \frac{p - 1}{p}.
\]
References


