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Abstract

We consider a certain Dirichlet series of Rankin-Selberg type associated with two Siegel cusp forms of the same integral weight with respect to $\mathrm{Sp}_n(\mathbb{Z})$. In particular, we give an explicit formula for the Dirichlet series associated with the Ikeda lifting of cuspidal Hecke eigenforms with respect to $\mathrm{SL}_2(\mathbb{Z})$. We also comment on a contribution to the Ikeda's conjecture on the period of the lifting.

Key words: Siegel modular forms, Jacobi forms, Periods.

1 Introduction

Let $n > 1$ and let F and G be Siegel modular forms of degree n and integral weight k. For any positive integer N, we denote by ϕ_N and ψ_N the N-th Fourier-Jacobi coefficients of F and G, respectively. If F and G are cusp forms, not necessarily Hecke eigenforms, then we define a certain Dirichlet series $D_1(s; F, G)$ associated with F and G, which can be viewed as a generalization of the Rankin-Selberg convolution series for elliptic cusp forms. Namely, it is defined by

$$D_1(s; F, G) := \zeta(2s - 2k + 2n) \sum_{N=1}^{\infty} \langle \phi_N, \psi_N \rangle N^{-s},$$

where $\zeta(s)$ is the Riemann zeta function and we denote by $\langle *, * \rangle$ the Petersson inner product defined on the space of Jacobi cusp forms of degree $n-1$, weight k and index N. We easily see by an analogy of the standard Hecke's method that $D_1(s; F, G)$ converges absolutely for $\mathrm{Re}(s) > k+1$.

Furthermore, T. Yamazaki ([13]) proved by using the Rankin-Selberg method with a certain non-holomorphic Eisenstein series of Klingen-Siegel type that $D_1(s; F, G)$ has the following analytic properties:

Theorem 1 (cf. Theorems 3.4 and 3.5 in [13]). *Let*

$$\Gamma_{n,k}(s) := \pi^{k-n}(2\pi)^{-2s}\Gamma(s)\Gamma(s-k+n),$$

where $\Gamma(s)$ is the gamma function. Then the function

$$\mathscr{D}_1(s; F, G) := \Gamma_{n,k}(s)\, D_1(s; F, G)$$

is holomorphic on the entire complex plane except for simple poles of residue $\langle F, G \rangle$ at $s = k$ and $s = k - n$, where we denote by $\langle *, * \rangle$ the Petersson inner product defined on the space of Siegel cusp forms of degree n and weight k. Furthermore, it satisfies a functional equation

$$\mathscr{D}_1(s; F, G) = \mathscr{D}_1(2k - n - s; F, G).$$

Here we note that the type of the functional equation of $\mathscr{D}_1(s; F, G)$ is same as that of the Hecke L-function $L(s, f)$ associated with a cuspidal Hecke eigenform f of degree 1 and weight $2k - n$.

On the other hand, let n and k be positive even integers such that $k > n+1$. For a normalized cuspidal Hecke eigenform f of degree 1 and weight $2k - n$, we consider the so-called Ikeda lifting of f into the space of Siegel cusp forms of degree n and weight k. Namely, there exists a cuspidal Hecke eigenform $I_{n,k}(f)$ of degree n and weight k whose standard L-function is equal to

$$\zeta(s)\prod_{i=1}^{n} L(s+k-i, f).$$

We note that the Ikeda lifting coincides with the Saito-Kurokawa lifting in case $n = 2$.

The main result in this paper is the following:

Main Theorem. *Let n and k be positive even integers such that $k > n+1$. If f is a normalized cuspidal Hecke eigenform of degree 1 and weight $2k - n$, then*

$$D_1(s; I_{n,k}(f), I_{n,k}(f)) = \langle \phi_1, \phi_1 \rangle\, \zeta(s-k+1)\zeta(s-k+n)L(s, f), \qquad (1)$$

where ϕ_1 is the first Fourier-Jacobi coefficient of $I_{n,k}(f)$.

We easily see that the gamma factor $\Gamma_{n,k}(s)$ is a constant multiple of

$$\prod_{i=0}^{n/2-1} (s - k + 2i + 1)\, \Gamma_{\mathbb{R}}(s - k + 1)\Gamma_{\mathbb{R}}(s - k + n)\Gamma_{\mathbb{C}}(s),$$

where we denote by $\Gamma_{\mathbb{R}}(s)$ and $\Gamma_{\mathbb{C}}(s)$ the gamma factors of $\zeta(s)$ and $L(s, f)$, respectively (cf. §6 below). Therefore the equation (1) agrees with Theorem 1.

By comparing residues at $s = k$ on the both sides of (1), we also obtain the following equation:

Corollary. *Under the same assumption as above, we have*

$$(-1)^{n/2+1}\, \pi^k \cdot \frac{2^{2k-n+1}\, n}{(k-1)!\, B_n} \cdot \frac{\langle I_{n,k}(f),\, I_{n,k}(f)\rangle}{\langle \phi_1,\, \phi_1 \rangle} = L(k,\, f), \qquad (2)$$

where B_n is the n-th Bernoulli number.

The equations (1) and (2) are generalizations of the well-known formulae for the Saito-Kurokawa lifting, which were obtained by W. Kohnen and N.-P. Skoruppa ([9]). We also obtain a new proof of them in case $n = 2$.

This paper consists of the 5 sections as follows: The first §2 and §3 are devoted to the reviews of the Ikeda lifting (cf. [6]) and some basic facts on Jacobi forms of integral index (cf. [11] and [12]), respectively. In §4, we study certain linear operators acting on the graded ring of Jacobi forms and their adjoints with respect to Petersson inner products. In particular, we prove some multiplicative relations between them, which play important roles in the proof of the main theorem in §5. Finally, in §6, we comment on a contribution to the Ikeda's conjecture on the period of the Ikeda lifting (cf. [7]).

Notation. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ the set of natural numbers, the ring of rational integers, the field of rational numbers, the field of real numbers and the field of complex numbers, respectively. For any commutative ring R, we denote by $\mathrm{M}_{m,n}(R)$ the set of $m \times n$ matrices with entries in R, and especially write $\mathrm{M}_n(R) = \mathrm{M}_{n,n}(R)$ and $R^n = \mathrm{M}_{1,n}(R)$. We denote by 1_n, $0_n \in \mathrm{M}_n(R)$ the unit matrix and the zero matrix of size n, respectively. Let $\mathrm{GL}_n(R)$ be the group of all invertible elements of $\mathrm{M}_n(R)$, and $\mathcal{S}_n(R)$ be the set of symmetric matrices of size n with entries in R. For any integral domain R, let $\mathcal{H}_n(R)$ be the set of half-integral symmetric matrices of size n over R, that is,

$$\mathcal{H}_n(R) := \{T = (t_{ij}) \in \mathcal{S}_n(Q(R)) \,|\, t_{ii} \in R\ (1 \leq i \leq n),\, 2\,t_{ij} \in R\ (1 \leq i \neq j \leq n)\},$$

where $Q(R)$ is the quotient field of R. If $R = \mathbb{Z}$, we denote by $\mathcal{H}_n(\mathbb{Z})_{\geq 0}$ and $\mathcal{H}_n(\mathbb{Z})_{>0}$ the subsets of $\mathcal{H}_n(\mathbb{Z})$ consisting of all positive semi-definite and definite half-integral symmetric matrices, respectively. For any commutative ring R, matrices $X \in \mathrm{M}_{m,n}(R)$ and $A \in \mathrm{M}_m(R)$, we write $A[X] = {}^t X A X \in \mathrm{M}_n(R)$, where ${}^t X$ denotes the transpose of X. For any $r_1, \cdots, r_n \in R$, we denote by $\mathrm{diag}(r_1, \cdots, r_n)$ the diagonal matrix with entries $r_1, \cdots, r_n$, that is,

$$\mathrm{diag}(r_1, \cdots, r_n) := \begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}.$$

For any $A \in \mathrm{M}_n(R)$, we denote by $\mathrm{tr}(A)$ and $\det(A)$ the trace and the determinant of A, respectively.

Let S_n, G_n, Γ_n be the subgroup of the real general symplectic group consisting of all elements with positive similitudes, the real symplectic group and the Siegel modular group, respectively. Namely,

$$S_n := \mathrm{GSp}_n^+(\mathbb{R}) = \{M \in \mathrm{M}_{2n}(\mathbb{R}) \,|\, {}^t M J_n M = \nu J_n \text{ for some } \nu > 0\},$$
$$G_n := \mathrm{Sp}_n(\mathbb{R}) = \{M \in \mathrm{M}_{2n}(\mathbb{R}) \,|\, {}^t M J_n M = J_n\},$$
$$\Gamma_n := \mathrm{Sp}_n(\mathbb{Z}) = G_n \cap \mathrm{M}_{2n}(\mathbb{Z}),$$

where $J_n = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}$. For any $M \in S_n$, we denote by $\nu(M)$ the similitude of M, that is, ${}^t M J_n M = \nu(M) J_n$. For any $N \in \mathbb{N}$, we denote by $\Gamma_0^{(n)}(N)$ a congruence subgroup of Γ_n defined by

$$\Gamma_0^{(n)}(N) := \{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \Gamma_n \,|\, C \equiv 0_n \pmod{N} \}.$$

We denote the Siegel upper-half space of degree n by $\mathbb{H}_n$, that is,

$$\mathbb{H}_n := \{Z = X + \sqrt{-1}\, Y \in \mathcal{S}_n(\mathbb{C}) \mid Y > 0 \text{ (positive definite)}\}.$$

For any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in S_n$ and $Z \in \mathbb{H}_n$, we put $M\langle Z \rangle := (AZ+B)(CZ+D)^{-1}$. As is well-known, this defines an action of S_n on $\mathbb{H}_n$. In particular, the group G_n acts transitively on $\mathbb{H}_n$. For any $k \in \mathbb{Z}$, a holomorphic function $F(Z)$ on $\mathbb{H}_n$ is called a *(holomorphic) Siegel modular form* of degree n and weight k if it satisfies the following two conditions:
(i) $F(M\langle Z \rangle) = \det(CZ + D)^k F(Z)$ for any $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right) \in \Gamma_n$,
(ii) F has a Fourier expansion of the form

$$F(Z) = \sum_{T \in \mathcal{H}_n(\mathbb{Z})_{\geq 0}} A(T) \exp(2\pi \sqrt{-1}\, \mathrm{tr}(TZ)).$$

If F satisfies the stronger condition $A(T) = 0$ unless $T > 0$ (positive definite), it is called a *Siegel cusp form*.

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We denote by $M_k(\Gamma_n)$ and $S_k(\Gamma_n)$ the $\mathbb{C}$-vector spaces of (holomorphic) Siegel modular forms and Siegel cusp forms of degree n and weight k, respectively. We note that if $n > 1$, then the condition on Fourier coefficients in (ii) follows from the condition (i) (*Koecher's principle*). If $F, G \in M_k(\Gamma_n)$ and $FG \in S_{2k}(\Gamma_n)$, then we can define *the Petersson inner product* of F and G by

$$\langle F, G \rangle := \int_{\Gamma_n \backslash \mathbb{H}_n} F(Z)\overline{G(Z)} \det(Y)^{k-n-1} dX dY,$$

where $Z = X + \sqrt{-1}\, Y \in \mathbb{H}_n$. As is well-known, the Petersson inner product defines a hermitian inner product on $S_k(\Gamma_n)$. For further details on the facts of Siegel modular forms set out above, see [1] or [4].

For any prime number p, let $\mathbb{Q}_p$ be the field of p-adic numbers, and let $\mathbb{Z}_p$ and $\mathbb{Z}_p^\times$ be the ring of p-adic integers and the group of p-adic units, respectively. Let $\mathrm{ord}_p(*)$ denote the p-adic order. For any complex number x, we put $e(x) := \exp(2\pi\sqrt{-1}x)$ and $e^m(x) := e(mx)$ ($m \in \mathbb{N}$), and for any p-adic number x, we put $e_p(x) = e(x')$, where x' denotes the fractional part of x.

2 Review of the Ikeda lifting

Let n be a positive even integer throughout this section.

For any $B \in \mathcal{H}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$, we denote by

$$\mathfrak{D}_B := (-1)^{n/2} \det(2B)$$

the discriminant of B. Then $\mathfrak{D}_B \equiv 0,\, 1 \pmod 4$ and we write

$$\mathfrak{D}_B = \mathfrak{d}_B \cdot \mathfrak{f}_B{}^2$$

with the corresponding fundamental discriminant $\mathfrak{d}_B \in \mathbb{Z}$ and $\mathfrak{f}_B \in \mathbb{N}$. Namely, $\mathfrak{d}_B$ is the discriminant of the quadratic extention $\mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q}$ and $\mathfrak{f}_B = \sqrt{\frac{\mathfrak{D}_B}{\mathfrak{d}_B}}$.

2.1 The Siegel series

For any $B \in \mathcal{H}_n(\mathbb{Z})$, we define *the Siegel series* by

$$b(B;\, s) := \sum_{R \in \mathcal{S}_n(\mathbb{Q})/\mathcal{S}_n(\mathbb{Z})} e(\mathrm{tr}(BR)) \cdot \mu(R)^{-s},$$

where $\mu(R)$ is the product of denominators of elementary divisors of R.

Let k be a non-negative even integer. For any $Z \in \mathbb{H}_n$ and $s \in \mathbb{C}$, we define the so-called non-holomorphic Siegel Eisenstein series of degree n and weight k by

$$E_k^{(n)}(Z, s) = \sum_{\{C, D\}} \det(CZ + D)^{-k} |\det(CZ + D)|^{-2s} \det(\mathrm{Im}(Z))^s,$$

where $\{C, D\}$ runs over a complete set of representatives of the equivalence classes of coprime symmetric pairs of size n. As is well-known, the non-holomorphic Siegel Eisenstein series can be expressed by using the Siegel series and the so-called confluent hypergeometric function.

Remark. If k is an even integer such that $k > n + 1$, then for any $B \in \mathcal{H}_n(\mathbb{Z})_{>0}$, the B-th Fourier coefficient $A_{n,k}(B)$ of the (holomorphic) Siegel Eisenstein series $E_{n,k}(Z) := E_k^{(n)}(Z, 0) \in M_k(\Gamma_n)$ is given by

$$A_{n,k}(B) = (-1)^{nk/2} 2^{nk-(n-1)n/2} \prod_{i=2k-n+1}^{2k} \frac{\pi^{i/2}}{\Gamma(i/2)} (\det B)^{(2k-n-1)/2} b(B; k).$$

To investigate the Siegel series, for a prime number p and any $B \in \mathcal{H}_n(\mathbb{Z}_p)$, we define *the local Siegel series* by

$$b_p(B; s) := \sum_{R \in \mathcal{S}_n(\mathbb{Q}_p)/\mathcal{S}_n(\mathbb{Z}_p)} e_p(\mathrm{tr}(BR)) \cdot \mu_p(R)^{-s},$$

where $\mu_p(R) = p^{\mathrm{ord}_p(\mu(R))}$. Then we easily see that

$$b(B; s) = \prod_{p : \mathrm{prime}} b_p(B; s)$$

for any $B \in \mathcal{H}_n(\mathbb{Z})$.

For any $B \in \mathcal{H}_n(\mathbb{Z}_p) \cap \mathrm{GL}_n(\mathbb{Q}_p)$, we define a polynomial $\gamma_p(B; X) \in \mathbb{Z}[X]$ by

$$\gamma_p(B; X) := (1 - X) \prod_{i=1}^{n/2} (1 - p^{2i} X^2) \cdot (1 - p^{n/2} \chi_B(p) X)^{-1},$$

where χ_B is the Kronecker character corresponding to $\mathbb{Q}(\sqrt{\mathfrak{D}_B})/\mathbb{Q}$. Then there exists a polynomial $F_p(B; X) \in \mathbb{Z}[X]$ whose constant term is equal to 1 such that

$$b_p(B; s) = \gamma_p(B; p^{-s}) \cdot F_p(B; p^{-s}).$$

Thus for any $B \in \mathcal{H}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$, we have

$$b(B; s) = \frac{L(s - n/2, \chi_B)}{\zeta(s) \prod_{i=1}^{n/2} \zeta(2s - 2i)} \prod_{p : \mathrm{prime}} F_p(B; p^{-s}),$$

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where $L(s, \chi_B)$ is the Dirichlet L-function associated with χ_B.

One of the authors ([8]) proved that $F_p(B; X)$ satisfies a certain induction formula via the theory of local densities. By using this, we can explicitly compute $F_p(B; X)$ for any $B \in \mathcal{H}_n(\mathbb{Z}) \cap \mathrm{GL}_n(\mathbb{Q})$ and any prime number p. He also proved in [8] that for any $B \in \mathcal{H}_n(\mathbb{Z})$, the polynomial $F_p(B; X)$ satisfies the functional equation

$$F_p(B; p^{-(n+1)} X^{-1}) = (p^{(n+1)/2} X)^{-2\mathrm{ord}_p(\mathfrak{f}_B)} F_p(B; X).$$

Thus the Laurent polynomial

$$\widetilde{F}_p(B; X) := X^{-\mathrm{ord}_p(\mathfrak{f}_B)} F_p(B; p^{-(n+1)/2} X)$$

is reciprocal, that is, $\widetilde{F}_p(B; X) \in \mathbb{C}[X + X^{-1}]$. In particular,

$$\widetilde{F}_p(B; X^{-1}) = \widetilde{F}_p(B; X).$$

It follows that $\deg F_p(B; X) = 2 \, \mathrm{ord}_p(\mathfrak{f}_B)$. Moreover, if $p \nmid \mathfrak{f}_B$ then

$$F_p(B; X) = \widetilde{F}_p(B; X) = 1.$$

2.2 The Ikeda lifting

Let k be an even integer such that $k > n + 1$. Let

$$f(z) = \sum_{N=1}^{\infty} a(N) \, e(Nz) \in S_{2k-n}(\Gamma_1) \quad (z \in \mathbb{H}_1),$$

be a normalized Hecke eigenform. Then the Hecke L-function associated with f is defined by

$$L(s, f) := \sum_{N=1}^{\infty} a(N) N^{-s} = \prod_{p:\mathrm{prime}} (1 - a(p) \, p^{-s} + p^{2k-(n+1)-2s})^{-1}.$$

For a prime number p, let $\alpha_p \in \mathbb{C}$ be the p-th Satake parameter of f. Namely, α_p is an algebraic number such that

$$\alpha_p + \alpha_p^{-1} = a(p) \, p^{-k+(n+1)/2}.$$

Then we have

$$L(s, f) = \prod_{p:\mathrm{prime}} \{(1 - \alpha_p \, p^{k-(n+1)/2-s})(1 - \alpha_p^{-1} \, p^{k-(n+1)/2-s})\}^{-1}.$$

We note that α_p is uniquely determined up to inversion.

Let

$$h(\tau) = \sum_{\substack{N \geq 1, \\ (-1)^{k-n/2} N \equiv 0, 1 \ (\mathrm{mod}\ 4)}} c(N)\, e(N\tau) \in S^+_{k-(n-1)/2}(\Gamma_0^{(1)}(4)) \quad (\tau \in \mathbb{H}_1)$$

be a Hecke eigenform which corresponds to f under the Shimura correspondence, where we denote by $S^+_{k-(n-1)/2}(\Gamma_0^{(1)}(4))$ the Kohnen's plus subspace of cusp forms of half-integral weight $k - (n-1)/2$ with respect to $\Gamma_0^{(1)}(4)$. We note that $h(\tau)$ is uniquely determined by f up to constant multiple. As for the details on elliptic modular forms of half-integral weight and the Shimura correspondence, see [10].

For any $B \in \mathcal{H}_n(\mathbb{Z})_{>0}$, we put

$$A_f(B) := c(|\mathfrak{d}_B|)\, \mathfrak{f}_B^{\,k-(n+1)/2} \prod_{p \mid \mathfrak{f}_B} \widetilde{F}_p(B;\, \alpha_p).$$

As mentioned above, the p-th Satake parameter α_p of f is determined up to inversion. But we have that $A_f(B)$ is independent of the choice of α_p since $\widetilde{F}_p(B;\, X)$ is invariant under $X \mapsto X^{-1}$.

Then we shall introduce the Ikeda lifting:

Theorem 2 (cf. Theorems 3.2 and 3.3 in [6]). *Assume that n and k are even integers such that $k > n+1$. If $f \in S_{2k-n}(\Gamma_1)$ is a normalized Hecke eigenform, then*

$$I_{n,k}(f)(Z) := \sum_{B \in \mathcal{H}_n(\mathbb{Z})_{>0}} A_f(B)\, e(\mathrm{tr}(BZ)) \quad (Z \in \mathbb{H}_n),$$

is a Hecke eigenform in $S_k(\Gamma_n)$ whose standard L-function is equal to

$$\zeta(s) \prod_{i=1}^{n} L(s + k - i,\, f).$$

We call it the Ikeda lifting of f.

Remark. (i) We note that the proof of Theorem 2 shows that the Ikeda lifting is injective. Indeed, if $f_1,\, f_2 \in S_{2k-n}(\Gamma_1)$ are distinct normalized Hecke eigenforms, then their eigenvalues of Hecke operators $T^{(1)}(p)$ are distinct for at least one prime number p. Hence $I_{n,k}(f_1)$ and $I_{n,k}(f_2)$ belong to different eigenspaces for the local Hecke algebra at p of degree n, and therefore they are orthogonal with respect to the Petersson inner product.

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(ii) The above construction has an analogy to the following relation between the elliptic Eisenstein series and the Siegel Eisenstein series: for the elliptic Eisenstein series $E_{1,\,2k-n} \in M_{2k-n}(\Gamma_1)$, the Cohen Eisenstein series $H_{k-(n-1)/2} \in M^+_{k-(n-1)/2}(\Gamma_0^{(1)}(4))$ is a Hecke eigenform corresponding to $E_{1,\,2k-n}$ under the Shimura correspondence. We denote the Fourier expansion of $H_{k-(n-1)/2}$ by

$$H_{k-(n-1)/2}(\tau) = \sum_{\substack{N \geq 0, \\ (-1)^{k-n/2}N \equiv 0,\,1 \ (\mathrm{mod}\ 4)}} c(N)\, e(N\tau) \quad (\tau \in \mathbb{H}_1).$$

Then for any $B \in \mathcal{H}_n(\mathbb{Z})_{>0}$, the B-th Fourier coefficient $A_{n,\,k}(B)$ of the Siegel Eisenstein series $E_{n,\,k} \in M_k(\Gamma_n)$ is described as

$$A_{n,\,k}(B) = \xi(n,\,k)\, c(|\mathfrak{d}_B|)\, \mathfrak{f}_B{}^{k-(n+1)/2} \prod_{p\,|\,\mathfrak{f}_B} \widetilde{F}_p(B;\, p^{k-(n+1)/2}), \qquad (3)$$

where

$$\xi(n,\,k) = 2^{n/2}\, \zeta(1-k)^{-1} \prod_{i=1}^{n/2} \zeta(1+2i-2k)^{-1}.$$

3 Jacobi forms of integral index

3.1 Jacobi groups

Let $\mathrm{H}_{1,n}(\mathbb{R})$ be the real Heisenberg group of characteristic $(1,\,n)$, that is, the set

$$\mathrm{H}_{1,n}(\mathbb{R}) := \mathbb{R}^{2n} \times \mathbb{R} = \{[X,\,\kappa]\,|\,X \in \mathbb{R}^{2n},\,\kappa \in \mathbb{R}\}$$

with the following group-structure: for $[X_i,\,\kappa_i] \in \mathrm{H}_{1,n}(\mathbb{R})$ $(i = 1, 2)$,

$$[X_1,\,\kappa_1] * [X_2,\,\kappa_2] := [X_1 + X_2,\,\kappa_1 + \kappa_2 + X_1 J_n{}^t X_2].$$

Since the group S_n acts on $\mathrm{H}_{1,n}(\mathbb{R})$ by

$$[X,\,\kappa] \cdot M := [\nu(M)^{-1} X M,\,\nu(M)^{-1}\kappa] \quad ([X,\,\kappa] \in \mathrm{H}_{1,n}(\mathbb{R}),\,M \in S_n),$$

we can define the semi-direct product $S_n^J := S_n \ltimes \mathrm{H}_{1,n}(\mathbb{R})$, that is, the set

$$S_n \ltimes \mathrm{H}_{1,n}(\mathbb{R}) := S_n \times \mathrm{H}_{1,n}(\mathbb{R})$$

with the following group-structure: for $g_i = (M_i,\,[X_i,\,\kappa_i]) \in S_n \ltimes \mathrm{H}_{1,n}(\mathbb{R})$ $(i = 1, 2)$,

$$\begin{aligned} g_1 g_2 &:= (M_1 M_2,\,([X_1,\,\kappa_1] \cdot M_2) * [X_2,\,\kappa_2]) \\ &= (M_1 M_2,\,[\nu(M_2)^{-1} X_1 M_2 + X_2,\,\nu(M_2)^{-1}\kappa_1 + \kappa_2 + \nu(M_2)^{-1} X_1 M_2 J_n{}^t X_2]). \end{aligned}$$

For simplicity, we denote any element of S_n^J by $[M, X, \kappa] = (M, [X, \kappa])$ with $M \in S_n$, $X \in \mathbb{R}^{2n}$ and $\kappa \in \mathbb{R}$.

Remark. For any $g = [M, X, \kappa] \in S_n^J$, we write $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $X = (\lambda, \mu)$, in which A, B, C, D are $n \times n$ matrices and λ, μ are n-vectors. Then we define g' by

$$g' := \begin{pmatrix} \nu & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{pmatrix} \begin{pmatrix} 1 & \lambda & \kappa & \mu \\ 0 & 1_n & {}^t\mu & 0_n \\ 0 & 0 & 1 & 0 \\ 0 & 0_n & -{}^t\lambda & 1_n \end{pmatrix},$$

where $\nu = \nu(M)$. Then we easily see that $g' \in S_{n+1}$ and the correspondence $g \mapsto g'$ defines an injective group-homomorphism.

We also define two subgroups of S_n^J by $G_n^J := G_n \ltimes \mathrm{H}_{1,n}(\mathbb{R})$ and $\Gamma_n^J := \Gamma_n \ltimes \mathrm{H}_{1,n}(\mathbb{Z})$, where $\mathrm{H}_{1,n}(\mathbb{Z}) := \mathrm{H}_{1,n}(\mathbb{R}) \otimes \mathbb{Z}$. we call G_n^J *the Jacobi group* of characteristic $(1, n)$.

Let k and m be non-negative integers. For any $[M, X, \kappa] \in S_n^J$, we decompose M and X into $n \times n$ blocks $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and n-vectors (λ, μ), respectively. For any function $\phi(\tau, z)$ on $\mathbb{H}_n \times \mathbb{C}^n$, we define

$$
\begin{aligned}
(\phi|_{k,m}&[M, X, \kappa])(\tau, z) \\
&:= e^{m\nu}(\kappa + \tau[{}^t\lambda] + 2\lambda \, {}^tz + \lambda \, {}^t\mu - (C\tau + D)^{-1}C \, [{}^t(z + \lambda\tau + \mu)]) \\
&\quad \times \det(C\tau + D)^{-k}\phi(M\langle\tau\rangle, \, \nu(z + \lambda\tau + \mu)(C\tau + D)^{-1}),
\end{aligned}
$$

where we write $\nu = \nu(M)$. Then for any $g_i = [M_i, X_i, \kappa_i] \in S_n^J$ ($i = 1, 2$), we have

$$(\phi|_{k,m}\, g_1)|_{k,m\nu}\, g_2 = \phi|_{k,m}\, (g_1 g_2),$$

where we write $\nu = \nu(M_1)$. Moreover, we denote the actions of $M \in S_n$ and $X \in \mathbb{Z}^{2n}$ by

$$\phi|_{k,m}\, M := \phi|_{k,m}\, [M, 0, 0],$$

and

$$\phi|_m\, X := \phi|_{k,m}\, [1_{2n}, X, 0],$$

respectively. Then for any $M, M' \in S_n$ and $X, X' \in \mathbb{Z}^{2n}$, we have

$$
\begin{cases}
(\phi|_{k,m}\, M)|_{k,m\nu}\, M' = \phi|_{k,m}\, (MM'), \\[2mm]
(\phi|_m\, X)|_m\, X' = \phi|_m\, (X + X'), \\[2mm]
(\phi|_{k,m}\, M)|_{m\nu}\, (\nu^{-1}XM) = (\phi|_m\, X)|_{k,m}\, M,
\end{cases}
$$

where we write $\nu = \nu(M)$.

3.2 Jacobi forms

Let k and m be positive integers.

Definition. A holomorphic function ϕ on $\mathbb{H}_n \times \mathbb{C}^n$ is called a *(holomorphic)* *Jacobi form* of degree n, weight k and index m if it satisfies the following two conditions:

(i) $\phi|_{k,m}\gamma = \phi$ for any $\gamma \in \Gamma_n^J$,

(ii) ϕ has a Fourier expansion of the form

$$\phi(\tau,\, z) = \sum_{T \in \mathcal{H}_n(\mathbb{Z}),\, r \in \mathbb{Z}^n} c_\phi(T,\, r) e(\mathrm{tr}(T\tau) + r\,{}^t z)$$

with $c_\phi(T,\, r) = 0$ unless $4mT - {}^t rr \geq 0$. If ϕ satisfies the stronger condition $c_\phi(T,\, r) = 0$ unless $4mT - {}^t rr > 0$, it is called a *Jacobi cusp form*.

We denote by $J_{k,m}(\Gamma_n^J)$ and $J_{k,m}^{\mathrm{cusp}}(\Gamma_n^J)$ the $\mathbb{C}$-vector spaces of the (holomorphic) Jacobi forms and Jacobi cusp forms of degree n, weight k and index m, respectively.

Remark (*Koecher's principle*). If $n > 1$, then the condition on Fourier coefficients in (ii) follows from the condition (i).

As an important example of Jacobi form, we consider Fourier-Jacobi coefficients of Siegel modular forms of degree $n + 1$. Let $F \in M_k(\Gamma_{n+1})$ have a Fourier expansion

$$F(Z) = \sum_{B \in \mathcal{H}_{n+1}(\mathbb{Z})_{\geq 0}} A(B) e(\mathrm{tr}(BZ)) \quad (Z \in \mathbb{H}_{n+1}),$$

and we put $Z = \begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}$ with $\tau \in \mathbb{H}_n$, $z \in \mathbb{C}^n$ and $\tau' \in \mathbb{H}_1$. Then we have the so-called Fourier-Jacobi expansion (of type $(1,\, n)$)

$$F\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\right) = \sum_{N=0}^{\infty} \phi_N(\tau,\, z) e(N\tau'),$$

where

$$\phi_N(\tau,\, z) = \sum_{\substack{T \in \mathcal{H}_n(\mathbb{Z}),\, r \in \mathbb{Z}^n, \\ 4NT - {}^t rr \geq 0}} A\left(\begin{pmatrix} N & r/2 \\ {}^t r/2 & T \end{pmatrix}\right) e(\mathrm{tr}(T\tau) + r\,{}^t z). \tag{4}$$

We easily see that the N-th coefficient $\phi_N \in J_{k,N}(\Gamma_n^J)$ for each $N \in \mathbb{N}$. In particular, if $F \in S_k(\Gamma_{n+1})$, then $\phi_N \in J_{k,N}^{\mathrm{cusp}}(\Gamma_n^J)$.

As another example, if k is an even integer such that $k > n+2$, then for any $N \in \mathbb{N}$, we define *the Jacobi Eisenstein series* of degree n, weight k and index N by

$$\mathfrak{E}_{k,N}^{(n)}(\tau, z) := \sum_{\gamma \in \Gamma_{n,0}^J \backslash \Gamma_n^J} (1|_{k,N}\, \gamma)(\tau, z) \quad (\tau \in \mathbb{H}_n,\, z \in \mathbb{C}^n),$$

where we denote by 1 the constant one function and we put

$$\Gamma_{n,0}^J := \left\{ [(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}),\, (\lambda, \mu),\, \kappa] \in \Gamma_n^J \,|\, C = 0_n,\, \lambda = 0 \right\}.$$

We easily see that the right-hand side of the above definition is absolutely convergent and $\mathfrak{E}_{k,N}^{(n)} \in J_{k,N}(\Gamma_n^J)$.

Remark. For any $N \in \mathbb{N}$, we denote by $e_{k,N}^{(n)} \in J_{k,N}(\Gamma_n^J)$ the N-th coefficient of the above Fourier-Jacobi expansion of the Siegel Eisenstein series $E_{n+1,k} \in M_k(\Gamma_{n+1})$. In the next section, we shall review the fact that there exists a certain relation between $\mathfrak{E}_{k,N}^{(n)}$ and $e_{k,N}^{(n)}$, which was proved by S. Böcherer ([2]).

If $\phi, \psi \in J_{k,m}(\Gamma_n^J)$ and $\phi\psi \in J_{2k,2m}^{\mathrm{cusp}}(\Gamma_n^J)$, then we can define the Petersson inner product of ϕ and ψ by

$$\langle \phi, \psi \rangle := \int_{\Gamma_n^J \backslash (\mathbb{H}_n \times \mathbb{C}^n)} \phi(\tau, z)\overline{\psi(\tau, z)} \det(v)^{k-n-2} \exp(-4\pi m v^{-1}[{}^t y])\, du\, dv\, dx\, dy,$$

where $\tau = u + \sqrt{-1}v \in \mathbb{H}_n$, $z = x + \sqrt{-1}y \in \mathbb{C}^n$. As is well-known, the Petersson inner product defines a hermitian inner product on $J_{k,m}^{\mathrm{cusp}}(\Gamma_n^J)$.

4   Certain linear operators acting on Jacobi forms

In this section, we assume throughout that k is even. Here we shall review certain linear operators acting on Jacobi forms, which shift indices by some integers.

4.1   Hecke operators

As discussed in [11] and [12], the Hecke ring of the pair $(\Gamma_n, \overline{S}_n)$ acts on the graded ring $\bigoplus_{m \in \mathbb{N}} J_{k,m}(\Gamma_n^J)$, where $\overline{S}_n = S_n \cap \mathrm{M}_{2n}(\mathbb{Z})$. Let $M \in \overline{S}_n$. Decompose

the double coset $\Gamma_n M \Gamma_n$ into the left cosets:

$$\Gamma_n M \Gamma_n = \bigsqcup_{i=1}^{d} \Gamma_n M_i \quad \text{(disjoint union)}.$$

For any $\phi \in J_{k,m}(\Gamma_n^J)$, we define the action

$$\phi|_{k,m}(\Gamma_n M \Gamma_n) := \nu(M)^{(n+1)k/2-n(n+1)/2} \sum_{i=1}^{d} \phi|_{k,m} M_i.$$

It is obvious that the right-hand side of the above is independent of the choice of representatives $\{M_i\}$.

Remark. The above action is equal to the one given in [11] and [12] up to normalizing factors.

Lemma 1. *If* $M \in \overline{S}_n$ *and* $\phi \in J_{k,m}(\Gamma_n^J)$, *then* $\phi|_{k,m}(\Gamma_n M \Gamma_n) \in J_{k,m\nu(M)}(\Gamma_n^J)$.

Proof. We write $\psi = \phi|_{k,m}(\Gamma_n M \Gamma_n)$ and $\nu = \nu(M)$. For any $[M', X, \kappa] \in \Gamma_n^J$, we can decompose it into the following form:

$$[M', X, \kappa] = [M', 0, 0][1_{2n}, X, 0][1_{2n}, 0, \kappa].$$

Since the action of $[1_{2n}, 0, \kappa]$ is trivial, it suffices to prove the following two transformation formulae:

$$\begin{cases} \text{(i) } \psi|_{k,m\nu} M' = \psi \ \text{ for any } M' \in \Gamma_n, \\[2mm] \text{(ii) } \psi|_{m\nu} X = \psi \ \text{ for any } X \in \mathbb{Z}^{2n}. \end{cases}$$

If $\{M_i\}$ is a complete set of representatives for $\Gamma_n \backslash \Gamma_n M \Gamma_n$, then so is the set $\{M_i M'\}$. Since

$$(\phi|_{k,m} M_i)|_{k,m\nu} M' = \phi|_{k,m} M_i M',$$

we have

$$\begin{aligned} \psi|_{k,m\nu} M' &= \nu^{(n+1)k/2-n(n+1)/2} \sum_{i=1}^{d} (\phi|_{k,m} M_i)|_{k,m\nu} M' \\ &= \nu^{(n+1)k/2-n(n+1)/2} \sum_{i=1}^{d} \phi|_{k,m} M_i M' \\ &= \psi. \end{aligned}$$

On the other hand, since $\nu X M_i^{-1} \in \mathbb{Z}^{2n}$ for any $X \in \mathbb{Z}^{2n}$, we have

$$(\phi|_{k,m} M_i)|_{m\nu} X = (\phi|_m \nu X M_i^{-1})|_{k,m} M_i = \phi|_{k,m} M_i.$$

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Therefore we have $\psi|_{m\nu} X = \psi$. When $n = 1$, the condition on Fourier coefficients follows by the explicit formulae for their actions on Fourier coefficients, which was given in [3]. $\square$

4.2 The operators $V_n(N)$, $U_n(N)$ and their adjoints with respect to Petersson inner products

$N \in \mathbb{N}$, we define two linear operators on $\phi \in J_{k,m}(\Gamma_n^J)$ by

$$V_n(N)\, \phi := \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)/\Gamma_n} \phi|_{k,m}(\Gamma_n M \Gamma_n) = N^{(n+1)k/2 - n(n+1)/2} \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)} \phi|_{k,m} M,$$

$$U_n(N)\, \phi := \phi|_{k,m}(\Gamma_n(N \cdot 1_{2n})\Gamma_n) = N^{(n+1)k - n(n+1)}\, \phi|_{k,m}(N \cdot 1_{2n}),$$

where $\overline{S}_n(N) := \{M \in \overline{S}_n \mid \nu(M) = N\}$. From Lemma 1, it is obvious that the above operators are linear mappings such that

$$V_n(N) : J_{k,m}(\Gamma_n^J) \to J_{k,mN}(\Gamma_n^J)$$

and

$$U_n(N) : J_{k,m}(\Gamma_n^J) \to J_{k,mN^2}(\Gamma_n^J).$$

Furthermore, we easily see that

$$V_n(N) : J_{k,m}^{\mathrm{cusp}}(\Gamma_n^J) \to J_{k,mN}^{\mathrm{cusp}}(\Gamma_n^J)$$

and

$$U_n(N) : J_{k,m}^{\mathrm{cusp}}(\Gamma_n^J) \to J_{k,mN^2}^{\mathrm{cusp}}(\Gamma_n^J)$$

by the explicit formulae for their actions on Fourier coefficients.

Remark. When $n = 1$, the operators $V_1(N)$ and $U_1(N)$ are equal to the operators V_N and U_N given in [3] up to normalizing factors.

Proposition 1. *For any N, $m \in \mathbb{N}$, let $V_n^*(N) : J_{k,mN}^{\mathrm{cusp}}(\Gamma_n^J) \to J_{k,m}^{\mathrm{cusp}}(\Gamma_n^J)$ be the adjoint of $V_n(N)$ with respect to Petersson inner products, that is,*

$$\langle V_n(N)\, \phi, \, \psi \rangle = \langle \phi, \, V_n^*(N)\, \psi \rangle$$

for any $\phi \in J_{k,m}^{\mathrm{cusp}}(\Gamma_n^J)$ and $\psi \in J_{k,mN}^{\mathrm{cusp}}(\Gamma_n^J)$. If $\psi \in J_{k,mN}^{\mathrm{cusp}}(\Gamma_n^J)$, then

$$V_n^*(N)\, \psi = N^{-(n-1)k/2 - n(n+5)/2} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)} \psi|_{k,mN}\left(\frac{1}{N}M\right)|_m X.$$

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Proof. By easy calculations, we have for $\phi \in J_{k,m}^{\mathrm{cusp}}(\Gamma_n^J)$,

$$V_n(N)\,\phi = N^{k/2 - n(n+1)/2} \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)} \phi_{\sqrt{N}}|_{k,\,mN}\left(\frac{1}{\sqrt{N}}M\right),$$

where $\phi_c(\tau,\,z) := \phi(\tau,\,cz)$ $(c \in \mathbb{C})$. We denote by $\overline{S}_n^*(N)$ the set of all primitive elements in $\overline{S}_n(N)$, that is,

$$\overline{S}_n^*(N) := \{M \in \overline{S}_n(N) \,|\, \gcd(M) = 1\},$$

then we can rewrite the above formula as

$$V_n(N)\,\phi = N^{k/2 - n(n+1)/2} \sum_{\substack{N' \mid N,\\ N/N' = \square}} \sum_{M \in \Gamma_n \backslash \overline{S}_n^*(N')} \phi_{\sqrt{N}}|_{k,\,mN}\left(\frac{1}{\sqrt{N'}}M\right),$$

where the notation "$\dfrac{N}{N'} = \square$" means that $\dfrac{N}{N'}$ is a perfect square. For any $d_i \in \mathbb{N}\,(1 \leq i \leq n)$ satisfying the conditions

$$d_i \mid d_{i+1} \ (1 \leq i < n), \ \ d_n \mid N,$$

we denote

$$[d_1, \cdots, d_n]_N := \mathrm{diag}(d_1, \cdots, d_n, N/d_1, \cdots, N/d_n)$$

and

$$\overline{S}_n(N;\, d_1, \cdots, d_n) := \{M \in \overline{S}_n(N) \,|\, \mathrm{sd}(M) = [d_1, \cdots, d_n]_N\},$$

where $\mathrm{sd}(M)$ is the symplectic divisor matrix of M. Then we can decompose $\overline{S}_n^*(N')$ into the form

$$\overline{S}_n^*(N') = \bigsqcup_{d_2 \mid \cdots \mid d_n \mid N'} \overline{S}_n(N';\, 1, d_2, \cdots, d_n).$$

We consider the map $\Gamma_n \to \overline{S}_n(N';\, 1, d_2, \cdots, d_n)$ defined by

$$M \mapsto [1, d_2, \cdots, d_n]_{N'} \cdot M.$$

We easily see that this map induces a bijection

$$K_n(N';\, 1, d_2, \cdots, d_n) \backslash \Gamma_n \xrightarrow{\simeq} \Gamma_n \backslash \overline{S}_n(N';\, 1, d_2, \cdots, d_n),$$

where

$$K_n(N';\, 1, d_2, \cdots, d_n) := \Gamma_n \cap [1, d_2, \cdots, d_n]_{N'}^{-1} \Gamma_n [1, d_2, \cdots, d_n]_{N'}$$

is a congruence subgroup of Γ_n. Hence we have

$$V_n(N)\,\phi = N^{k/2-n(n+1)/2} \sum_{\substack{N'\mid N,\\ N/N'=\square}} \sum_{d_2\mid\cdots\mid d_n\mid N'}$$

$$\sum_{M\in K_n(N';\,1,d_2,\cdots,d_n)\backslash\Gamma_n} \phi_{\sqrt{N}}|_{k,\,mN}\left(\frac{1}{\sqrt{N'}}[1,d_2,\cdots,d_n]_{N'}\cdot M\right).$$

Here we note that

$$\phi_{\sqrt{N}}|_{k,\,mN}\left(\frac{1}{\sqrt{N'}}[1,d_2,\cdots,d_n]_{N'}\right) \in J_{k,\,mN}^{\mathrm{cusp}}(K_n(N';\,1,d_2,\cdots,d_n)^J),$$

where

$$K_n(N';\,1,d_2,\cdots,d_n)^J := K_n(N';\,1,d_2,\cdots,d_n)\ltimes \mathrm{H}_{1,n}(\mathbb{Z}).$$

The above argument shows for any $\phi\in J_{k,\,m}^{\mathrm{cusp}}(\Gamma_n^J)$, $\psi\in J_{k,\,mN}^{\mathrm{cusp}}(\Gamma_n^J)$,

$$\langle V_n(N)\,\phi,\,\psi\rangle = N^{k/2-n(n+1)/2} \sum_{\substack{N'\mid N,\\ N/N'=\square}} \sum_{d_2\mid\cdots\mid d_n\mid N'} \sum_{M\in K_n(N';\,1,d_2,\cdots,d_n)\backslash\Gamma_n}$$

$$\langle \phi_{\sqrt{N}}|_{k,\,mN}\left(\frac{1}{\sqrt{N'}}[1,d_2,\cdots,d_n]_{N'}\right)|_{k,\,mN}M,\,\psi\rangle$$

$$= N^{k/2-n(n+1)/2} \sum_{\substack{N'\mid N,\\ N/N'=\square}} \sum_{d_2\mid\cdots\mid d_n\mid N'} [\Gamma_n:\,K_n(N';\,1,d_2,\cdots,d_n)]$$

$$\times\langle \phi_{\sqrt{N}}|_{k,\,mN}\left(\frac{1}{\sqrt{N'}}[1,d_2,\cdots,d_n]_{N'}\right),\,\psi\rangle,$$

where in the last line, we have made use of the fact that

$$\langle \phi|_{k,\,m'}M,\,\psi\rangle = \langle \phi,\,\psi|_{k,\,m'}M^{-1}\rangle$$

for any $m'\in\mathbb{N}$ and any $M\in\mathrm{Sp}_n(\mathbb{Q})\ltimes\mathrm{H}_{1,n}(\mathbb{Q})$. It is easy to check the above formula by using the standard techniques as in the case of ordinary modular forms. Since

$$\psi_{\sqrt{N}^{-1}}|_{k,\,m}\left(\frac{1}{\sqrt{N'}}[1,d_2,\cdots,d_n]_{N'}\right)^{-1} \in J_{k,\,m}^{\mathrm{cusp}}(K_n'(N';\,1,d_2,\cdots,d_n)^J),$$

where

$$K_n'(N';\,1,d_2,\cdots,d_n) := \Gamma_n\cap[1,d_2,\cdots,d_n]_{N'}\Gamma_n[1,d_2,\cdots,d_n]_{N'}^{-1},$$

and

$$\langle \phi_{\sqrt{N}}|_{k,\, mN} \Big(\frac{1}{\sqrt{N'}} [1, d_2, \cdots, d_n]_{N'} \Big),\, \psi \rangle$$
$$= \langle \phi,\, \psi_{\sqrt{N}^{-1}}|_{k,\, m} \Big(\frac{1}{\sqrt{N'}} [1, d_2, \cdots, d_n]_{N'} \Big)^{-1} \rangle,$$

we have

$$\langle \phi_{\sqrt{N}}|_{k,\, mN} \Big(\frac{1}{\sqrt{N'}} [1, d_2, \cdots, d_n]_{N'} \Big),\, \psi \rangle$$
$$= N'^{-2n} [\Gamma_n :\, K'_n(N';\, 1, d_2, \cdots, d_n)]^{-1} \sum_{X \in \mathbb{Z}^{2n}/N'\mathbb{Z}^{2n}}$$
$$\sum_{M \in K'_n(N';\, 1, d_2, \cdots, d_n) \backslash \Gamma_n} \langle \phi,\, \psi_{\sqrt{N}^{-1}}|_{k,\, m} \Big(\frac{1}{\sqrt{N'}} [1, d_2, \cdots, d_n]_{N'} \Big)^{-1} M|_m X \rangle.$$

Hence, by a similar argument as above, we have

$$\langle V_n(N)\, \phi,\, \psi \rangle = \langle \phi,\, N^{k/2 - n(n+5)/2} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)} \psi_{\sqrt{N}^{-1}}|_{k,\, m} \Big(\frac{1}{\sqrt{N}} M \Big)|_m X \rangle.$$

Here the function standing on the right-hand side in the Petersson inner product in the above formula is, in fact, in $J_{k,\, m}^{\mathrm{cusp}}(\Gamma_n^J)$. Therefore we have proved that

$$V_n^*(N)\, \psi = N^{k/2 - n(n+5)/2} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)} \psi_{\sqrt{N}^{-1}}|_{k,\, m} \Big(\frac{1}{\sqrt{N}} M \Big)|_m X.$$

Finally, we note that

$$\psi_{\sqrt{N}^{-1}} = N^{-nk/2}\, \psi|_{k,\, mN} \Big(\frac{1}{\sqrt{N}} \cdot 1_{2n} \Big),$$

we complete the proof of Proposition 1. $\quad\square$

Proposition 2. *For any* $N, m \in \mathbb{N}$, *let* $U_n^*(N) :\, J_{k,\, mN^2}^{\mathrm{cusp}}(\Gamma_n^J) \to J_{k,\, m}^{\mathrm{cusp}}(\Gamma_n^J)$ *be the adjoint of* $U_n(N)$ *with respect to Petersson inner products. If* $\psi \in J_{k,\, mN^2}^{\mathrm{cusp}}(\Gamma_n^J)$, *then*

$$U_n^*(N)\, \psi = N^{-(n-1)k - n(n+3)} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \psi|_{k,\, mN^2} \Big(\frac{1}{N} \cdot 1_{2n} \Big)|_m X.$$

Proof. By an argument similar to that in the proof of Proposition 1, we can give $U_n^*(N)$ by the following: for any $\psi \in J_{k,\, mN^2}^{\mathrm{cusp}}(\Gamma_n^J)$,

$$U_n^*(N)\, \psi = N^{k - n(n+3)} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \psi_{N^{-1}}|_m X,$$

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where $\psi_{N^{-1}}(\tau, z) = \psi(\tau, N^{-1}z)$. Finally, we note that

$$\psi_{N^{-1}} = N^{-nk}\,\psi|_{k,\,mN^2}\left(\frac{1}{N}\cdot 1_{2n}\right),$$

we complete the proof of Proposition 2. □

Remark. Renewing the definitions of $V_n^*(N)$ and $U_n^*(N)$ as the operators given by the formulae in Proposition 1 and 2, we also obtain

$$V_n^*(N) : J_{k,\,mN^2}(\Gamma_n^J) \to J_{k,\,m}(\Gamma_n^J)$$

and

$$U_n^*(N) : J_{k,\,mN^2}(\Gamma_n^J) \to J_{k,\,m}(\Gamma_n^J).$$

For the subsequent use, we shall give the action of $U_n^*(N)$ on Fourier coefficients, explicitly.

Corollary. *For*

$$\psi(\tau, z) = \sum_{\substack{T \in \mathcal{H}_n(\mathbb{Z}),\, r \in \mathbb{Z}^n, \\ 4mN^2 T - {}^t r r \geq 0}} c_\psi(T, r) e(\mathrm{tr}(T\tau) + r\,{}^t z) \in J_{k,\,mN^2}(\Gamma_n^J),$$

we have

$$N^{-k+n(n+1)}\,U_n^*(N)\,\psi(\tau, z)$$

$$= \sum_{\substack{T \in \mathcal{H}_n(\mathbb{Z}),\, r \in \mathbb{Z}^n, \\ 4mT - {}^t r r \geq 0}} \left\{ N^{-n} \sum_{\substack{\lambda \in \mathbb{Z}^n/2mN\mathbb{Z}^n, \\ \lambda \equiv r \pmod{2m\mathbb{Z}^n}}} c_\psi\left(T - \frac{1}{4m}({}^t r r - {}^t\lambda\lambda),\, N\lambda\right) \right\} e(\mathrm{tr}(T\tau) + r\,{}^t z).$$

Here we note that ${}^t r r - {}^t\lambda\lambda \in 4m\mathcal{H}_n(\mathbb{Z})$ *if* $\lambda \equiv r \pmod{2m\mathbb{Z}^n}$.

Indeed, we have

$$U_{n-1}^*(N)\,\psi(\tau, z) = N^{k-n(n+3)} \sum_{\lambda,\,\mu \in \mathbb{Z}^n/N\mathbb{Z}^n} e^m(\lambda\tau\,{}^t\lambda + 2\lambda\,{}^t z)\,\psi\left(\tau, \frac{z + \lambda\tau + \mu}{N}\right)$$

$$= N^{k-n(n+3)} \sum_{\lambda \in \mathbb{Z}^n/N\mathbb{Z}^n} \sum_{4mN^2 T - {}^t r r \geq 0} \sum_{\mu \in \mathbb{Z}^n/N\mathbb{Z}^n} e(N^{-1}r\,{}^t\mu)$$

$$\times c_\psi(T, r)\,e\left(\mathrm{tr}\left(\{T + {}^t\lambda(N^{-1}r + m\lambda)\}\tau\right) + (N^{-1}r + 2m\lambda)\,{}^t z\right).$$

Here the sum $\sum_{\mu \bmod N\mathbb{Z}^n} e(N^{-1}r\,{}^t\mu)$ has the value N^n or 0 according as $r \in N\mathbb{Z}^n$ or not. Replacing $N^{-1}r$ by r, we have

$$U_n^*(N)\,\psi(\tau,\,z) = N^{k-n(n+2)} \sum_{\lambda \in \mathbb{Z}^n/N\mathbb{Z}^n}\ \sum_{4mT-{}^trr \geq 0} c_\psi(T,\,Nr)$$

$$\times e(\mathrm{tr}\big(\{T + {}^t\lambda(r + m\lambda)\}\tau\big) + (r + 2m\lambda)\,{}^tz)$$

$$= N^{k-n(n+2)} \sum_{\lambda \in \mathbb{Z}^n/N\mathbb{Z}^n}\ \sum_{4mT-{}^trr \geq 0} c_\psi(T,\,Nr)$$

$$\times e(\mathrm{tr}\big(\{T + \frac{1}{4m}\big({}^t(r + 2m\lambda)(r + 2m\lambda) - {}^trr\big)\}\tau\big) + (r + 2m\lambda)\,{}^tz)$$

$$= N^{k-n(n+2)} \sum_{\lambda \in \mathbb{Z}^n/N\mathbb{Z}^n}\ \sum_{4mT-{}^t(r+2m\lambda)(r+2m\lambda) \geq 0}$$

$$c_\psi\big(T - \frac{1}{4m}\big({}^t(r + 2m\lambda)(r + 2m\lambda) - {}^trr\big),\,Nr\big)$$

$$\times e(\mathrm{tr}(T\tau) + (r + 2m\lambda)\,{}^tz)$$

$$= N^{k-n(n+2)} \sum_{4mT-{}^trr \geq 0}\ \sum_{\lambda \in \mathbb{Z}^n/N\mathbb{Z}^n}$$

$$c_\psi\big(T - \frac{1}{4m}\big({}^trr - {}^t(r - 2m\lambda)(r - 2m\lambda)\big),\,N(r - 2m\lambda)\big)$$

$$\times e(\mathrm{tr}(T\tau) + r\,{}^tz),$$

where in the last line, we have replaced r by $r - 2m\lambda$. Replacing $r - 2m\lambda$ by λ, we complete the proof of Corollary. $\square$

The operators $V_n(N)$, $U_n(N)$, $V_n^*(N)$ and $U_n^*(N)$ satisfy the following multiplicative relations:

Proposition 3. *For any N, $N' \in \mathbb{N}$,*

(i) $U_n(N) \cdot U_n(N')\,\phi = U_n(NN')\,\phi$,

(ii) $V_n(N) \cdot V_n(N')\,\phi = V_n(N') \cdot V_n(N)\,\phi$ if $\gcd(N,\,N') = 1$,

(iii) $U_n(N) \cdot V_n(N')\,\phi = V_n(N') \cdot U_n(N)\,\phi$,

(iv) $U_n^*(N) \cdot U_n(N)\,\phi = N^{2k-2n(n+1)}\,\phi$,

(v) $U_n^*(N) \cdot V_n(N^2)\,\phi = V_n^*(N^2) \cdot U_n(N)\,\phi$,

(vi) $U_n^*(N) \cdot V_n(N)\,\psi = N^{k-n(n+1)}\,V_n^*(N)\,\psi$,

where $\phi \in J_{k,m}(\Gamma_n^J)$ and $\psi \in J_{k,mN}(\Gamma_n^J)$.

Proof. The equations (i), (ii) and (iii) are trivial by the definitions. Furthermore, the equation (v) follows by the equations (iv) and (vi). Hence it suffices to prove that the equations (iv) and (vi) hold. By Proposition 2, for $\phi \in J_{k,m}(\Gamma_n^J)$, we have

$$U_n^*(N) \cdot U_n(N)\,\phi = N^{2k-2n(n+2)} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \phi|_{k,m}(N \cdot 1_{2n})|_{k,mN^2}\left(\frac{1}{N} \cdot 1_{2n}\right)|_m X$$

$$= N^{2k-2n(n+2)} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \phi|_m X$$

$$= N^{2k-2n(n+1)}\,\phi.$$

By Proposition 2, for $\psi \in J_{k,mN}(\Gamma_n^J)$, we have

$$U_n^*(N) \cdot V_n(N)\,\psi = N^{-(n-3)k/2-n(3n+7)/2} \times$$

$$\sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)} \psi|_{k,mN}M|_{k,mN^2}\left(\frac{1}{N} \cdot 1_{2n}\right)|_m X$$

$$= N^{-(n-3)k/2-n(3n+7)/2} \sum_{X \in \mathbb{Z}^{2n}/N\mathbb{Z}^{2n}} \sum_{M \in \Gamma_n \backslash \overline{S}_n(N)} \psi|_{k,m}\left(\frac{1}{N}M\right)|_m X.$$

By Proposition 1, we have

$$U_n^*(N) \cdot V_n(N)\,\psi = N^{k-n(n+1)}\,V_n^*(N)\,\psi.$$

Therefore we complete the proof of Proposition 3.   □

4.3   Fourier-Jacobi coefficients of the Siegel Eisenstein series and the operator $U_n(N)$

In this subsection, we shall give some observations for Fourier-Jacobi coefficients of the Siegel Eisenstein series and the Ikeda lifting.

Let k be an even integer such that $k > n + 2$. For any $N \in \mathbb{N}$, we denote by $e_{k,N}^{(n)} \in J_{k,N}(\Gamma_n^J)$ the N-th Fourier-Jacobi coefficient of the Siegel Eisenstein series $E_{n+1,k} \in M_k(\Gamma_{n+1})$, that is,

$$E_{n+1,k}\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\right) = \sum_{N=0}^{\infty} e_{k,N}^{(n)}(\tau, z)e(N\tau'),$$

where $\tau \in \mathbb{H}_n$, $z \in \mathbb{C}^n$ and $\tau' \in \mathbb{H}_1$. As mentioned in §3.2, S. Böcherer ([2]) proved that there exists a certain relation between $e_{k,N}^{(n)}$ and the Jacobi Eisenstein series $\mathfrak{E}_{k,N}^{(n)}$. Here we review such a relation and express it in terms of the operator $U_n(N)$:

Theorem 3 (cf. Satz 7 in [2] and Theorem 5.5 in [11])**.** *For any $N \in \mathbb{N}$, we*

have

$$e_{k,N}^{(n)} = \sum_{d^2 \mid N} \sigma_{k-1}(N/d^2) \sum_{a \mid d} \mu(a)(d/a)^{-k+n(n+1)} U_n(d/a)\, \mathfrak{E}_{k,N/(d/a)^2}^{(n)},$$

where $\mu(*)$ is the Möbius function and $\sigma_{k-1}(m) := \sum_{d \mid m} d^{k-1}$ for any $m \in \mathbb{N}$.

By using Theorem 3, we obtain the following fact on the Fourier-Jacobi coefficients of the Siegel Eisenstein series and the Ikeda lifting:

Lemma 2. *Let n and k be even integers such that $k > n + 1$. We denote by $\phi_m \in J_{k,m}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$ the m-th Fourier-Jacobi coefficient of the Ikeda lifting $I_{n,k}(f) \in S_k(\Gamma_n)$ of $f \in S_{2k-n}(\Gamma_1)$. If m and N are relatively prime, then we have*

$$N^{-k+(n-1)n}\, U_{n-1}^*(N)\, e_{k,mN^2}^{(n-1)} = N^{k-(n+1)/2} \prod_{p \mid N} \Psi_p(N;\, p^{k-(n+1)/2})\, e_{k,m}^{(n-1)},$$

and

$$N^{-k+(n-1)n}\, U_{n-1}^*(N)\, \phi_{mN^2} = N^{k-(n+1)/2} \prod_{p \mid N} \Psi_p(N;\, \alpha_p)\, \phi_m,$$

where $\Psi_p(N;\, X) = \Psi_p^{(n-1)}(N;\, X)$ is a Laurent polynomial in X defined by

$$\Psi_p^{(n-1)}(N;\, X) := \frac{X^{\delta+1} - X^{-(\delta+1)}}{X - X^{-1}} + p^{-(n-1)/2} \cdot \frac{X^\delta - X^{-\delta}}{X - X^{-1}}$$

if $\mathrm{ord}_p(N) = \delta$, and α_p is the p-th Satake parameter of f.

Proof. By (i) of Proposition 3, it suffices to consider the case of $N = p^\delta$ ($\delta > 0$) for any prime number p. By Theorem 3, we have

$$e_{k,m}^{(n-1)} = \sigma_{k-1}(m)\mathfrak{E}_{k,m}^{(n-1)} + \sum_{\substack{d^2 \mid m, \\ d > 1}} \sigma_{k-1}(m/d^2) \sum_{a \mid d} \mu(a)$$
$$\times (d/a)^{-k+(n-1)n} U_{n-1}(d/a)\, \mathfrak{E}_{k,m/(d/a)^2}^{(n-1)}.$$

On the other hand, by Theorem 3 and (iv) of Proposition 3, we also have

$$p^{\delta\{-k+(n-1)n\}} U_{n-1}^*(p^\delta)\, e_{k,\,mp^{2\delta}}^{(n-1)}$$

$$= \sigma_{k-1}(m)\left\{\sum_{i=0}^{\delta} \sigma_{k-1}(p^{2i}) p^{i\{-k+(n-1)n\}} U_{n-1}^*(p^i)\, \mathfrak{E}_{k,\,mp^{2i}}^{(n-1)}\right.$$

$$\left.-\sum_{i=1}^{\delta} \sigma_{k-1}(p^{2i-2}) p^{i\{-k+(n-1)n\}} U_{n-1}^*(p^i)\, \mathfrak{E}_{k,\,mp^{2i}}^{(n-1)}\right\}$$

$$+ \sum_{\substack{d^2\,|\,m,\\ d>1}} \sigma_{k-1}(m/d^2) \sum_{a\,|\,d} \mu(a)(d/a)^{-k+(n-1)n} U_{n-1}(d/a)$$

$$\times \sigma_{k-1}(p^{2\delta}) p^{\delta\{-k+(n-1)n\}} U_{n-1}^*(p^\delta)\, \mathfrak{E}_{k,\,\{m/(d/a)^2\}p^{2\delta}}^{(n-1)}.$$

Here, by the definition, we easily see

$$p^{i\{-k+(n-1)n\}} U_{n-1}^*(p^i)\, \mathfrak{E}_{k,\,mp^{2i}}^{(n-1)} = p^{-i(n-1)}\, \mathfrak{E}_{k,\,m}^{(n-1)}$$

and therefore we obtain

$$p^{\delta\{-k+(n-1)n\}} U_{n-1}^*(p^\delta)\, e_{k,\,mp^{2\delta}}^{(n-1)} = p^{\delta\{k-(n+1)/2\}} \Psi_p(p^\delta;\, p^{k-(n+1)/2})\, e_{k,\,m}^{(n-1)}.$$

By (3), (4) and Corollary to Proposition 2, the above equation implies the fact that the Laurent polynomial $\widetilde{F}_p(B;\, X)$ introduced in §2 satisfies the equation

$$p^{-\delta(n-1)} \sum_{\substack{\lambda\in\mathbb{Z}^{n-1}/2mp^\delta\mathbb{Z}^{n-1},\\ \lambda\equiv r\ (\mathrm{mod}\ 2m\mathbb{Z}^{n-1})}} \widetilde{F}_p\left(\begin{pmatrix} mp^{2\delta} & p^\delta\lambda/2 \\ * & T-\frac{1}{4m}({}^t rr - {}^t\lambda\lambda) \end{pmatrix};\, p^{k-(n+1)/2}\right)$$

$$= p^{\delta\{k-(n+1)/2\}} \Psi_p(p^\delta;\, p^{k-(n+1)/2})\, \widetilde{F}_p\left(\begin{pmatrix} m & r/2 \\ * & T \end{pmatrix};\, p^{k-(n+1)/2}\right)$$

for $T\in\mathcal{H}_{n-1}(\mathbb{Z})$ and $r\in\mathbb{Z}^{n-1}$ such that $4mT-{}^t rr > 0$. Since the above equation holds for infinitely many $k\,(>n+1)$, it is also valid as Laurent polynomials in X. Therefore, by substituting in $X=\alpha_p$, we obtain

$$p^{\delta\{-k+(n-1)n\}} U_{n-1}^*(p^\delta)\, \phi_{mp^{2\delta}} = p^{\delta\{k-(n+1)/2\}} \Psi_p(p^\delta;\, \alpha_p)\, \phi_m$$

and we complete the proof of Lemma 2. $\square$

5 Proof of Main Theorem

As a preperation for the proof of Main Theorem, we shall introduce a certain linear operator acting on Jacobi forms of "odd" degree, which was defined by S. Hayashida.

Let n be a positive even integer. For any $N \in \mathbb{N}$, we define a linear operator $D_{n-1}(N) = D_{n-1}(N, \{c_p\})$ through the following Dirichlet series with the Euler product:

$$\sum_{N=1}^{\infty} D_{n-1}(N) N^{-s}$$
$$= \prod_{p:\text{prime}} \left\{ 1 - G_p(c_p) V_{n-1}(p) p^{(n/2-1)(n/2+2)/2-s} + U_{n-1}(p) p^{(n-1)n-1-2s} \right\}^{-1},$$

where $G_p(X) = G_p^{(n-1)}(X)$ is a Laurent polynomial in X defined by

$$G_p^{(n-1)}(X) := \begin{cases} \displaystyle\prod_{i=1}^{n/2-1} \left\{ (1 + X p^{-(2i-1)/2})(1 + X^{-1} p^{-(2i-1)/2}) \right\}^{-1} & \text{if } n > 2, \\ \\ 1 & \text{if } n = 2, \end{cases}$$

and $c_p \in \mathbb{C}$ is an arbitrary constant for each p. It follows by (i) and (ii) of Proposition 3 that the above operator is well-defined. For simplicity, we omit the set of constants $\{c_p\}$ as above except for a few special cases.

Remark. When $n = 2$, the operator $D_1(N)$ is obviously independent of the set of constants $\{c_p\}$ by the definition. More precisely, we have that $D_1(N) = V_1(N)$ for any $N \in \mathbb{N}$.

By the properties of operators $V_{n-1}(p)$ and $U_{n-1}(p)$, we have

$$D_{n-1}(N) : J_{k,m}(\Gamma_{n-1}^J) \to J_{k,mN}(\Gamma_{n-1}^J)$$

and

$$D_{n-1}(N) : J_{k,m}^{\text{cusp}}(\Gamma_{n-1}^J) \to J_{k,mN}^{\text{cusp}}(\Gamma_{n-1}^J).$$

By Proposition 3, we also have the following multiplicative relations for $D_{n-1}(N)$ and its adjoint $D_{n-1}^*(N)$ with respect to Petersson inner products:

Proposition 4. *For any $N, N' \in \mathbb{N}$,*

(i) $D_{n-1}(N) \cdot U_{n-1}(N') \phi = U_{n-1}(N') \cdot D_{n-1}(N) \phi,$

(ii) $D_{n-1}(N) \cdot D_{n-1}(N') \phi = \displaystyle\sum_{d \mid \gcd(N, N')} d^{(n-1)n-1} U_{n-1}(d) \cdot D_{n-1}(NN'/d^2) \phi,$

(iii) $U_{n-1}^*(N) \cdot D_{n-1}(N^2) \psi = D_{n-1}^*(N^2) \cdot U_{n-1}(N) \psi,$

(iv) $U_{n-1}^*(N) \cdot D_{n-1}(N) \psi' = N^{k-(n-1)n} D_{n-1}^*(N) \psi',$

where $\phi \in J_{k,m}(\Gamma_{n-1}^J)$, $\psi \in J_{k,m}^{\text{cusp}}(\Gamma_{n-1}^J)$ and $\psi' \in J_{k,mN}^{\text{cusp}}(\Gamma_{n-1}^J)$. In particular, the equations (i) and (ii) imply that $D_{n-1}(N)$ and $U_{n-1}(N)$ all commute.

Remark. The above equations (i) and (ii) are generalizations of the well-known multiplicative relations for $V_1(N)$ and $U_1(N)$, which were obtained in [3].

Proof. By the definition, it suffices to consider the case of $N = p^\delta$ ($\delta \geq 0$) for any prime number p. Here we note that it satisfies the following induction formula:

$$\begin{cases} D_{n-1}(1) = 1, \\[2mm] D_{n-1}(p) = G_p(c_p)\, p^{(n/2-1)(n/2+2)/2}\, V_{n-1}(p), \\[2mm] D_{n-1}(p^\delta) = D_{n-1}(p) \cdot D_{n-1}(p^{\delta-1}) - p^{(n-1)n-1} U_{n-1}(p) \cdot D_{n-1}(p^{\delta-2}) \quad (\delta \geq 2). \end{cases}$$

Hence, by using Proposition 3, we easily see by induction that the equations (i), (iii) and (iv) hold. Therefore it suffices to prove that the equation (ii) holds. At first, by the above induction formula, we have

$$D_{n-1}(p) \cdot D_{n-1}(p^\delta) = \sum_{i=0}^{\min(\delta, 1)} p^{i\{(n-1)n-1\}} U_{n-1}(p^i) \cdot D_{n-1}(p^{\delta+1-2i}).$$

On the other hand, since

$$D_{n-1}(p^\delta) = \sum_{i=0}^{[\delta/2]} (-1)^i\, p^{i\{(n-1)n-1\}} \binom{\delta-i}{i} U_{n-1}(p^i) \cdot D_{n-1}(p)^{\delta-2i},$$

we have

$$D_{n-1}(p) \cdot D_{n-1}(p^\delta) = D_{n-1}(p^\delta) \cdot D_{n-1}(p).$$

By using these two relations, we have

$$D_{n-1}(p^\delta) \cdot D_{n-1}(p^\varepsilon) = \sum_{i=0}^{\min(\delta, \varepsilon)} p^{i\{(n-1)n-1\}} U_{n-1}(p^i) \cdot D_{n-1}(p^{\delta+\varepsilon-2i})$$

for any $\varepsilon \geq 0$. Indeed, if $\varepsilon \geq 2$, then

$$\begin{aligned} D_{n-1}(p^\delta) \cdot D_{n-1}(p^\varepsilon) &= D_{n-1}(p^\delta) \cdot \{D_{n-1}(p) \cdot D_{n-1}(p^{\varepsilon-1}) \\ &\qquad\qquad -p^{(n-1)n-1} U_{n-1}(p) \cdot D_{n-1}(p^{\varepsilon-2})\} \\ &= D_{n-1}(p) \cdot \{D_{n-1}(p^\delta) \cdot D_{n-1}(p^{\varepsilon-1})\} \\ &\qquad\qquad -p^{(n-1)n-1} U_{n-1}(p) \cdot \{D_{n-1}(p^\delta) \cdot D_{n-1}(p^{\varepsilon-2})\}. \end{aligned}$$

Therefore, by induction on ε, we prove that the desired relation holds. $\square$

S. Hayashida proved in his unpublished paper that all Fourier-Jacobi coefficients of the Ikeda lifting are related by a linear operator which contains some information of an original Hecke eigenform of degree 1. With his permission, we shall introduce it together with his proof:

Theorem 4 (S. Hayashida, 2004)**.** *Let n and k be even integers such that $k > n+1$, and let $f \in S_{2k-n}(\Gamma_1)$ be a normalized Hecke eigenform. For each $N \in \mathbb{N}$, we denote by $\phi_N \in J_{k,N}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$ the N-th Fourier-Jacobi coefficient of the Ikeda lifting $I_{n,k}(f) \in S_k(\Gamma_n)$ of f, and we put $D_{n-1,f}(N) := D_{n-1}(N, \{\alpha_p\})$, where $\{\alpha_p\}$ is the set of all Satake parameters of f. Then*

$$\phi_N = D_{n-1,f}(N)\,\phi_1.$$

Proof. T. Yamazaki ([11]) proved that the equation

$$e_{k,N}^{(n-1)} = D_{n-1}(N, \{p^{k-(n+1)/2}\})\, e_{k,1}^{(n-1)}$$

holds for infinitely many $k\,(> n+1)$. By an argument similar to the last argument in the proof of Lemma 2, we can show that the values of Laurent polynomials $\widetilde{F}_p(B; c_p)$ satisfy certain equations for $D_{n-1}(N)$ with any set of constants $\{c_p\}$. Therefore, by choosing $\{\alpha_p\}$ as $\{c_p\}$, we see that the figure of the above equation is also valid for ϕ_N, that is,

$$\phi_N = D_{n-1}(N, \{\alpha_p\})\,\phi_1 = D_{n-1,f}(N)\,\phi_1.$$

Thus we complete the proof of Theorem 4. □

Finally, we shall prove Main Theorem.

Proof of Main Theorem. Under the same notations as above, by Theorem 4, we have the Fourier-Jacobi expansion

$$I_{n,k}(f)\left(\begin{pmatrix} \tau' & z \\ {}^t z & \tau \end{pmatrix}\right) = \sum_{N=1}^{\infty} D_{n-1,f}(N)\,\phi_1(\tau, z)\,e(N\tau'),$$

where $\tau \in \mathbb{H}_{n-1}$, $z \in \mathbb{C}^{n-1}$ and $\tau' \in \mathbb{H}_1$. Hence, for $\mathrm{Re}(s) \gg 0$, the Dirichlet series of Rankin-Selberg type associated with $I_{n,k}(f)$ is given by

$$
\begin{aligned}
&D_1\big(s; I_{n,k}(f), I_{n,k}(f)\big) \\
&= \zeta(2s - 2k + 2n) \sum_{N=1}^{\infty} \langle D_{n-1,f}(N)\,\phi_1,\, D_{n-1,f}(N)\,\phi_1 \rangle\, N^{-s} \\
&= \zeta(2s - 2k + 2n) \sum_{N=1}^{\infty} \langle D_{n-1,f}^*(N) \cdot D_{n-1,f}(N)\,\phi_1,\, \phi_1 \rangle\, N^{-s}.
\end{aligned}
$$

By (ii) and (iv) of Proposition 4 and by (iv) of Proposition 3,

$$
\begin{aligned}
D_{n-1,\,f}^{*}(N) &\cdot D_{n-1,\,f}(N)\,\phi_1 \\
&= N^{-k+(n-1)n}\,U_{n-1}^{*}(N)\cdot D_{n-1,\,f}(N)^2\,\phi_1 \\
&= N^{-k+(n-1)n}\sum_{d\,|\,N} d^{(n-1)n-1}\,U_{n-1}^{*}(N)\cdot U_{n-1}(d)\cdot D_{n-1,\,f}((N/d)^2)\,\phi_1 \\
&= N^{-k+(n-1)n}\sum_{d\,|\,N} d^{2k-(n-1)n-1}\,U_{n-1}^{*}(N/d)\cdot D_{n-1,\,f}((N/d)^2)\,\phi_1 \\
&= \sum_{d\,|\,N} d^{k-1}\,(N/d)^{-k+(n-1)n}\,U_{n-1}^{*}(N/d)\cdot D_{n-1,\,f}((N/d)^2)\,\phi_1.
\end{aligned}
$$

Therefore, by Theorem 4 and Lemma 2, we obtain

$$
\begin{aligned}
D_{n-1,\,f}^{*}(N)\cdot D_{n-1,\,f}(N)\,\phi_1
&= \sum_{d\,|\,N} d^{k-1}\left\{ (N/d)^{-k+(n-1)n}\,U_{n-1}^{*}(N/d)\,\phi_{(N/d)^2}\right\} \\
&= \sum_{d\,|\,N} d^{k-1}\,(N/d)^{k-(n+1)/2}\prod_{p\,|\,(N/d)}\Psi_p(N/d;\,\alpha_p)\,\phi_1.
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
D_1&\big(s;\,I_{n,\,k}(f),\,I_{n,\,k}(f)\big) \\
&= \langle\,\phi_1,\,\phi_1\,\rangle\,\zeta(2s-2k+2n)\,\zeta(s-k+1)\sum_{N=1}^{\infty}N^{k-(n+1)/2}\prod_{p\,|\,N}\Psi_p(N;\,\alpha_p)\,N^{-s} \\
&= \langle\,\phi_1,\,\phi_1\,\rangle\,\zeta(2s-2k+2n)\,\zeta(s-k+1)\prod_{p\,:\,\mathrm{prime}}\sum_{\delta=0}^{\infty}p^{\delta\{k-(n+1)/2\}}\,\Psi_p(p^\delta;\,\alpha_p)\,p^{-\delta s}.
\end{aligned}
$$

Furthermore, for any prime number p, we have

$$
\begin{aligned}
\sum_{\delta=0}^{\infty}&\,p^{\delta\{k-(n+1)/2\}}\,\Psi_p(p^\delta;\,\alpha_p)\,p^{-\delta s} \\
&= \frac{1+p^{-s+k-n}}{(1-\alpha_p\,p^{k-(n+1)/2}\,p^{-s})(1-\alpha_p^{-1}\,p^{k-(n+1)/2}\,p^{-s})} \\
&= \frac{1-p^{-2s+2k-2n}}{(1-p^{-s+k-n})(1-\alpha_p\,p^{k-(n+1)/2}\,p^{-s})(1-\alpha_p^{-1}\,p^{k-(n+1)/2}\,p^{-s})}.
\end{aligned}
$$

Therefore

$$
\prod_{p\,:\,\mathrm{prime}}\sum_{\delta=0}^{\infty}p^{\delta\{k-(n+1)/2\}}\,\Psi_p(p^\delta;\,\alpha_p)\,p^{-\delta s}=\frac{\zeta(s-k+n)L(s,\,f)}{\zeta(2s-2k+2n)}
$$

and we complete the proof of Main Theorem. □

6 A contribution to the Ikeda's conjecture

At first, we shall introduce some notations of L-functions.

Let l be a positive even integer. For a normalized Hecke eigenform $f \in S_l(\Gamma_1)$, we put

$$\xi(s) := \Gamma_{\mathbb{R}}(s)\zeta(s),$$
$$\Lambda(s,\, f) := \Gamma_{\mathbb{C}}(s)L(s,\, f),$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. Let $L(s,\, f,\, \mathrm{Ad})$ be the adjoint L-function associated with f, which is defined by

$$L(s,\, f,\, \mathrm{Ad}) := \prod_{p:\, \mathrm{prime}} \{(1 - p^{-s})(1 - \alpha_p^2\, p^{-s})(1 - \alpha_p^{-2}\, p^{-s})\}^{-1},$$

where α_p is the p-th Satake parameter of f. Then we put

$$\Lambda(s,\, f,\, \mathrm{Ad}) := \Gamma_{\mathbb{R}}(s + 1)\Gamma_{\mathbb{C}}(s + l - 1)L(s,\, f,\, \mathrm{Ad}).$$

Here we note that the following functional equations hold:

$$\xi(1 - s) = \xi(s),$$
$$\Lambda(l - s,\, f) = (-1)^{l/2}\Lambda(s,\, f),$$
$$\Lambda(1 - s,\, f,\, \mathrm{Ad}) = \Lambda(s,\, f,\, \mathrm{Ad}).$$

We also consider certain modifications of $\xi(s)$ and $\Lambda(s,\, f,\, \mathrm{Ad})$ as

$$\widetilde{\xi}(s) := \Gamma_{\mathbb{R}}(s + 1)\xi(s) = \Gamma_{\mathbb{C}}(s)\zeta(s),$$
$$\widetilde{\Lambda}(s,\, f,\, \mathrm{Ad}) := \Gamma_{\mathbb{R}}(s)\Lambda(s,\, f,\, \mathrm{Ad}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s + l - 1)L(s,\, f,\, \mathrm{Ad}).$$

Now T. Ikeda ([7]) gave the following conjecture on the period of the Ikeda lifting:

Conjecture 1 (cf. Conjecture 5.1 in [7]). *Let n and k be even integers such that $k > n + 1$. Under the same situation as in §2.2, then there exists an integer $\alpha(n,\, k)$ depending only on n and k such that*

$$\Lambda(k,\, f)\prod_{i=1}^{n/2} \widetilde{\Lambda}(2i - 1,\, f,\, \mathrm{Ad})\, \widetilde{\xi}(2i) = 2^{\alpha(n,\, k)} \frac{\langle f,\, f\rangle\langle I_{n,\, k}(f),\, I_{n,\, k}(f)\rangle}{\langle h,\, h\rangle}. \qquad (5)$$

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Remark. By some computer calculations, he also gave the following conjectural value of $\alpha(n, k)$:

$$\alpha(n, k) = (n - 1)(k - n/2 + 1)$$

for general n.

By combining the equations (2), (5) and the facts that $\widetilde{\Lambda}(1, f, \mathrm{Ad}) = 2^{2k-n}\langle f, f \rangle$ and $\widetilde{\xi}(n) = (-1)^{n/2+1}B_n/n$, we obtain

$$2^{3k-2n+2} \prod_{i=1}^{n/2-1} \widetilde{\xi}(2i)\, \widetilde{\Lambda}(2i + 1, f, \mathrm{Ad}) = 2^{\alpha(n, k)} \frac{\langle \phi_1, \phi_1 \rangle}{\langle h, h \rangle},$$

where we denote by $\phi_1 \in J_{k,1}^{\mathrm{cusp}}(\Gamma_{n-1}^J)$ the first Fourier-Jacobi coefficient of $I_{n,k}(f)$.

Now we note that there exists a certain linear isomorphism between Jacobi forms of even integral weight k and index 1, and Siegel modular forms of half-integral weight $k - 1/2$, which was discovered by W. Kohnen, M. Eichler and D. Zagier ([3]) in the case of degree 1 and by T. Ibukiyama ([5]) in the case of higher degree:

Theorem 5 (cf. Theorem 1 in [5])**.** *For any n, $k \in \mathbb{N}$, we denote by $M_{k-1/2}^+(\Gamma_0^{(n)}(4))$ and $S_{k-1/2}^+(\Gamma_0^{(n)}(4))$ the generalized Kohnen's plus subspaces of Siegel modular forms and Siegel cusp forms of weight $k - 1/2$ with respect to $\Gamma_0^{(n)}(4)$, respectively. If k is even, then there exists a $\mathbb{C}$-linear isomorphism*

$$J_{k,1}(\Gamma_n^J) \cong M_{k-1/2}^+(\Gamma_0^{(n)}(4))$$

and its restriction to the space of Jacobi cusp forms also induces a $\mathbb{C}$-linear isomorphism

$$J_{k,1}^{\mathrm{cusp}}(\Gamma_n^J) \cong S_{k-1/2}^+(\Gamma_0^{(n)}(4)).$$

Moreover, the above isomorphisms are compatible with the actions of Hecke operators.

Let $H \in S_{k-1/2}^+(\Gamma_0^{(n-1)}(4))$ be a Hecke eigenform corresponding to ϕ_1 under the isomorphism in Theorem 5. Then by using an analogous method to the proof of Theorem 5.4 in [3], we have

$$\langle \phi_1, \phi_1 \rangle = 2^{(2k-2)(n-1)} \langle H, H \rangle, \tag{6}$$

where we denote by $\langle H, H \rangle$ the Petersson inner product of $H \in S_{k-1/2}^+(\Gamma_0^{(n-1)}(4))$ which is normalized by $[\Gamma_{n-1} : \Gamma_0^{(n-1)}(4)]^{-1}$.

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Remark. When $n = 2$, since $H = h \in S^+_{k-1/2}(\Gamma^{(1)}_0(4))$ and $\langle \phi_1, \phi_1 \rangle = 2^{2k-2}\langle h, h \rangle$, we have already proved that Conjecture 1 is true with $\alpha(2, k) = k$.

Therefore we can reduce Conjecture 1 to the following conjecture on the quotient of Petersson inner products of two cusp forms of half-integral weights:

Conjecture 2. *Assume the same situation as above, that is, consider the diagram of liftings*

$$S^+_{k-(n-1)/2}(\Gamma^{(1)}_0(4)) \cong S_{2k-n}(\Gamma_1) \to S_k(\Gamma_n) \to J^{\mathrm{cusp}}_{k,1}(\Gamma^J_{n-1}) \cong S^+_{k-1/2}(\Gamma^{(n-1)}_0(4))$$
$$h \qquad\qquad \leftrightarrow \qquad f \qquad \mapsto I_{n,k}(f) \mapsto \qquad \phi_1 \qquad \leftrightarrow \qquad H.$$

Then there exists an integer $\beta(n, k)$ depending only on n and k such that

$$\prod_{i=1}^{n/2-1} \widetilde{\xi}(2i)\,\widetilde{\Lambda}(2i + 1, f, \mathrm{Ad}) = 2^{\beta(n, k)}\frac{\langle H, H \rangle}{\langle h, h \rangle}. \tag{7}$$

Furthermore, by using the Rankin-Selberg method for $H \in S^+_{k-1/2}(\Gamma^{(n-1)}_0(4))$ obtained by $I_{n,k}(f)$, we can prove that Conjecture 2 is true in case $n = 4$, and therefore so is Conjecture 1. We would like to explain it in the subsequent paper.

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