# Exact standard zeta-values of Siegel modular forms

# Exact standard zeta-values of Siegel modular forms



| | |
|---|---|
| 著者 | KATSURADA Hidenori |
| journal or publication title | Experimental Mathematics |
| volume | 19 |
| number | 1 |
| page range | 65-77 |
| year | 2010 |
| URL | http://hdl.handle.net/10258/482 |

# Exact standard zeta-values of Siegel modular forms


Hidenori Katsurada
Muroran Institute of Technology
27-1 Mizumoto Muroran, 050-8585, Japan
(e-mail: hidenori@mmm.muroran-it.ac.jp)



**Abstract** In this paper, we give exact values of the standard zeta function for cuspidal Hecke eigenforms with respect to $Sp_2(\mathbf{Z})$.


## 1 Introduction

For a cuspidal Hecke eigenform $f$ of weight $k$ with respect to $Sp_n(\mathbf{Z})$, let $L(f, s, \underline{\mathrm{St}})$ be the standard zeta function of $f$. Then for some positive integer, the value $\dfrac{L(f, m, \underline{\mathrm{St}})}{< f, f > \pi^{-n(n+1)/2+nk+(n+1)m}}$ is an algebraic number if all the Fourier coefficients of $f$ are algebraic, where $< f, f >$ is the (non-normalized) Petersson product (cf. Böcherer [Böcherer 1985], Mizumoto [Mizumoto 1991].) In [Katsurada 2008], we gave an explicit formula to compute this value in terms of Fourier coefficients of $f$ and some other elementary quantities. In this paper, we compute this value exactly by using it in case $n = 2$. The main tool we use is the pullback formula for the Siegel-Eisenstein series due to Garrett [Garrett 1984] and Böcherer [Böcherer 1985]. This method has been applied to the case of elliptic modular forms in [Katsurada 2005]. To carry out the computation in the case of Sigel modular forms of degree 2, we need an explicit form of differential operators on the space of Siegel modular forms of degree 4 due to Ibukiyama [Ibukiyama 1999], and an explicit formula for global Siegel series for a half-integral matrix of degree 4. The generating function of the differential operators has been given in

[Ibukiyama 1999], and by a direct but rather elaborate computation we can get an explicit form of them. An explicit formula for local Siegel series has been given in [Katsurada 1999]. However it seems rather difficult to use the formula directly for a practical computation. In this paper we show a trick which enables us to reduce the computation of global Siegel series of degree 4 to that of degree 2.

The contents of this paper are as follows. In Section 2, first we review a result concerning Fourier coefficients of Siegel Eisenstein series following [Katsurada 2008], and explain the relation between the Siegel series and local densities. Next, we review a result concerning the pullback formula of Siegel Eisenstein series due to Böcherer to obtain an exact value the standard zeta function following [Katsurada 2008]. In Section 3, we restrict ourselves to the case of Siegel modular forms of degree 2. First we show a trick for the computation of global Siegel series stated above. Next we give an explicit formula of the differential operators acting on the space Siegel modular forms of degree 4 due to Ibukiyama [Ibukiyama 1999], and give a main result of this paper, which enables us to get exact standard zeta values (cf. Theorem 3.6.) In Section 4, we give numerical examples of such values and give some comments on the conjecture proposed in [Katsurada 2008].

The author would like to thank the referee for many valuable comments.

**Notation.** For a commutative ring $R$, we denote by $M_{mn}(R)$ the set of $(m,n)$-matrices with entries in $R$. In particular put $M_n(R) = M_{nn}(R)$. For an $(m,n)$-matrix $X$ and an $(m,m)$-matrix $A$, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of $X$. Let $a$ be an element of $R$. Then for an element $X$ of $M_{mn}(R)$ we often use the same symbol $X$ to denote the coset $X \bmod aM_{mn}(R)$. Put $GL_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of a square matrix $A$, and $R^*$ denotes the unit group of $R$. Let $S_n(R)$ denote the set of symmetric matrices of degree $n$ with entries in $R$. Furthermore, for an integral domain $R$ of characteristic different from 2, let $\mathcal{H}_n(R)$ denote the set of half-integral matrices of degree $n$ over $R$, that is, $\mathcal{H}_n(R)$ is the set of symmetric matrices of degree $n$ whose $(i,j)$-component belongs to $R$ or $\frac{1}{2}R$ according as $i = j$ or not. For a subset $S$ of $M_n(R)$ we denote by $S^\times$ the subset of $S$ consisting of non-degenerate matrices. In particular, if $S$ is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of $S$ consisting of positive definite (resp. semi-positive definite) matrices. Let $R'$ be a subring of $R$. Two symmetric matrices $A$ and $A'$ with entries in $R$ are called equivalent over $R'$

with each other and write $A \underset{R'}{\sim} A'$ if there is an element $X$ of $GL_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices $X$ and $Y$ we write $X \perp Y = \begin{pmatrix} X & O \\ O & Y \end{pmatrix}$.

# 2 Pullback formula of Siegel-Eisenstein series

In this section, first we review the Fourier coefficients of Siegel-Eisenstein series following [Katsurada 2008]. Furthermore, for later purpose, we consider the relation between Siegel series and local densities.

Put $J_n = \begin{pmatrix} O_n & -1_n \\ 1_n & O_n \end{pmatrix}$, where $1_n$ denotes the unit matrix of degree $n$. For a subring $K$ of $\mathbf{R}$ put

$$\Gamma^{(n)} = Sp_n(\mathbf{Z}) = \{M \in GL_{2n}(\mathbf{Z}) \mid J_n[M] = J_n\}.$$

Let $\mathbf{H}_n$ be Siegel's upper half-space. We denote by $\mathfrak{M}_k(\Gamma^{(n)})$ (resp. $\mathfrak{M}_k^{\infty}(\Gamma^{(n)})$ the space of holomorphic (resp. $C^\infty$-) modular forms of weight $k$ with respect to $\Gamma^{(n)}$. We denote by $\mathfrak{S}_k(\Gamma^{(n)})$ the submodule of $\mathfrak{M}_k(\Gamma^{(n)})$ consisting of cusp forms. For two $C^\infty$-modular forms $f$ and $g$ of weight $k$ with respect to $\Gamma^{(n)}$ we define the Petersson scalar product $< f, g >$ as in [Katsurada 2008].

For a positive integer $k \geq (n+1)/2$ we define the Siegel Eisenstein series $E_{n,k}(Z,s)$ of degree $n$ as

$$E_{n,k}(Z,s)$$

$$= \zeta(1-k-2s) \prod_{i=1}^{[n/2]} \zeta(1-2k-4s+2i) \sum_{M \in \Gamma_\infty^{(n)} \backslash \Gamma^{(n)}} j(M,Z)^{-k} (\det(\mathrm{Im}(M(Z))))^s$$

( $Z \in \mathbf{H}_n, s \in \mathbf{C}$), where $\zeta(*)$ is Riemann's zeta function, and $\Gamma_\infty^{(n)} = \{\begin{pmatrix} * & * \\ O_n & * \end{pmatrix} \in \Gamma^{(n)}\}$. Then $E_{n,k}(Z,s)$ is holomorphic at $s=0$ as a function of $s$, and $E_{n,k}(Z,0)$ is holomorphic as a function of $Z$ unless $k = (n+2)/2 \equiv 2 \bmod 4$, or $k = (n+3)/2 \equiv 2 \bmod 4$ (cf. [Shimura 1983].) From now on we assume that $E_{n,k}(Z,0)$ is holomorphic as a function of $Z$, and write $E_{n,k}(Z) = E_{n,k}(Z,0)$. To see the Fourier expansion of $E_{n,k}(Z,0)$, for a prime number $p$ and a half-integral matrix $B$ of degree $n$ over $\mathbf{Z}_p$ define the local Siegel series $b_p(B,s)$ as in [Katsurada 2008]. Let $m,n$ be non-negative integers such that $m \geq n \geq 1$. For $A \in \mathcal{H}_m(\mathbf{Z}_p)$ and $B \in S_n(\mathbf{Q}_p)$ define the local density $\alpha_p(A, B)$ and the primitive local density $\beta_p(A, B)$ by

$$\alpha_p(A, B) = 2^{\delta_{mn}} \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \# \mathcal{A}_e(A, B),$$

and

$$\beta_p(A, B) = 2^{\delta_{mn}} \lim_{e \to \infty} p^{(-mn+n(n+1)/2)e} \# \mathcal{B}_e(A, B),$$

where $\delta_{mn}$ is Kronecker's delta,

$$\mathcal{A}_e(A, B) = \{X \in M_{mn}(\mathbf{Z}_p)/p^e M_{mn}(\mathbf{Z}_p) \mid A[X] - B \in p^e \mathcal{H}_n(\mathbf{Z}_p)\},$$

and

$$\mathcal{B}_e(A, B) = \{X \in \mathcal{A}_e(A, B) \mid \mathrm{rank}_{\mathbf{Z}_p/p\mathbf{Z}_p}(X) = n\}.$$

We define $\chi_p(a)$ for $a \in \mathbf{Q}_p \backslash \{0\}$ as follows;

$$\chi_p(a) = \begin{cases} +1 & \text{if } \mathbf{Q}_p(\sqrt{a}) = \mathbf{Q}_p \\ -1 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic unramified} \\ 0 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic ramified.} \end{cases}$$

For a half-integral matrix $B$ of even degree $n$ define $\xi_p(B)$ by

$$\xi_p(B) = \chi_p((-1)^{n/2} \det B).$$

Let $B \in \mathcal{H}_n(\mathbf{Z})_{>0}$ with $n$ even. Then we can write $(-1)^{n/2} 2^n \det B = \mathfrak{d}_B \mathfrak{f}_B^2$ with $\mathfrak{d}_B$ a fundamental discriminant and $\mathfrak{f}_B \in \mathbf{Z}_{>0}$. Furthermore, let $\chi_B = (\frac{\mathfrak{d}_B}{*})$ be the Kronecker character corresponding to $\mathbf{Q}(\sqrt{(-1)^{n/2} \det B})/\mathbf{Q}$. We note that we have $\chi_B(p) = \xi_p(B)$ for any prime $p$. Let $H_k = \overbrace{H \perp ... \perp H}^{k}$ with $H = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

For a non-degenerate half-integral matrix $B$ of degree $n$ over $\mathbf{Z}_p$ define a polynomial $\gamma_p(B, X)$ in $X$ by

$$\gamma_p(B, X) = \begin{cases} (1 - X) \prod_{i=1}^{n/2} (1 - p^{2i} X^2)(1 - p^{n/2} \xi_p(B) X)^{-1} & \text{if } n \text{ is even} \\ (1 - X) \prod_{i=1}^{(n-1)/2} (1 - p^{2i} X^2) & \text{if } n \text{ is odd.} \end{cases}$$

Then the following lemma is well known (e.g. [Kitaoka 1984], Lemma 1)

**Lemma 2.1.** *For a non-degenerate half-integral matrix $B$ of degree $n$ over $\mathbf{Z}_p$ there exists a unique polynomial $F_p(B,X)$ in $X$ over $\mathbf{Z}$ with constant term 1 such that*

$$b_p(B,s) = \gamma_p(B,p^{-s})F_p(B,p^{-s}).$$

*Furthermore for any positive integer $k \geq n/2$ and a half integral matrix $A$ of degree $2k$ over $\mathbf{Z}_p$ such that $2A$ is unimodular, we have*

$$\alpha_p(A,B) = F_p(B,\xi_p(A)p^{-k})\gamma_p(B,\xi_p(A)p^{-k})$$

*and, in particular,*

$$\alpha_p(H_k,B) = F_p(B,p^{-k})\gamma_p(B,p^{-k}).$$

**Remark.** For an element $B \in \mathcal{H}_n(\mathbf{Z}_p)$ of rank $m \geq 0$, there exists an element $\tilde{B} \in \mathcal{H}_m(\mathbf{Z}_p) \cap GL_m(\mathbf{Q}_p)$ such that $B \sim \tilde{B} \perp O_{n-m}$. We note that $b_p(\tilde{B},s)$ does not depend on the choice of $\tilde{B}$ (cf. [Kitaoka 1984].) Thus we write this as $b_p^{(m)}(B,s)$. Furthermore, $F_p(\tilde{B},X)$ does not depend on the choice of $\tilde{B}$. Then we put $F_p^{(m)}(B,X) = F_p(\tilde{B},X)$. For an element $B \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}$ of rank $m \geq 0$, there exist an element $\tilde{B} \in \mathcal{H}_m(\mathbf{Z})_{>0}$ such that $B \sim \tilde{B} \perp O_{n-m}$. Then, $\det \tilde{B}$ does not depend on the choice of $B$. Thus we put $\det^{(m)} B = \det \tilde{B}$. Similarly, we write $\chi_B^{(m)} = \chi_{\tilde{B}}$ if $m$ is even.

Now for a semi-positive definite half-integral matrix $B$ of degree $2n$ and of rank $m$, we put

$$c_{2n,l}(B) = 2^{[(m+1)/2]} \prod_p F_p^{(m)}(B,p^{l-m-1})$$

$$\times \begin{cases} \prod_{i=m/2+1}^{n} \zeta(1+2i-2l)L(1+m/2-l,\chi_B^{(m)}) & \text{if } m \text{ is even} \\ \prod_{i=(m+1)/2}^{n} \zeta(1+2i-2l) & \text{if } m \text{ is odd} \end{cases}.$$

Here we make the convention $F_p^{(m)}(B,p^{l-m-1}) = 1$ and $L(1+m/2-l,\chi_B^{(m)}) = \zeta(1-l)$ if $m = 0$. Then we have

**Theorem 2.2.** *Let $l$ be a positive even integer. Assume that $l \geq n+3$ or $l \geq n+1$ according as $n \equiv 1 \bmod 4$ or not. Then we have*

$$E_{2n,l}(Z) = \sum_{B \in \mathcal{H}_{2n}(\mathbf{Z})_{\geq 0}} c_{2n,l}(B)\mathbf{e}(\mathrm{tr}(BZ)),$$

*where* $\mathbf{e}(x) = \exp(2\pi\sqrt{-1}x)$ *for a complex number* $x$*, and* $\mathrm{tr}$ *denotes the trace of a matrix.*

Now we review the pullback formula of Siegel Eisenstein series following Sections 3 and 4 of [Katsurada 2008]. Let $\mathbf{L}_n' = \mathbf{L}_{\mathbf{Q}}(GSp_n(\mathbf{Q})^+ \cap M_{2n}(\mathbf{Z}), \Gamma^{(n)})$ denote the Hecke ring over $\mathbf{Z}$ associated with the Hecke pair $(GSp_n(\mathbf{Q})^+ \cap M_{2n}(\mathbf{Z}), \Gamma^{(n)})$. Furthermore for a prime number $p$ put

$$GSp_n(\mathbf{Q}_p) = \{M \in GL_{2n}(\mathbf{Q}_p); J_n[M] = \kappa(M)J_n \text{ with some } \kappa(M) \in \mathbf{Q}_p^\times\},$$

and let $\mathbf{L}_{np} = \mathbf{L}(GSp_n(\mathbf{Q}_p), GSp_n(\mathbf{Q}_p) \cap GL_{2n}(\mathbf{Z}_p))$ be the Hecke algebra associated with the pair $(GSp_n(\mathbf{Q}_p), GSp_n(\mathbf{Q}_p) \cap GL_{2n}(\mathbf{Z}_p))$. Now assume that $f$ is a Hecke eigenform, namely, a common eigenfunction of all Hecke operators. For each prime number $p$, let $\alpha_0(p), \alpha_1(p), ...., \alpha_n(p)$ be the Satake parameters of $\mathbf{L}_{np}$ determined by $f$. We then define the standard zeta function $L(f, s, \underline{\mathrm{St}})$ by

$$L(f, s, \underline{\mathrm{St}}) = \prod_p \prod_{i=1}^n \{(1 - p^{-s})(1 - \alpha_i(p)p^{-s})(1 - \alpha_i(p)^{-1}p^{-s})\}^{-1}.$$

Now let $\overset{\circ}{\mathcal{D}}{}_{n,l}^{\nu}$ be the differential operator from $\mathfrak{M}_l^\infty(\Gamma^{(2n)})$ to $\mathfrak{M}_{l+\nu}^\infty(\Gamma^{(n)}) \otimes \mathfrak{M}_{l+\nu}^\infty(\Gamma^{(n)})$ in [Böcherer and Schmidt 2000].

Let $E_{2n,l}(Z) = E_{2n,l}(Z, 0)$ be the Eisenstein series as above. We then define $\mathcal{E}_{2n,l,k}(z_1, z_2)$ as

$$\mathcal{E}_{2n,l,k}(z_1, z_2) = (-1)^{l/2+1}2^{-n}(2\pi\sqrt{-1})^{(l-k)n}(l - n)\ \overset{\circ}{\mathcal{D}}{}_{n,l}^{k-l}\ (E_{2n,l})(z_1, z_2),$$

where $z_1, z_2 \in \mathbf{H}_n$. Let $f(z) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{>0}} a(A)\mathbf{e}(\mathrm{tr}(Az))$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$. For a positive integer $m \le k - n$ such that $m \equiv n \bmod 2$ put

$$\Lambda(f, m, \underline{\mathrm{St}}) = (-1)^{n(-m+1)/2+1}2^{-4kn+3n^2+n+(n-1)m+2}$$
$$\times\Gamma(m + 1)\prod_{i=1}^n \Gamma(2k - n - i)\frac{L(f, m, \underline{\mathrm{St}})}{< f, f > \pi^{-n(n+1)/2+nk+(n+1)m}}.$$

For two semi-positive definite half-integral matrices $A_1, A_2$ of degree $n$, put

$$\epsilon_{l,k}(A_1, A_2) = \sum_{A_2 - \frac{1}{4}A_1^{-1}[R] \ge 0} \tilde{c}_{2n,l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix})Q_{n,l}^{k-l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}),$$

where $\tilde{c}_{2n,l}(A) = (-1)^{l/2+1}2^{-n}(l-n)c_{2n,l}(A)$ for $A \in \mathcal{H}_{2n}(\mathbf{Z})_{\geq 0}$, and $Q_{n,l}^{k-l}$ is the polynomial defined in Section 3 of [Katsurada 2008]. Furthermore, for each semi-positive definite half-integral matrix $A_1$ put

$$\mathcal{F}_{l,k;A_1}(z_2) = \sum_{A_2 \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} \epsilon_{l,k}(A_1, A_2)\mathbf{e}(\mathrm{tr}(A_2 z_2)).$$

We note that we have

$$\mathcal{E}_{2n,l,k}(z_1, z_2) = \sum_{A_1 \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} \sum_{A_2 \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} \epsilon_{l,k}(A_1, A_2)\mathbf{e}(\mathrm{tr}(A_1 z_1 + A_2 z_2)).$$

Thus $\mathcal{F}_{l,k;A_1}(z_2)$ belongs to $\mathfrak{M}_k(\Gamma^{(n)})$, and

$$\mathcal{E}_{2n,l,k}(z_1, z_2) = \sum_{A_1 \in \mathcal{H}_n(\mathbf{Z})_{\geq 0}} \mathcal{F}_{l,k;A_1}(z_2)\mathbf{e}(\mathrm{tr}(A_1 z_1))$$

(cf. Section 3 of [Katsurada 2008].) In particular, if $l < k$, $\mathcal{F}_{l,k;A_1}(z_2)$ belongs to $\mathfrak{S}_k(\Gamma^{(n)})$, and

$$\mathcal{E}_{2n,l,k}(z_1, z_2) = \sum_{A_1 \in \mathcal{H}_n(\mathbf{Z})_{>0}} \mathcal{F}_{l,k;A_1}(z_2)\mathbf{e}(\mathrm{tr}(A_1 z_1)).$$

Take an orthogonal basis $\{f_i\}_{i=1}^{d_1}$ of $\mathfrak{S}_k(\Gamma^{(n)})$ consisting of Hecke eigenforms. Write

$$f_i(z) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{>0}} a_i(A)\mathbf{e}(\mathrm{tr}(Az)).$$

Now we have the following:

**Proposition 2.3.** (cf. [Katsurada 2008, Theorem 4.4]) *Let $l, k$ and $n$ be a positive integers. Assume that $k$ and $l+n$ is even, and $3 \leq l \leq k-n-2$ or $1 \leq l \leq k-n-2$ according as $n \equiv 1 \bmod 4$ or not. Then for any positive definite half-integral matrix $A_1$ of degree $n$ we have*

$$\mathcal{F}_{l+n,k;A_1}(z) = \sum_{i=1}^{d_1} \Lambda(f_i, l, \underline{\mathrm{St}})a_i(A_1)\overline{f_i(-\bar{z})}.$$

**Remark.** There are some misprints in [Katsurada 2008].
Page 105, line 9: For "$(2\pi\sqrt{-1})^{l-k}$", read "$(2\pi\sqrt{-1})^{(l-k)n}$".
Page 105, line 12: For "$(-1)^{n(m+1)/2+1}$", read "$(-1)^{n(-m+1)/2+1}$".
Page 105, line 13: For "$\Gamma(m)$", read "$\Gamma(m+1)$".

Now, for a prime number $p$, let $T(p)$ be the element of $\mathbf{L}'_n$ defined by $T(p) = \Gamma^{(n)}(1_n \perp p1_n)\Gamma^{(n)}$. For each non-negative integer $i$ and $A_1 \in \mathcal{H}_n(\mathbf{Z})_{>0}$, write $\mathcal{F}_{l+n,k;A_1}|T(p)^i(z)$ as

$$\mathcal{F}_{l+n,k;A_1}|T(p)^i(z) = \sum_{A \in \mathcal{H}_n(\mathbf{Z})_{>0}} \epsilon_{l+n,k}(i, A_1, A)\mathbf{e}(\mathrm{tr}(Az)).$$

Furthermore write

$$f_j|T(p)(z) = \lambda_j f_j(z).$$

Thus by Proposition 2.3 we have the following.

**Lemma 2.4.** *Under the above notation and the assumption, we have*

$$\epsilon_{l+n,k}(i, A_1, A) = \sum_{j=1}^{d} \lambda_j^i \Lambda(f_j, l, \underline{\mathrm{St}}) a_j(A_1)\overline{a_j(A)}$$

*for any non-zero integer $i$ and $A \in \mathcal{H}_n(\mathbf{Z})_{>0}$.*

From now on assume that the $\mathfrak{S}_k(\Gamma^{(n)})$ has the multiplicity one condition. Namely assume that a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ is uniquely determined, up to constant multiple, by its eigenvalues of Hecke operators. First we normalize the standard zeta value $\Lambda(f, l, \underline{\mathrm{St}})$ for a Hecke eigenform $f$ in $\mathfrak{S}_k(\Gamma^{(n)})$ following [Katsurada 2008]. We define the following quantities: for a Hecke eigenform $f(z) = \sum_A a_f(A)\mathbf{e}(\mathrm{tr}(Az))$ in $\mathfrak{S}_k(\Gamma^{(n)})$, let $\mathfrak{I}_f$ be the $\mathfrak{O}_{\mathbf{Q}(f)}$-module generated by all $a_f(A)'s$. Then, by multiplying a suitable constant $c$ we may assume all $a_f(A)'s$ are elements of $\mathbf{Q}(f)$ with bounded denominator. Then $\mathfrak{I}_f$ is a fractional ideal in $\mathbf{Q}(f)$, and therefore, so is $\Lambda(f, l, \underline{\mathrm{St}})\mathfrak{I}_f^2$ if $l$ satisfies the conditions in Proposition 2.3. We note that this fractional ideal does not depend on the choice of $c$. In particular, these values are uniquely determined by the system of eigenvalues of $f$. We also note that the value $N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\mathrm{St}}))N(\mathfrak{I}_f)^2$ does not depend on the choice of $c$, where $N(\mathfrak{I}_f)$ is the norm of the ideal $\mathfrak{I}_f$.

**Theorem 2.5.** *In addition to the above assumption, let* $f = f_1, \lambda = \lambda_1, a(A_1) = a_1(A_1), a(A) = a_1(A), K = \mathbf{Q}(f)$ *and* $e_i = \epsilon_{l+n,k}(i, A_1, A)$. *Furthermore, let* $\Phi(X) = \Phi_{T(p)}(X) = \sum_{i=0}^{d} b_{d-i}X^i$ *be the characteristic polynomial of* $T(p)$ *in* $\mathfrak{S}_k(\Gamma^{(n)})$. *Put* $\Lambda^*(f, l, \mathrm{St}) = N_{\mathbf{Q}(f)}(\Lambda(f, l, \underline{\mathrm{St}}))N(\mathfrak{J}_f)^2$ *Assume that* $\Phi'(\lambda) \neq 0$, *and* $a(A_1)a(A) \neq 0$. *Then for any positive integer* $l$ *satisfying the conditions in Proposition 2.3, we have*

$$\Lambda^*(f, l, \underline{\mathrm{St}}) = N_{K/\mathbf{Q}}\left(\frac{\sum_{i=0}^{d-1}\sum_{j=i}^{d-1} e_{d-1-j}b_{j-i}\lambda^i}{\Phi'(\lambda)}\right)\frac{N(\mathfrak{J}_f)^2}{N_{K/\mathbf{Q}}(a(A_1)a(A))}.$$

*Proof.* By Lemma 2.4, we have

$$e_i = \sum_{j=1}^{d} \lambda_j^i \Lambda(f_j, l, \underline{\mathrm{St}})a_j(A_1)\overline{a_j(A)}$$

for each $i = 0, ..., d-1$. Then by Lemma 2.2 of Goto [Goto 1998], we have

$$\Lambda(f, l, \underline{\mathrm{St}})a(A_1)a(A) = \frac{\sum_{i=0}^{d-1}\sum_{j=i}^{d-1} e_{d-1-j}b_{j-i}\lambda^i}{\Phi'(\lambda)}.$$

Thus the assertion immediately follows.

By using the above theorem, we can get standard zeta values of a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$ in principle. However to make the computation explicit, we need to compute the Fourier coefficients of the Siegel Eisenstein series of degree $2n$ and the differential operators explicitly. We will do this in case $n = 2$ in the next sections.

# 3 Exact standard zeta values in case $n = 2$

In this section we obtain a useful formula for computing exact standard zeta values in the case of degree 2. The following lemma can easily be proved (e.g. [Katsurada 1999, Proposition 2.2].)

**Lemma 3.1.** *Let* $n = n_1 + n_2$ *with* $n_1$ *even. Let* $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p) \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ *and* $A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p) \cap GL_{n_2}(\mathbf{Q}_p)$. *Then for any* $l \geq n$ *we have*

$$\alpha_p(H_l, A_{11}\perp A_{22}) = \beta_p(H_l, A_{11})\alpha_p(H_{l-n_1}\perp(-A_{11}), A_{22}).$$

**Proposition 3.2.** *Let $n_1$ be an even integer. Let $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p) \cap \frac{1}{2}GL_{n_1}(\mathbf{Z}_p)$ and $A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p)$. Let $m$ be the rank of $A_{22}$. Then we have*

$$F_p^{(n_1+m)}(A_{11}\perp A_{22}, X) = F_p^{(m)}(A_{22}, \xi_p(A_{11})p^{n_1/2}X).$$

*Proof.* We may assume that $A_{22}$ is non-degenerate. By Lemma 2.1 for any $l \geq n_1 + n_2$ we have

$$\alpha_p(H_l, A_{11}\perp A_{22}) = \gamma_p(A_{11}\perp A_{22}, p^{-l})F_p(A_{11}\perp A_{22}, p^{-l}).$$

By Lemma 3.1, we have

$$\alpha_p(H_l, A_{11}\perp A_{22}) = \beta_p(H_l, A_{11})\alpha_p(H_{l-n_1}\perp(-A_{11}), A_{22}).$$

Again by Lemma 2.1, we have

$$\alpha_p(H_{l-n_1}\perp(-A_{11}), A_{22}) = \gamma_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l})F_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l}).$$

Furthermore we have

$$\beta_p(H_l, A_{11}) = (1-p^{-l})\prod_{i=1}^{n_1/2}(1-p^{2i-2l})(1-p^{n_1/2-l}\xi_p(A_{11}))^{-1}$$

(eg. [Kitaoka 1993]), and by definition we have

$$\gamma_p(A_{11}\perp A_{22}, p^{-l}) = \beta_p(H_l, A_{11})\gamma_p(A_{22}, \xi_p(A_{11})p^{n_1/2-l}).$$

Thus the assertion holds. $\square$

**Corollary 1.** *Let $A = \begin{pmatrix} A_{11} & A_{12}/2 \\ {}^tA_{12}/2 & A_{22} \end{pmatrix} \in \mathcal{H}_{n_1+n_2}(\mathbf{Z}_p) \cap GL_{n_1+n_2}(\mathbf{Q}_p)$ with $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z}_p), A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z}_p)$, and $A_{12} \in M_{n_1,n_2}(\mathbf{Z}_p)$. Let $m$ be the rank of $A$. Assume $2A_{11} \in GL_{n_1}(\mathbf{Z}_p)$. Then we have*

$$F_p^{(m)}(A, X) = F_p^{(m-n_1)}(A_{22} - \frac{1}{4}A_{11}^{-1}[A_{12}], \xi_p(A_{11})p^{n_1/2}X).$$

**Corollary 2.** *Let $n_1$ and $n_2$ be positive even integers. Let* $A = \begin{pmatrix} A_{11} & A_{12}/2 \\ {}^tA_{12}/2 & A_{22} \end{pmatrix} \in \mathcal{H}_{n_1+n_2}(\mathbf{Z})_{>0}$ *with* $A_{11} \in \mathcal{H}_{n_1}(\mathbf{Z})_{>0}, A_{22} \in \mathcal{H}_{n_2}(\mathbf{Z})_{>0},$ *and* $A_{12} \in M_{n_1,n_2}(\mathbf{Z})$. *Let $p_0$ be a prime number. Let $m$ be the rank of $A$. Assume $2A_{11} \in GL_{n_1}(\mathbf{Z}_p)$ for any prime number $p \neq p_0$, and $2A_{22} \in GL_{n_2}(\mathbf{Z}_{p_0})$. Then we have*

$$\prod_p F_p^{(m)}(A, X) = F_{p_0}^{(m-n_2)}(A_{11} - \frac{1}{4}A_{22}^{-1}[{}^tA_{12}], \chi_{A_{22}}(p_0)p_0^{n_2/2}X)$$

$$\times \prod_{p \neq p_0} F_p^{(m-n_1)}(A_{22} - \frac{1}{4}A_{11}^{-1}[A_{12}], \chi_{A_{11}}(p)p^{n_1/2}X).$$

Now to make the computation in Section 4 smooth, we give an explicit form of $F_p^{(1)}(A, X)$ and $F_p^{(2)}(A, X)$ in case $\deg A = 2$ (e.g. [Katsurada 2005].)

**Proposition 3.3.** *Let* $A = \begin{pmatrix} a_{11} & a_{12}/2 \\ a_{12}/2 & a_{22} \end{pmatrix} \in \mathcal{H}_2(\mathbf{Z})_{\geq 0}$. *Put* $e = e_A =$ $\mathrm{GCD}(a_{11}, a_{12}, a_{22})$

(1) *Assume* rank $A = 1$. *Then we have*

$$F_p^{(1)}(A, X) = \sum_{i=0}^{\mathrm{ord}_p(e_A)} (pX)^i.$$

(2) *Assume $A > 0$. Then we have*

$$F_p^{(2)}(A, X) = \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i} (p^3X^2)^j$$

$$-\chi_A(p)pX \sum_{i=0}^{\mathrm{ord}_p(e_A)} (p^2X)^i \sum_{j=0}^{\mathrm{ord}_p(\mathfrak{f}_A)-i-1} (p^3X^2)^j.$$

Now we give an explicit form of differential operator in the case of degree 2 due to Ibukiyama. Let $y_1, y_2, y_3$ be variables, and for a positive even integer $l$ put

$$G_l(y_1, y_2, y_3; t)$$
$$= \frac{1}{R(y_1, y_2, y_3; t)^{(2l-5)/2}(\Delta_0(y_1, y_2; t)^2 - 4y_3t^2)^{1/2}},$$

where

$$\Delta_0(y_1,y_2;t)=1-y_1t+y_2t^2$$

and

$$R(y_1,y_2,y_3;t)=(\Delta_0(y_1,y_2;t)+(\Delta_0(y_1,y_2;t)^2-4y_3t^2)^{1/2})/2.$$

Write

$$G_l(y_1,y_2,y_3;t)=\sum_{m=0}^{\infty} G_{l,m}(y_1,y_2,y_3)t^m,$$

and define a polynomial map $Q_{l,m}(\begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix})$ from $S_4(\mathbf{C})$ to $\mathbf{C}$ by

$$Q_{l,m}(\begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix})=G_{l,m}(\det W_2, \det W_1 \det W_4, \det \begin{pmatrix} W_1 & W_2 \\ {}^tW_2 & W_4 \end{pmatrix}),$$

where $W_1,W_4\in S_2(\mathbf{C})$, and $W_2\in M_2(\mathbf{C})$. Furthermore define a polynomial map $P_{l,m}(X_1,X_2)$ from $M_2(\mathbf{C})\times M_2(\mathbf{C})$ to $\mathbf{C}$ by

$$P_{l,m}(X_1,X_2)=Q_{l,m}(\begin{pmatrix} X_1{}^tX_1 & X_1{}^tX_2 \\ X_2{}^tX_1 & X_2{}^tX_2 \end{pmatrix}).$$

Then by [Ibukiyama 1999]

**Proposition 3.4.** *$P_{l,m}(X_1,X_2)$ satisfies the conditions D-1 $\sim$ D-3 in [Katsurada 2008, Section 3].*

Furthermore, by a direct but rather elaborate calculation we have

**Proposition 3.5.**

$$G_{l,m}(y_1,y_2,y_3)=\sum_{n=0}^{[m/2]} \begin{pmatrix} 2n+l-5/2 \\ n \end{pmatrix} y_3^n$$

$$\times \sum_{\nu=0}^{[(m-2n)/2]} (-y_2)^{\nu} \begin{pmatrix} l+m-\nu-5/2 \\ m-2n-\nu \end{pmatrix} \begin{pmatrix} m-2n-\nu \\ \nu \end{pmatrix} (2y_1)^{m-2n-2\nu},$$

*where* $\begin{pmatrix} s \\ m \end{pmatrix}=\dfrac{\prod_{i=1}^{m}(s-i+1)}{m!}$.

We note that $G_{l,m}(y_1,y_2,y_3) \in 2^{-m}\mathbf{Z}[y_1,y_2,y_3]$. Let

$$\mathcal{G}_{l,m} = G_{l,m}(\frac{1}{4}\det(\frac{\partial}{\partial z_{ij}})_{1\le i\le 2, 3\le j\le 4}, \det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1\le i,j\le 2}\det(\frac{\tilde{\partial}}{\partial z_{ij}})_{3\le i,j\le 4}, \det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1\le i,j\le 4}),$$

and for $f \in C^{\infty}(\mathbf{H}_4)$ we define $\tilde{\mathcal{G}}_{l,m}(f)$ by

$$\tilde{\mathcal{G}}_{l,m}(f) = \mathcal{G}_{l,m}(f)_{Z_{12}=0},$$

where we write $Z = \begin{pmatrix} Z_1 & Z_{12} \\ {}^tZ_{12} & Z_2 \end{pmatrix}$ with $Z_1, Z_2 \in \mathbf{H}_2$ and $Z_{12} \in M_2(\mathbf{C})$. By [Ibukiyama 1999], $\tilde{\mathcal{G}}_{l,m}$ is a constant multiple of $\overset{\circ}{\mathcal{D}}{}_{2,l}^{m}$. Namely we have

$$\tilde{\mathcal{G}}_{l,m} = d_{l,m}\,\overset{\circ}{\mathcal{D}}{}_{2,l}^{m}$$

with some $d_{l,m}$. To obtain an exact value of $d_{l,m}$, put $w = z_{13}z_{24} - z_{14}z_{23}$. Then for any integer $s$ we have

$$(\frac{1}{4}\det(\frac{\partial}{\partial z_{ij}})_{1\le i\le 2, 3\le j\le 4})(w^s) = C_2(s/2)w^{s-1},$$

$$\det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1\le i,j\le 4})(w^s) = C_2(s/2)C_2((s-1)/2)w^{s-2},$$

and

$$\det(\frac{\tilde{\partial}}{\partial z_{ij}})_{1\le i,j\le 2}\det(\frac{\tilde{\partial}}{\partial z_{ij}})_{3\le i,j\le 4}(w^s) = 0,$$

where $C_2(s) = s(s+1/2)$. Thus for a positive even integer $m$ we have

$$\tilde{\mathcal{G}}_{l,m}(w^m) = \prod_{\mu=1}^{m} C_2(\mu/2)\sum_{n=0}^{m/2}\begin{pmatrix} 2n+l-5/2 \\ n \end{pmatrix}\begin{pmatrix} l+m-5/2 \\ m-2n \end{pmatrix}2^{m-2n}$$

$$= \prod_{\mu=1}^{m} C_2(\mu/2)\begin{pmatrix} 2l+2m-5 \\ m \end{pmatrix}.$$

Here we have used the formula

$$\sum_{n=0}^{m/2}\begin{pmatrix} s-m+2n \\ n \end{pmatrix}\begin{pmatrix} s \\ m-2n \end{pmatrix}2^{m-2n} = \begin{pmatrix} 2s \\ m \end{pmatrix}$$

for $s \in \mathbf{C}$. On the other hand we have

$$\overset{\circ}{\mathcal{D}}{}^{m}_{2,l}\,(w^m) = \prod_{\mu=1}^{m} C_2(\mu/2)C_2(l-2+m-\mu/2)$$

(cf. Section 3 of [Katsurada 2008],) and therefore we have

$$d_{l,m} = \frac{\begin{pmatrix} 2l+2m-5 \\ m \end{pmatrix}}{\prod_{\mu=1}^{m} C_2(l-2+m-\mu/2)}.$$

Now, for a positive even integer $l \leq k-2$ put

$$\tilde{\Lambda}(f,l,\underline{\mathrm{St}}) = \frac{\begin{pmatrix} 2k-5 \\ k-l-2 \end{pmatrix}}{2^{4k-2l-9}}\Gamma(l+1)\Gamma(k+l-2)\Gamma(k+l-1)\frac{L(f,l,\underline{\mathrm{St}})}{< f,f > (2\pi)^{2k+3l-3}}.$$

Now let $l$ be an even integer such that $4 \leq l \leq k-2$, and put

$$\tilde{\mathcal{E}}_{4,l,k}(z_1,z_2) = (-1)^{l/2+1}2^{-2}(2\pi\sqrt{-1})^{2(l-k)}(l-2)\tilde{\mathcal{G}}_{l,k-l}(E_{4,l})(z_1,z_2),$$

where $z_1, z_2 \in \mathbf{H}_2$. Then we note that $E_{4,l}(Z,0)$ belongs to $\mathfrak{M}_l(\Gamma^{(4)})$, and $\tilde{\mathcal{E}}_{4,l,k}(z_1,z_2)$ belongs to $\mathfrak{S}_k(\Gamma^{(2)}) \otimes \mathfrak{S}_k(\Gamma^{(2)})$. We note that

$$\tilde{\Lambda}(f,l-2,\underline{\mathrm{St}}) = d_{l,k-l}\Lambda(f,l-2,\underline{\mathrm{St}}),$$

and

$$\tilde{\mathcal{E}}_{4,l,k}(z_1,z_2) = d_{l,k-l}\mathcal{E}_{4,l,k}(z_1,z_2).$$

Now for a positive definite half-integral matrices $A_1$ and $A_2$ of degree 2, let $\epsilon_{l,k}(A_1,A_2)$ be the one in Section 2, and put $\tilde{\epsilon}_{l,k}(A_1,A_2) = d_{l,k}\epsilon_{l,k-l}(A_1,A_2)$. Furthermore, for each positive definite half-integral matrix $A_1$ put

$$\tilde{\mathcal{F}}_{l,k;A_1}(z_2) = \sum_{A_2 \in \mathcal{H}_2(\mathbf{Z})_{>0}} \tilde{\epsilon}_{l,k}(A_1,A_2)\mathbf{e}(\mathrm{tr}(A_2z_2)).$$

Then we have $\tilde{\mathcal{F}}_{l,k;A_1}(z_2) = d_{l,k-l}\mathcal{F}_{l,k;A_1}(z_2)$, and therefore we have

$$\tilde{\mathcal{E}}_{4,l,k}(z_1,z_2) = \sum_{A_1 \in \mathcal{H}_2(\mathbf{Z})_{>0}} \tilde{\mathcal{F}}_{l,k;A_1}(z_2)\mathbf{e}(\mathrm{tr}(A_1z_1)).$$

Let $p_0$ be a prime number. Assume that $2A_1 \in GL_2(\mathbf{Z}_p)$ for any prime number $p \neq p_0$ and $2A_2 \in GL_2(\mathbf{Z}_{p_0})$. Then we have

$$\tilde{\epsilon}_{l,k}(A_1, A_2) = \sum_{R \in M_2(\mathbf{Z})} \tilde{c}_{4,l}(\begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix})$$

$$\times G_{l,k-l}(\frac{1}{4} \det R, \det A_1 \det A_2, \det \begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}),$$

where

$$\tilde{c}_{4,l}(A) = (-1)^{l/2+1}(l-2)$$

$$\times F_{p_0}^{(m-2)}(A_1 - \frac{1}{4}A_2^{-1}[{}^tR], \chi_{A_2}(p_0)p_0^{l-m}) \prod_{p \neq p_0} F_p^{(m-2)}(A_2 - \frac{1}{4}A_1^{-1}[R], \chi_{A_1}(p)p^{l-m})$$

$$\times \begin{cases} L(3-l, \chi_A) & \text{if } m = 4 \\ \zeta(5-2l) & \text{if } m = 3, \end{cases}$$

for $A = \begin{pmatrix} A_1 & R/2 \\ {}^tR/2 & A_2 \end{pmatrix}$ with $\mathrm{rank}(A) = m$. We note that $\tilde{\epsilon}_{l,k}(A_1, A_2)$ is rational number and any prime divisor of its denominator is not greater than $(2l-1)!$.

Fix an $A_1 \in \mathcal{H}_2(\mathbf{Z})_{>0}$ and a prime number $p$. We define $\epsilon_{l,k}(i, A_1, A)$ as follows:

$$\epsilon_{l,k}(0, A_1, A) = \tilde{\epsilon}_{l,k}(A_1, A),$$

$$\epsilon_{l,k}(i, A_1, A) = \epsilon_{l,k}(i-1, A_1, pA) + p^{2k-3}\epsilon_{l,k}(i-1, A_1, A/p)$$

$$+p^{k-2} \sum_{D \in GL_2(\mathbf{Z})U_pGL_2(\mathbf{Z})/GL_2(\mathbf{Z})} \epsilon_{l,k}(i-1, A_1, A[D]/p),$$

where $U_p = \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}$. Let $\{f_j\}_{j=1}^d$ be an orthogonal basis of $\mathfrak{S}_k(\Gamma^{(2)})$ consisting of Hecke eigenforms, and $\lambda_j$ be the eigenvalue of $T(p)$ for $f_j$. Then by the Hecke theory of Siegel modular forms (e.g. Andrianov [Andrianov 1987]) we have

$$\mathcal{F}_{l,k;A_1}|T(p)^i(z_2) = \sum_{A_2 \in \mathcal{H}_2(\mathbf{Z})_{>0}} \epsilon_{l,k}(i, A_1, A_2)\mathbf{e}(\mathrm{tr}(A_2 z_2))$$

for any non-negative integer $i$ and $A_1 \in \mathcal{H}_2(\mathbf{Z})_{>0}$. Thus by Theorem 2.5 we have the following.

**Theorem 3.6.** *In addition to the above assumption, let $f = f_1, \lambda = \lambda_1, a(A_1) = a_1(A_1), a(A) = a_1(A), K = \mathbf{Q}(f)$, and for a positive even integer $l \le k-4$ put $e_i = \epsilon_{l+2,k}(i, A_1, A)$. Furthermore, let $\Phi(X) = \Phi_{T(p)}(X) = \sum_{i=0}^{d} b_{d-i}X^i$ be the characteristic polynomial of $T(p)$ in $\mathfrak{S}_k(\Gamma^{(2)})$. Put $\tilde{\Lambda}^*(f,l,\underline{\mathrm{St}}) = N_{K/\mathbf{Q}}(\tilde{\Lambda}(f,l,\underline{\mathrm{St}}))N(\mathfrak{J}_f)^2$. Assume that $\Phi'(\lambda) \neq 0$, and $a(A_1)a(A) \neq 0$. Then we have*

$$\tilde{\Lambda}^*(f,l,\underline{\mathrm{St}}) = N_{K/\mathbf{Q}}\left(\frac{\sum_{i=0}^{d-1}\sum_{j=i}^{d-1} e_{d-1-j}b_{j-i}\lambda^i}{\Phi'(\lambda)}\right)\frac{N(\mathfrak{J}_f)^2}{N_{K/\mathbf{Q}}(a(A_1)a(A))}.$$

# 4 Numerical examples and comments

We compute the special values of the standard zeta functions by using Mathematica. Let $\phi_{10,1}(\tau,z)$ and $\phi_{12,1}(\tau,z)$ be the Jacobi cusp forms in $J_{10,1}^{\mathrm{cusp}}$ and $J_{12,1}^{\mathrm{cusp}}$ in Page 40 of [Eichler and Zagier 1985], respectively. Here $\tau \in \mathbf{H}_1$ and $z \in \mathbf{C}$. Furthermore let $E_{1,k}(\tau)$ be the Eisenstein series of weight $k$ with respect to $\Gamma^{(1)}$ defined in Section 2, and put $E_k(\tau) = \zeta(1-k)^{-1}E_{1,k}(\tau)$. Then it is well known that $E_4^a(\tau)E_6(\tau)^b\phi_{j,1}(\tau,z)$ $(a,b \ge 0, j = 10,12, 4a+6b+j = k)$ form a basis of $J_{k,1}^{\mathrm{cusp}}$. Let $A_0 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $A_2 = \begin{pmatrix} 2 & 1/2 \\ 1/2 & 2 \end{pmatrix}$. Furthermore, we denote by $\mathfrak{S}_k(\Gamma^{(2)})^*$ the Maass subspace of $\mathfrak{S}_k(\Gamma^{(2)})$.

(1) We have dim $\mathfrak{S}_{20}(\Gamma^{(2)}) = 3$, and dim $\mathfrak{S}_{38}(\Gamma^{(1)}) = 2$. Let $f_1, f_2$ be the basis of $\mathfrak{S}_{38}(\Gamma^{(1)})$ consisting of primitive forms. For $i = 1,2$ let $\lambda_i = 48(-2025 + \sqrt{D})$ and $48(-2025 - \sqrt{D})$ with $D = 63737521$. Then $\lambda_i$ is the eigenvalues of the $T(2)$ with respect to $f_i$. Then they satisfy the equation $X^2 + 194400X^2 - 137403408384 = 0$, and $\mathbf{Q}(f_i) = \mathbf{Q}(\lambda_i) = K$ with $K = \mathbf{Q}(\sqrt{D})$ (cf. [Hida and Maeda 1997].) Put $\theta_i = \lambda_i/96$. Then $\theta_i$ satisfies the following equation $g(X) := X^2 + 2025X - 14909224 = 0$. The discriminant of $g(X)$ is $D$. Thus the discriminant of $\mathbf{Q}(\sqrt{D})$ is $D$, and the ring of integers in $\mathbf{Q}(\sqrt{D})$ is $\mathbf{Z}(\theta_1)$. Let $h_1(\tau,z) = E_4(\tau)E_6(\tau)\phi_{10,1}(\tau,z)$, and $h_2(\tau,z) = E_4(\tau)^2\phi_{12,1}(\tau,z)$. These form a basis of $J_{20,1}^{\mathrm{cusp}}$. Put $g_i = \mathcal{V}h_i$ for $i = 1,2$ (cf. [Katsurada 08, p.108]). Then these form a basis of $\mathfrak{S}_{20}(\Gamma^{(2)})^*$ whose $A_0$-th Fourier coefficient is 1. Furthermore for $i = 1,2$ put $\hat{f}_i =$

$-230g_1 + (-4862 - \theta_i)g_2$. Then $\hat{f}_i$ is the Saito-Kurokawa lift of $f_i$ whose $A_0$-th Fourier coefficient is $a_{\hat{f}_i}(A_0) = -5092 - \theta_i$. We note that we have $\hat{f}_i = \chi_{20}^{(i)}/2$ for $i = 1, 2$, where $\chi_{20}^{(1)}$ and $\chi_{20}^{(2)}$ are the eigenforms in Kurokawa [Kurokawa 1978]. Then we have $a_{\hat{f}_i}(A_1) = -10(4816 + \theta_i)$. Furthermore we have $\lambda_{\hat{f}_i}(T(2)) = \lambda_i + 3 \cdot 2^{18}$. Then $N_{K/\mathbf{Q}}(a_{\hat{f}_i}(A_0)) = 2^2 \cdot 3^4 \cdot 5 \cdot 19 \cdot 23$, and $N_{K/\mathbf{Q}}(a_{\hat{f}_i}(A_1)) = -2^5 \cdot 3 \cdot 5^2 \cdot 23 \cdot 2659$. By a simple computation we have $N(\mathfrak{J}_{\hat{f}_i}) = 2^5 \cdot 3^2 \cdot 5 \cdot 23$. Let $\Upsilon 20$ be the cuspidal Hecke eigenform in Skoruppa [Skoruppa 1992]. It is a unique (up to constant) Hecke eigenform in $\mathfrak{S}_{20}(\Gamma^{(2)})$ which is not a Saito-Kurokawa lift. We note that $\Upsilon 20 = \chi_{20}^{(3)}/2$, where $\chi_{20}^{(3)}$ is the Hecke eigenform in [Kurokawa 1978]. Then $\hat{f}_1, \hat{f}_2$ and $\Upsilon 20$ form an orthogonal basis of $\mathfrak{S}_{20}(\Gamma^{(2)})$. We have $\mathfrak{J}_{\Upsilon 20} = 1$ and $a_{\Upsilon 20}(A_0) = 1$ and $a_{\Upsilon 20}(A_1) = 2^2$. Furthermore we have $\lambda_{\Upsilon 20}(T(2)) = -2^8 \cdot 3^2 \cdot 5 \cdot 73$. Thus by Theorem 3.6, we have the following tables:

Table 1.1.

| $l$ | $N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}})N(\mathfrak{J}_{\hat{f}_i})^2$ |
|---|---|
| 2 | $3^5 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 23^2 \cdot 29 \cdot 31/2^{46} \cdot D$ |
| 4 | $3^7 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17^2 \cdot 23 \cdot 29 \cdot 31 \cdot 173 \cdot 443/2^{39} \cdot D$ |
| 6 | $3^6 \cdot 5^3 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 29 \cdot 31 \cdot 227 \cdot 1381069/2^{31} \cdot D$ |
| 8 | $3^9 \cdot 5^5 \cdot 7 \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 23 \cdot 29 \cdot 31 \cdot 21347 \cdot 58169/2^{22} \cdot D$ |
| 10 | $3^8 \cdot 5^6 \cdot 7^2 \cdot 13^3 \cdot 17^2 \cdot 19 \cdot 23 \cdot 31 \cdot 863 \cdot 3673 \cdot 3426433/2^{17} \cdot D$ |
| 12 | $3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 17^2 \cdot 23 \cdot 29 \cdot 37 \cdot 293 \cdot 691^2 \cdot 33721 \cdot 96875477/2^4 \cdot D$ |
| 14 | $2^7 \cdot 3^8 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13^2 \cdot 17 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 467196139 \cdot 541368271/D$ |
| 16 | $2^{25} \cdot 3^{14} \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 67 \cdot 1699 \cdot 3617^2 \cdot 296551/D$ |

Table 1.2.

| $l$ | $\tilde{\Lambda}(\Upsilon 20, l, \underline{\mathrm{St}})$ |
|---|---|
| 2 | $3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29 \cdot 31/2^{28}$ |
| 4 | $3^2 \cdot 5^2 \cdot 13 \cdot 23 \cdot 29 \cdot 31 \cdot 113/2^{25}$ |
| 6 | $3^4 \cdot 5 \cdot 7 \cdot 29 \cdot 31 \cdot 7549/2^{17}$ |
| 8 | $3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 29 \cdot 31 \cdot 37 \cdot 4861/2^{16}$ |
| 10 | $3 \cdot 5 \cdot 7 \cdot 31 \cdot 283 \cdot 617 \cdot 4098371/2^{13}$ |
| 12 | $3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 29 \cdot 31 \cdot 337 \cdot 91909/2^6$ |
| 14 | $2^4 \cdot 3^4 \cdot 7^2 \cdot 13 \cdot 29 \cdot 12893 \cdot 2166127$ |
| 16 | $2^{11} \cdot 3^6 \cdot 5^3 \cdot 13 \cdot 23 \cdot 29 \cdot 347162819$ |

(2) We have dim $\mathfrak{S}_{22}(\Gamma^{(2)}) = 4$, and dim $\mathfrak{S}_{42}(\Gamma^{(1)}) = 3$. Let $f_1, f_2, f_3$ be the basis of $\mathfrak{S}_{42}(\Gamma^{(1)})$ consisting of primitive forms. For $i = 1, 2, 3$ let $\lambda_i$ be

the eigenvalues of the $T(2)$ with respect to $f_i$. Then they satisfy the equation

$$X^3 + 344688X^2 - 6374982426624X - 520435526440845312 = 0,$$

and $\mathbf{Q}(f_i) = \mathbf{Q}(\lambda_i)$ (cf. [Hida and Maeda 1997].) Put $\theta_i = \lambda_i/48$ for $i = 1, 2, 3$. Then $\theta_i$ is also an algebraic integer and satisfy the following equation:

$$g(X) := X^3 + 7181X^2 - 2766919456X - 4705905729536 = 0.$$

The discriminant of $g(X)$ is $-2^{10} \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 1465869841 \cdot 578879197969$. Let $h_1(\tau, z) = E_4(\tau)^3\phi_{10,1}(\tau, z), h_2(\tau, z) = E_6(\tau)^2\phi_{10,1}(\tau, z)$, and $h_3(\tau, z) = E_4(\tau)E_6(\tau)\phi_{12,1}(\tau, z)$. Then these form a basis of $J_{22,1}^{\mathrm{cusp}}$. Put $g_i = \mathcal{V}h_i$ for $i = 1, 2, 3$. Then these form a basis of $\mathfrak{S}_{22}(\Gamma^{(2)})^*$ whose $A_0$-th Fourier coefficient is 1. Furthermore for $i = 1, 2, 3$ put

$$\hat{f}_i = 1155(435776+31\theta_i)g_1-220(4760624+79\theta_i)g_2+(286270336-605563\theta_i+\theta_i^2)g_3.$$

Then $\hat{f}_i$ is the Saito-Kurokawa lift of $f_i$ whose $A_0$-th Fourier coefficient $a_{\hat{f}_i}(A_0)$ is $-257745664 - 421138\theta_i + \theta_i^2$. Then we have $a_{\hat{f}_i}(A_1) = 10(395073536 - 642248\theta_i + \theta_i^2)$, and $a_{\hat{f}_i}(A_2) = -352(-3767171584 - 182733\theta_i + \theta_i^2)$. Let $\theta = \theta_1, \hat{f} = \hat{f}_1, \lambda = \lambda_1$, and $K = \mathbf{Q}(f_1)$. Put $R_1 = 1155(435776 + 31\theta), R_2 = -220(4760624 + 79\theta)$, and $R_3 = 286270336 - 605563\theta + \theta^2$. For a while, for elements $u_1, u_2, ..., u_r$ of $K$ we denote by $< u_1, u_2, ..., u_r >$ the $\mathbf{Z}$-module generated by $u_1, u_2, ..., u_r$. Then

$$[< 1, \theta, \theta^2 >:< R_1, R_2, R_3 >] = 2^6 \cdot 3^3 \cdot 5^3 \cdot 7^2 \cdot 11^3 \cdot 13 \cdot 157,$$

and

$$[< R_1, R_2, R_3 >:< a_{\hat{f}}(A_0), a_{\hat{f}}(A_1), a_{\hat{f}}(A_2) >] = 2^8 3^4.$$

By observing the Fourier coefficients of $h_1(\tau, z), h_2(\tau, z)$, and $h_3(\tau, z)$, we see that $\det \begin{pmatrix} a_{g_1}(A) & a_{g_1}(B) & a_{g_1}(C) \\ a_{g_2}(A) & a_{g_2}(B) & a_{g_2}(C) \\ a_{g_3}(A) & a_{g_3}(B) & a_{g_3}(C) \end{pmatrix}$ is divided by $2^8 3^4$ for any $A, B, C \in \mathcal{H}_2(\mathbf{Z})_{>0}$. Thus $[< R_1, R_2, R_3 >: \mathfrak{I}_{\hat{f}}]$ is divided by $2^8 3^4$. Thus we have $< a_{\hat{f}}(A_0), a_{\hat{f}}(A_1), a_{\hat{f}}(A_2) >= \mathfrak{I}_{\hat{f}}$, and

$$[< R_1, R_2, R_3 >: \mathfrak{I}_{\hat{f}}] = 2^8 \cdot 3^4.$$

Now by using, "round 2 method", we can find an element $\eta = (5984+5805\theta+\theta^2)/10080$ in $\mathfrak{O}_K$ so that $1, \theta, \eta$ form an integral basis of a $p$-maximal order of $\mathfrak{O}_K$ for $p = 2, 3, 5, 7$ (cf. H. Cohen [Cohen 1993].) We have

$$[< 1, \theta, \eta >:< 1, \theta, \theta^2 >] = 2^5 \cdot 3^2 \cdot 5 \cdot 7,$$

and therefore the discriminant $D_K$ of $K$ is not divisible by $2 \cdot 3 \cdot 5 \cdot 7$. Thus, we have $D_K = -D$ with $D = 1465869841 \cdot 578879197969$ and $\mathfrak{O}_K =< 1, \theta, \eta >$. This has been also examined with Magma [Bosma, Cannon, and Playoust 1997] by M. Kida. The author thanks him for his kind help. Thus we have

$$N(\mathfrak{J}_{\hat{f}}) = 2^{19} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 157.$$

Furthermore we have

$$\lambda_{\hat{f}}(T(2)) = \lambda + 3 \cdot 2^{20},$$

$$N_{K/\mathbf{Q}}(a_{\hat{f}}(A_0)) = -2^{14} \cdot 3^{13} \cdot 5^4 \cdot 7^4 \cdot 11^3 \cdot 13 \cdot 157 \cdot 1213$$

and

$$N_{K/\mathbf{Q}}(a_{\hat{f}}(A_1)) = -2^{24} \cdot 3^8 \cdot 5^5 \cdot 7^3 \cdot 11^3 \cdot 13 \cdot 157 \cdot 1447 \cdot 2437.$$

Let $\Upsilon 22$ be the cuspidal Hecke eigenform in Skoruppa [Skoruppa 1992]. It is a unique (up to constant) Hecke eigenform in $\mathfrak{S}_{22}(\Gamma^{(2)})$ which is not a Saito-Kurokawa lift. Then $\hat{f}_1, \hat{f}_2, \hat{f}_3$, and $\Upsilon 22$ form an orthogonal basis of $\mathfrak{S}_{22}(\Gamma^{(2)})$ and $\mathbf{Q}(\hat{f}_i) = \mathbf{Q}(f_i)$. We note that $N_{K_i/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}}))N(\mathfrak{J}_{\hat{f}_i})^2 = N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}, l, \underline{\mathrm{St}}))N(\mathfrak{J}_{\hat{f}})^2$ for any $i$. We have $\mathfrak{J}_{\Upsilon 22} = 1$ and $a_{\Upsilon 22}(A_0) = 1$ and $a_{\Upsilon 22}(A_1) = -2^2 \cdot 3$. Furthermore we have $\lambda_{\Upsilon 22}(T(2)) = -2^8 \cdot 3 \cdot 5 \cdot 577$. Thus by Theorem 3.6, we have the following tables:

Table 2.1.

| $l$ | $N_{K_i/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}}))N(\mathfrak{I}_{\hat{f}_i})^2$ |
|---|---|
| 2 | $3^9 \cdot 5^4 \cdot 7 \cdot 11^3 \cdot 13^5 \cdot 17^2 \cdot 19^3 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 37/2^{78} \cdot D$ |
| 4 | $3^{12} \cdot 5^2 \cdot 7^4 \cdot 11^2 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 151 \cdot 1601 \cdot 6551$ |
| | $\times 7951/2^{69} \cdot 1423 \cdot D$ |
| 6 | $3^{12} \cdot 5^9 \cdot 11^3 \cdot 13^3 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 137 \cdot 809$ |
| | $\times 38029874887/2^{57} \cdot 7 \cdot 1423 \cdot D$ |
| 8 | $3^9 \cdot 5 \cdot 7^5 \cdot 11 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23^2 \cdot 29 \cdot 31^2 \cdot 37 \cdot 84521 \cdot 8947751$ |
| | $\times 699588169271/2^{41} \cdot 1423 \cdot D$ |
| 10 | $3^{10} \cdot 5^9 \cdot 7^3 \cdot 11^4 \cdot 13^4 \cdot 17^2 \cdot 19^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37^2$ |
| | $\times 1423469629 \cdot 27864526583393/2^{28} \cdot 1423 \cdot D$ |
| 12 | $3^{12} \cdot 5 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19^2 \cdot 23^2 \cdot 29^2 \cdot 31 \cdot 37 \cdot 691^3 \cdot 953$ |
| | $\times 243911 \cdot 4251563 \cdot 6617174324030971171/2^{10} \cdot 7 \cdot 1423 \cdot D$ |
| 14 | $2^6 \cdot 3^{12} \cdot 5^5 \cdot 7^8 \cdot 11^3 \cdot 13^4 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37$ |
| | $\times 150197 \cdot 318467 \cdot 1465187 \cdot 13894099 \cdot 63630191/1423 \cdot D$ |
| 16 | $2^{26} \cdot 3^{19} \cdot 5^5 \cdot 7^2 \cdot 11^3 \cdot 13^3 \cdot 19 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37 \cdot 3617^3$ |
| | $\times 1465869841 \cdot 2775014078857939 \cdot 22683897890722493/1423 \cdot D$ |
| 18 | $2^{59} \cdot 3^{25} \cdot 5^{15} \cdot 11 \cdot 13^2 \cdot 17^2 \cdot 19^2 \cdot 23 \cdot 29^2 \cdot 31^2 \cdot 37^2$ |
| | $\times 107 \cdot 43867^3 \cdot 365257 \cdot 13553776667/1423 \cdot D$ |

Table 2.2.

| $l$ | $\tilde{\Lambda}(\Upsilon 22; l, \underline{\mathrm{St}})$ |
|---|---|
| 2 | $3^3 \cdot 5 \cdot 11 \cdot 23 \cdot 29 \cdot 31 \cdot 37/2^{32}$ |
| 4 | $3^4 \cdot 5 \cdot 11 \cdot 13 \cdot 29 \cdot 31 \cdot 37 \cdot 103 \cdot 157/2^{27} \cdot 1423$ |
| 6 | $3^6 \cdot 11 \cdot 29 \cdot 31 \cdot 37^2 \cdot 485363/2^{24} \cdot 1423$ |
| 8 | $3^2 \cdot 29 \cdot 31 \cdot 37 \cdot 149 \cdot 3361493719/2^{18} \cdot 1423$ |
| 10 | $3^3 \cdot 5 \cdot 11 \cdot 37 \cdot 89 \cdot 1039 \cdot 2741 \cdot 3616027/2^{15} \cdot 1423$ |
| 12 | $3^4 \cdot 11^2 \cdot 31 \cdot 37 \cdot 421 \cdot 254725279909/2^8 \cdot 1423$ |
| 14 | $3^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 31 \cdot 37 \cdot 733 \cdot 2131 \cdot 82625047/2 \cdot 1423$ |
| 16 | $2^5 \cdot 3^7 \cdot 5 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 30293340159041/1423$ |
| 18 | $2^{16} \cdot 3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 17 \cdot 31 \cdot 37 \cdot 101 \cdot 439 \cdot 1049 \cdot 49991/1423$ |

Finally we give some comments. First, observing Tables 1.1 and 2.1, we note that prime factors of the $l$-th Bernoulli number and the norm of the algebraic part of $L(f_i, l+k-2)L(f_i, l+k-1)$ appear in the numerator of $N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}})N(\mathfrak{I}_{\hat{f}_i})^2$. For examples, the prime factor 43867 of $N_{K/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, 18, \underline{\mathrm{St}})N(\mathfrak{I}_{\hat{f}_i})^2$ in Table 2.1 is a prime factor of the numerator of 18-th Bernoulli number, and the prime factors 13553776667 and 365257 of it appear in the norm of the algebraic parts of $L(f_i, 38)$ and $L(f_i, 39)$, respectively (cf. [Stein 2004].) This is not so surprising because we have

$$L(\hat{f}_i, l, \underline{\mathrm{St}}) = \zeta(l)L(f_i, l+k-2)L(f_i, l+k-1)$$

for $f_i \in \mathfrak{S}_{2k-2}(\Gamma^{(1)})$. Next we give a comment on our conjecture in [Katsurada 2008]. Let $f$ be a Hecke eigenform in $\mathfrak{S}_k(\Gamma^{(n)})$, and $M$ be a subspace of $\mathfrak{S}_k(\Gamma^{(n)})$ stable under Hecke operators $T \in \mathbf{L}'_n$. Assume that $M$ is contained in $(\mathbf{C}f)^{\perp}$, where $(\mathbf{C}f)^{\perp}$ is the orthogonal complement of $\mathbf{C}f$ in $\mathfrak{S}_k(\Gamma^{(n)})$ with respect to the Petersson product. A prime ideal $\mathfrak{P}$ of $\mathfrak{O}_{\mathbf{Q}(f)}$ is called a congruence prime of $f$ with respect to $M$ if there exists a Hecke eigenform $g \in M$ such that

$$\lambda_f(T) \equiv \lambda_g(T) \bmod \tilde{\mathfrak{P}}$$

for any $T \in \mathbf{L}'_n$, where $\tilde{\mathfrak{P}}$ is some prime ideal of $\mathfrak{O}_{\mathbf{Q}(f)\mathbf{Q}(g)}$ lying above $\mathfrak{P}$. If $M = (\mathbf{C}f)^{\perp}$, we simply call $\mathfrak{P}$ a congruence prime of $f$.

Now to explain our conjecture, for a normalized Hecke eigenform $f$ in $\mathfrak{S}_{2k-2}(\Gamma^{(1)})$ and a Dirichlet character $\chi$, let $L(f, s, \chi)$ be the Hecke L-function of $f$ twisted by $\chi$ define by as follows:

$$L(f, s, \chi) = \sum_{m=1}^{\infty} a_f(m)\chi(m)m^{-s}.$$

In particular, if $\chi$ is the principal character we write $L(f, s, \chi)$ as $L(f, s)$. Put $\Omega_f^{(+)} = (2\pi\sqrt{-1})^{-1}L(f, 1)$, and $\Omega_f^{(-)} = (2\pi\sqrt{-1})^{-2}L(f, 2)$. For $j = \pm, 1 \le l \le 2k-3$, and a Dirichlet character $\chi$ such that $\chi(-1) = j(-1)^{l-1}$, put

$$\mathbf{L}(f, l, \chi) = \frac{(2\pi\sqrt{-1})^{-l}\Gamma(l)L(f, l, \chi)}{\Omega^{(j)}}.$$

In particular, put $\mathbf{L}(f, l) = \mathbf{L}(f, l, \chi)$ if $\chi$ is the principal character. Then, in [Katsurada 2008], we proposed the following conjecture:

**Conjecture.** *Let* $\mathfrak{P}$ *be a prime ideal of* $\mathbf{Q}(f)$ *not dividing* $(2k-1)!$. *Then* $\mathfrak{P}$ *is a congruence prime of* $\hat{f}$ *with respect to* $(\mathfrak{S}_k(\Gamma^{(2)})^*)^{\perp}$ *if and only if* $\mathfrak{P}$ *divides the numerator of* $\mathbf{L}(f,k)$.

We note that that the "if part" of the above conjecture is a special case of Harder's conjecture (cf. [Harder 2003].)

Now look at Table 2.1. in this case, the prime number 1423 appear in the denominator of $N_{K_i/\mathbf{Q}}(\tilde{\Lambda}(\hat{f}_i, l, \underline{\mathrm{St}}))N(\mathfrak{J}_{\hat{f}_i})^2$ for $l = 4, \cdots, 18$. We have

$$1423 = \mathfrak{P}_i \mathfrak{P}_i'$$

in $\mathfrak{O}_{\mathbf{Q}(f_i)}$, where $\mathfrak{P}_i =< \lambda_i + 967, 1423 >$ and $\mathfrak{P}_i' =< \lambda_i^2 + 778\lambda_i + 660, 1423 >$. We have deg $\mathfrak{P}_i = 1$ and deg $\mathfrak{P}_i' = 2$. Thus by Theorem 5.2 of [Katsurada 2008], $\mathfrak{P}_i$ is a congruence prime of $\hat{f}_i$, and by a more careful analysis, we see that it is a congruence prime of $\hat{f}_i$ with respect to $\mathbf{C}\Upsilon 22$. In fact, we have

$$\lambda_{\hat{f}_i}(T(2)) \equiv \lambda_{\Upsilon 22}(T(2)) \bmod \mathfrak{P}_i.$$

Conversely, by a direct calculation, we see that there is no other congruence primes $> 43$ of $\hat{f}_i$ with respect to $\mathbf{C}\Upsilon 22$. On the other hand, according to the numerical table in Stein [Stein 2004], we have

$$|N_{\mathbf{Q}(f)/\mathbf{Q}}(\mathbf{L}(f, 22))| = \frac{11^3 \cdot 17 \cdot 1423}{2^{23} \cdot 3^{18} \cdot 5^{10} \cdot 13 \cdot 29 \cdot 31 \cdot 37^2 \cdot 137 \cdot 7481}.$$

This implies that the conjecture in Section 6 of [Katsurada 2008] is true in this case.

We note that the "if" part has been proved by Brown [Brown 2007] and Katsurada [Katsurada 2008] independently under certain conditions.

**References**

[Andrianov 1987] A. N. Andrianov, Quadratic forms and Hecke operators, Springer, 1987.

[Böcherer 1985] S. Böcherer, Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen II, Math. Z. 189(1985), 81-110.

[Böcherer and Schmidt 2000] S. Böcherer and C.G. Schmidt, p-adic measures attached to Siegel modular forms, Ann. Inst. Fourier, 50(2000), 1375-1443.

[Bosma, Cannon, and Playoust 1997] W. Bosma, J. Cannon, and C. Playoust. The Magma algebra system. I. The user language. J. Symbolic Comput., 24(1997),235-265

[Brown 2007] J. Brown, Saito-Kurokawa lifts and applications to the Bloch-Kato conjecture, Compositio Math. 143(2007), 290-322.

[Cohen 1993] H. Cohen, A course in computational algebraic number theory, Springer, 1993.

[Eichler and Zagier 1985] M. Eichler and D. Zagier, The theory of Jacobi forms, Birkhäser, Boston, 1985.

[Garrett 1984] P. Garrett, Pullbacks of Eisenstein series, Automorphic forms of several variables, Progress in Math. 46, Birkhäuser, 1984, 114-137.

[Goto 1998] K. Goto, A twisted adjoint L-value of an elliptic modular form, J. Number Theory 73(1998), 34-46.

[Harder 2003] G. Harder, A congruence between a Siegel modular form and an elliptic modular form, Preprint 2003.

[Hida and Maeda 1997] H. Hida and Y. Maeda, Non abelian base change for totally real fields, Pacific J. Math. Olga Taussky-Todd Memorial Issue(1997), 189-218.

[Ibukiyama 1999] T. Ibukiyama, On Differential operators on automorphic forms and invariant pluri-harmonic polynomials, Comm. Math. Univ. St. Pauli, 48(1999), 103-118.

[Katsurada 1999] H. Katsurada, An explicit formula for Siegel series, Amer. J. Math. 121(1999), 415-452.

[Katsurada 2005] H. Katsurada, Special values of the standard zeta functions for elliptic modular forms, Experiment. Math. 14(2005), 27-45.

[Katsurada 2008] H. Katsurada, Congruence of Siegel modular forms and special values of their standard zeta functions, Math. Z. 259(2008), 97-111.

[Kitaoka 1984] Y. Kitaoka, Dirichlet series in the theory of quadratic forms, Nagoya Math. J. 92(1984), 73-84.

[Kitaoka 1993] Y. Kitaoka, Arithmetic of quadratic forms, Cambridge. Tracts Math. 106, Cambridge Univ. Press, Cambridge, 1993.

[Kurokawa 1978] N. Kurokawa, Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two, Invent. Math. 49(1978), 149-165.
[Mizumoto 1991] S. Mizumoto, Poles and residues of standard L-functions attached to Siegel modular forms, Math. Ann. 289(1991) 589-612.
[Shimura 1983] G. Shimura, On Eisenstein series, Duke Math. J. 50(1983), 417-476.
[Skoruppa 1992] N-P. Skoruppa, Computations of Siegel modular forms of genus two, Math. Comp. 58(1992), 381-398.
[Stein 2004] W. A. Stein, The modular forms data base,
http://modular.fas.harvard.edu/index.html