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journal or publication title	Journal of geometry
volume	104
number	2
page range	297-307
year	2013-08
URL	http://hdl.handle.net/10258/2732

doi: info:doi/10.1007/s00022-013-0162-6

Existence and uniqueness for Legendre curves

Tomonori Fukunaga and Masatomo Takahashi

Dedicated to Professor Masahiko Suzuki on the occasion of his 60th birthday

Abstract. We give a moving frame of a Legendre curve (or, a frontal) in the unit tangent bundle and define a pair of smooth functions of a Legendre curve like as the curvature of a regular plane curve. It is quite useful to analyse the Legendre curves. The existence and uniqueness for Legendre curves hold similarly to the case of regular plane curves. As an application, we consider contact between Legendre curves and the arc-length parameter of Legendre immersions in the unit tangent bundle.

Mathematics Subject Classification (2010). Primary 58K05; Secondary 53A04, 53D35.

Keywords. existence, uniqueness, Legendre curve, frontal, curvature of Legendre curve, congruent as Legendre curve.

1. Introduction

A regular plane curve determines a curvature function, providing valuable geometric information about the original curve by using a moving frame of the curve. The existence and uniqueness results are fundamental theorems for regular plane curves, see below Theorems 1.1 and 1.2.

Let I be an interval or $\mathbb{R}$. Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular curve, that is, $\dot{\gamma}(t) \neq 0$ for any $t \in I$. If s is the arc-length parameter of γ, we denote $\boldsymbol{t}(s)$ by the unit tangent vector $\boldsymbol{t}(s) = \gamma'(s) = d\gamma/ds(s)$ and $\boldsymbol{n}(s)$ by the unit normal vector $\boldsymbol{n}(s) = J(\boldsymbol{t}(s))$ of $\gamma(s)$, where J is the anticlockwise rotation by $\pi/2$. Then we have the Frenet formula as follows:

$$\begin{pmatrix} \boldsymbol{t}'(s) \\ \boldsymbol{n}'(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \end{pmatrix},$$

where $\kappa(s) = \boldsymbol{t}'(s) \cdot \boldsymbol{n}(s)$ is the curvature of γ and $\cdot$ is the inner product on $\mathbb{R}^2$.

Even if t is not the arc-length parameter, we have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$, the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$ and the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where $\dot{\gamma}(t) = d\gamma/dt(t)$, $|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$ and $\kappa(t) = \det(\dot{\gamma}(t), \ddot{\gamma}(t))/|\dot{\gamma}(t)|^3 = \dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)/|\dot{\gamma}(t)|$. Note that $\kappa(t)$ is independent on the choice of a parametrization.

Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2$ be regular curves. We say that γ and $\widetilde{\gamma}$ are *congruent* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t))$ for all $t \in I$, where the *congruence* C is a composition of a rotation and a translation on $\mathbb{R}^2$.

As well-known results, the existence and uniqueness for regular plane curves are as follows (cf. [5, 6]):

Theorem 1.1. (The Existence Theorem) *Let $\kappa : I \to \mathbb{R}$ be a smooth function. There exists a regular parametrized curve $\gamma : I \to \mathbb{R}^2$ whose associated curvature function is κ.*

Theorem 1.2. (The Uniqueness Theorem) *Let γ and $\widetilde{\gamma} : I \to \mathbb{R}^2$ be regular curves whose speeds $s = |\dot{\gamma}(t)|$ and $\widetilde{s} = |\dot{\widetilde{\gamma}}(t)|$, and also curvatures κ and $\widetilde{\kappa}$ each coincide. Then γ and $\widetilde{\gamma}$ are congruent.*

If γ has a singular point, we can not construct a moving frame of γ. In the analytic category, there is a construction of a moving frame of an analytic curve under a mild condition, see in [9]. However, we can define a moving frame of a frontal for a Legendre curve in the unit tangent bundle in the smooth category. By using the moving frame, we define a pair of smooth functions like as the curvature of a regular curve. We call the pair the *curvature of the Legendre curve*. It is quite useful to analyse the Legendre curves (or, frontals). In this paper, we give the existence and uniqueness for Legendre curves similarly to the case of regular plane curves, see Theorems 1.4 and 1.5. These results are elementary, however, they might be new results, as far as we know.

We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$ for all $t \in I$, where θ is a canonical contact form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ (cf. [1, 2]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) *a Legendre immersion*. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* (respectively, *a front* or *a wave front*) if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

Let $\gamma = (\gamma_1, \gamma_2) : (\mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a plane curve germ. Then it can be easily shown that, if γ is not infinitely flat, namely, if either γ_1 or γ_2 does not belong to $\mathfrak{m}_1^\infty$ (the ideal of infinitely flat function germs), then γ is a frontal. In fact, there exists a smooth function germ α such that $\dot{\gamma}_1(t) = \alpha(t)\dot{\gamma}_2(t)$ (or, $\dot{\gamma}_2(t) = \alpha(t)\dot{\gamma}_1(t)$). Thus if $\nu(t) = (1/\sqrt{\alpha^2(t)+1})(-\alpha(t), 1)$

(or, $\nu(t) = (1/\sqrt{\alpha^2(t)+1})(1, -\alpha(t)))$, then (γ, ν) is a Legendre curve. On the other hand, constant maps are frontals (fronts), which do not satisfy the above sufficient condition. In particular an analytic curve germ is always frontal, because if it is infinitely flat, then it is constant. We give an explicit example of plane curve which is not a frontal, using a pair of infinitely flat C^∞ functions, see Example 3 in §2.

If γ is a regular curve around a point t_0, then we have the Frenet formula of γ. On the other hand, if γ is singular at a point t_0, then we can not, in general, define such a frame. However, for a Legendre curve (γ, ν), the frame of γ is defined by ν even if t is a singular point of γ.

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. Then we have the Frenet formula of a frontal γ as follows. We put on $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame of a frontal* $\gamma(t)$ in $\mathbb{R}^2$ and we have the Frenet formula of a frontal (or, Legendre curve) which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{1.1}$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, if $\dot{\gamma}(t) = \alpha(t)\nu(t) + \beta(t)\boldsymbol{\mu}(t)$ for some smooth functions $\alpha(t), \beta(t)$, then $\alpha(t) = 0$ follows from the condition $\dot{\gamma}(t) \cdot \nu(t) = 0$. Hence, there exists a smooth function $\beta(t)$ such that

$$\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t). \tag{1.2}$$

The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals). We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t).

Definition 1.3. Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t)) = A(\gamma(t)) + \boldsymbol{b}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$, where C is given by the rotation A and the translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

The main results in this paper are the existence and uniqueness for Legendre curves in the unit tangent bundle similarly to the case of regular plane curves, see Theorems 1.1 and 1.2.

Theorem 1.4. (The Existence Theorem) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).*

Theorem 1.5. (The Uniqueness Theorem) *Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves (ℓ, β) and $(\widetilde{\ell}, \widetilde{\beta})$ coincide. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are congruent as Legendre curves.*

We shall prove these theorems in §2. Moreover, we consider properties of the curvatures of Legendre curves. As an application, we consider contact between Legendre curves in §3 and give a special parameter, so-called the arc-length parameter, of Legendre immersions in the unit tangent bundle in §4. As further applications, we give the evolute of a front by using the moving frame of a front and the curvature of the Legendre immersion, for more detail in [4].

All maps and manifolds considered here are differential of class C^∞.

2. Properties of Legendre curves

First we prove the existence theorem (Theorem 1.4).

Proof of Theorem 1.4. Let $\theta : I \to \mathbb{R}$ be any function with the property that $\dot\theta(t) = \ell(t)$ for all $t \in I$. Furthermore, let

$$\nu(t) = (\cos\theta(t), \sin\theta(t)),\ \boldsymbol{\mu}(t) = (-\sin\theta(t), \cos\theta(t))$$

be the curves in the unit circle. Define smooth functions $x(t)$ and $y(t)$ with $\dot x(t) = -\beta(t)\sin\theta(t)$ and $\dot y(t) = \beta(t)\cos\theta(t)$. Then $\gamma : I \to \mathbb{R}^2$ is given by $\gamma(t) = (x(t), y(t))$, that is,

$$\gamma(t) = \left(-\int\left(\beta(t)\sin\int\ell(t)\ dt\right)dt,\ \int\left(\beta(t)\cos\int\ell(t)\ dt\right)dt\right).$$

It follows that $\dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t)$, $\dot\nu(t) = \ell(t)\boldsymbol{\mu}(t)$ and $\dot\gamma(t)\cdot\nu(t) = 0$ for all $t \in I$. Therefore, there exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is $(\ell(t), \beta(t))$.

□

In order to prove the uniqueness theorem (Theorem 1.5), we need two Lemmas.

Lemma 2.1. *Let (γ,ν) and $(\widetilde\gamma,\widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be congruent as Legendre curves. Then (γ,ν) and $(\widetilde\gamma,\widetilde\nu)$ have the same curvatures of Legendre curves (ℓ,β) and $(\widetilde\ell,\widetilde\beta)$ respectively.*

Proof. Since (γ,ν) and $(\widetilde\gamma,\widetilde\nu)$ are congruent as Legendre curves, there exist a rotation A and a fixed vector $\boldsymbol{b}$ with the property that

$$\widetilde\gamma(t) = A(\gamma(t)) + \boldsymbol{b},\ \widetilde\nu(t) = A(\nu(t))$$

for all $t \in I$. Since the definition of $\boldsymbol{\mu}$ and $JA = AJ$, we have $\widetilde{\boldsymbol{\mu}}(t) = A(\boldsymbol{\mu}(t))$ for all $t \in I$. By $\dot\gamma(t) = \beta(t)\boldsymbol{\mu}(t)$ and $\dot\nu(t) = \ell(t)\boldsymbol{\mu}(t)$,

$$\frac{d}{dt}\widetilde\gamma(t) = A(\dot\gamma(t)) = A(\beta(t)\boldsymbol{\mu}(t)) = \beta(t)A(\boldsymbol{\mu}(t)) = \beta(t)\widetilde{\boldsymbol{\mu}}(t),$$
$$\frac{d}{dt}\widetilde\nu(t) = A(\dot\nu(t)) = A(\ell(t)\boldsymbol{\mu}(t)) = \ell(t)A(\boldsymbol{\mu}(t)) = \ell(t)\widetilde{\boldsymbol{\mu}}(t).$$

Hence we have $\beta(t) = \widetilde\beta(t)$ and $\ell(t) = \widetilde\ell(t)$. □

Lemma 2.2. *Let (γ,ν) and $(\widetilde\gamma,\widetilde\nu) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves having equal curvatures of Legendre curves, that is, $(\ell(t),\beta(t)) = (\widetilde\ell(t),\widetilde\beta(t))$ for all $t \in I$. If there exists a parameter $t = t_0$ for which $(\gamma(t_0),\nu(t_0)) = (\widetilde\gamma(t_0),\widetilde\nu(t_0))$, then (γ,ν) and $(\widetilde\gamma,\widetilde\nu)$ coincide.*

Proof. Let $f(t) = \nu(t) \cdot \widetilde{\nu}(t) + \boldsymbol{\mu}(t) \cdot \widetilde{\boldsymbol{\mu}}(t)$ be a smooth function on I. Then

$$\begin{aligned}
\dot{f}(t) &= \dot{\nu}(t) \cdot \widetilde{\nu}(t) + \nu(t) \cdot \dot{\widetilde{\nu}}(t) + \dot{\boldsymbol{\mu}}(t) \cdot \widetilde{\boldsymbol{\mu}}(t) + \boldsymbol{\mu}(t) \cdot \dot{\widetilde{\boldsymbol{\mu}}}(t) \\
&= (\ell(t)\boldsymbol{\mu}(t)) \cdot \widetilde{\nu}(t) + \nu(t) \cdot (\widetilde{\ell}(t)\widetilde{\boldsymbol{\mu}}(t)) \\
&\qquad + (-\ell(t)\nu(t)) \cdot \widetilde{\boldsymbol{\mu}}(t) + \boldsymbol{\mu}(t) \cdot (-\widetilde{\ell}(t)\widetilde{\nu}(t)) \\
&= (\ell(t) - \widetilde{\ell}(t))\boldsymbol{\mu}(t) \cdot \widetilde{\nu}(t) + (\widetilde{\ell}(t) - \ell(t))\nu(t) \cdot \widetilde{\boldsymbol{\mu}}(t) \\
&= 0,
\end{aligned}$$

since $\ell(t) = \widetilde{\ell}(t)$ by the assumption. It follows that f is constant. Moreover, setting $t = t_0$ and $\nu(t_0) = \widetilde{\nu}(t_0)$, then $\boldsymbol{\mu}(t_0) = \widetilde{\boldsymbol{\mu}}(t_0)$ and hence $f(t_0) = |\nu(t_0)|^2 + |\boldsymbol{\mu}(t_0)|^2 = 2$. The function f is the constant value 2. By the Cauchy-Schwarz inequality, we have

$$\nu(t) \cdot \widetilde{\nu}(t) \leq |\nu(t)||\widetilde{\nu}(t)| = 1, \ \boldsymbol{\mu}(t) \cdot \widetilde{\boldsymbol{\mu}}(t) \leq |\boldsymbol{\mu}(t)||\widetilde{\boldsymbol{\mu}}(t)| = 1.$$

If either of these inequalities were strict, the value of $f(t)$ would be less than 2. It follows that both these inequalities are equalities, and we have $\nu(t) \cdot \widetilde{\nu}(t) = 1,\ \boldsymbol{\mu}(t) \cdot \widetilde{\boldsymbol{\mu}}(t) = 1$ for all $t \in I$. Then we have

$$|\nu(t) - \widetilde{\nu}(t)|^2 = \nu(t) \cdot \nu(t) - 2\nu(t) \cdot \widetilde{\nu}(t) + \widetilde{\nu}(t) \cdot \widetilde{\nu}(t) = 0,$$

and also $|\boldsymbol{\mu}(t) - \widetilde{\boldsymbol{\mu}}(t)|^2 = 0$. Hence $\nu(t) = \widetilde{\nu}(t)$ and $\boldsymbol{\mu}(t) = \widetilde{\boldsymbol{\mu}}(t)$ for all $t \in I$. Since $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$, $\dot{\widetilde{\gamma}}(t) = \widetilde{\beta}(t)\widetilde{\boldsymbol{\mu}}(t)$ and the assumption $\beta(t) = \widetilde{\beta}(t)$, $(d/dt)(\gamma(t) - \widetilde{\gamma}(t)) = 0$. It follows that $\gamma(t) - \widetilde{\gamma}(t)$ is constant. By the condition $\gamma(t_0) = \widetilde{\gamma}(t_0)$, we have $\gamma(t) = \widetilde{\gamma}(t)$ for all $t \in I$. □

Proof of Theorem 1.5. Choose any fixed value $t = t_0$ of the parameter. By using a rotation A and a translation $\boldsymbol{b}$, we can assume that $\widetilde{\gamma}(t_0) = A(\gamma(t_0)) + \boldsymbol{b}$ and $\widetilde{\nu}(t_0) = A(\nu(t_0))$. By Lemma 2.1, the curvatures of the Legendre curves (γ, ν) and $(A(\gamma(t)) + \boldsymbol{b}, A(\nu(t)))$ coincide. By Lemme 2.2, $\widetilde{\gamma}(t) = A(\gamma(t)) + \boldsymbol{b}$, $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$. It follows that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are congruent as Legendre curves.

□

Remark 2.3. Both Theorems 1.4 and 1.5 can be proved also by using the theory of the existence and uniqueness for the system of ordinary differential equations.

Let I and $\overline{I}$ be intervals. A smooth function $s : \overline{I} \to I$ is a *(positive) change of parameter* when s is surjective and has a positive derivative at every point. It follows that s is a diffeomorphism map by calculus.

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ and $(\overline{\gamma}, \overline{\nu}) : \overline{I} \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves are (ℓ, β) and $(\overline{\ell}, \overline{\beta})$ respectively. Suppose (γ, ν) and $(\overline{\gamma}, \overline{\nu})$ are parametrically equivalent via the change of parameter $s : \overline{I} \to I$. Thus $(\overline{\gamma}(t), \overline{\nu}(t)) = (\gamma(s(t)), \nu(s(t)))$ for all $t \in \overline{I}$. By differentiation, we have

$$\overline{\ell}(t) = \ell(s(t))\dot{s}(t), \ \overline{\beta}(t) = \beta(s(t))\dot{s}(t).$$

Hence the curvature of the Legendre curve is depended on a parametrization. However, for a Legendre immersion $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$, we can define the normalized curvature and the arc-length parameter. Then the normalized curvature of the Legendre curve independent on the change of a parametrization, see §4. Note that (γ, ν) is a Legendre immersion if and only if $(\ell(t), \beta(t)) \neq (0, 0)$ for all $t \in I$.

Remark 2.4. By the definition of the Legendre curve, if (γ, ν) is a Legendre curve, then $(\gamma, -\nu)$ is also. In this case, $\ell(t)$ does not change, but $\beta(t)$ changes to $-\beta(t)$.

Now we give examples of Legendre curves.

Example 1. One of the typical example of a front (and hence a frontal) is a regular plane curve. Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. In this case, we may take $\nu : I \to S^1$ by $\nu(t) = \boldsymbol{n}(t)$. Then it is easy to check that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion (a Legendre curve).

By a direct calculation, we give a relationship between the curvature of the Legendre curve $(\ell(t), \beta(t))$ and the curvature $\kappa(t)$ if γ is a regular curve.

Proposition 2.5. ([4, Lemma 3.1]) *Under the above notions, if γ is a regular curve, then $\ell(t) = |\beta(t)|\kappa(t)$.*

Example 2. Let n, m and k be natural numbers with $m = n + k$. Let $(\gamma, \nu) : \mathbb{R} \to \mathbb{R}^2 \times S^1$ be

$$\gamma(t) = \left(\frac{1}{n}t^n, \frac{1}{m}t^m\right), \ \nu(t) = \frac{1}{\sqrt{t^{2k}+1}}\left(-t^k, 1\right).$$

It is easy to see that (γ, ν) is a Legendre curve, and a Legendre immersion when $k = 1$. We call γ is of *type* (n, m). For example, the frontal of type $(2, 3)$ has the $3/2$-cusp (A_2 singularity) at $t = 0$, of type $(3, 4)$ has the $4/3$-cusp (E_6 singularity) at $t = 0$ and of type $(2, 5)$ has the $5/2$-cusp (A_4 singularity) at $t = 0$ (cf. [2, 3, 7]). By definition, we have $\boldsymbol{\mu}(t) = (1/\sqrt{t^{2k}+1})(-1, -t^k)$ and

$$\ell(t) = \frac{kt^{k-1}}{t^{2k}+1}, \ \beta(t) = -t^{n-1}\sqrt{t^{2k}+1}.$$

We also give an explicit example of plane curve which is not a frontal.

Example 3. Let $\gamma : \mathbb{R} \to \mathbb{R}^2$ be

$$\gamma(t) = \begin{cases} (0, e^{-1/t^2}) & \text{if } t > 0, \\ (0, 0) & \text{if } t = 0, \\ (e^{-1/t^2}, 0) & \text{if } t < 0. \end{cases}$$

Then one can show that γ is not a frontal.

3. Contact between Legendre curves

In this section, we discuss contact between Legendre curves. Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t),\nu(t))$ and $(\widetilde{\gamma},\widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u),\widetilde{\nu}(u))$ be Legendre curves, respectively and let k be a natural number. We say that *(γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have k-th order contact at $t = t_0, u = u_0$* if

$$(\gamma,\nu)(t_0) = (\widetilde{\gamma},\widetilde{\nu})(u_0), \frac{d}{dt}(\gamma,\nu)(t_0) = \frac{d}{du}(\widetilde{\gamma},\widetilde{\nu})(u_0),$$
$$\cdots, \frac{d^{k-1}}{dt^{k-1}}(\gamma,\nu)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma},\widetilde{\nu})(u_0)$$

and

$$\frac{d^k}{dt^k}(\gamma,\nu)(t_0) \neq \frac{d^k}{du^k}(\widetilde{\gamma},\widetilde{\nu})(u_0).$$

Moreover, we say that *(γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least k-th order contact at $t = t_0, u = u_0$* if

$$(\gamma,\nu)(t_0) = (\widetilde{\gamma},\widetilde{\nu})(u_0), \ \frac{d}{dt}(\gamma,\nu)(t_0) = \frac{d}{du}(\widetilde{\gamma},\widetilde{\nu})(u_0),$$
$$\cdots, \frac{d^{k-1}}{dt^{k-1}}(\gamma,\nu)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\gamma},\widetilde{\nu})(u_0).$$

Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t),\nu(t))$ and $(\widetilde{\gamma},\widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u),\widetilde{\nu}(u))$ be Legendre curves. In general, we may assume that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least first order contact at any point $t = t_0, u = u_0$, up to congruence as Legendre curves. We denote the curvatures of the Legendre curves $(\ell(t),\beta(t))$ of (γ,ν) and $(\widetilde{\ell}(u),\widetilde{\beta}(u))$ of $(\widetilde{\gamma},\widetilde{\nu})$, respectively.

Theorem 3.1. *Let $(\gamma,\nu): I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t),\nu(t))$ and $(\widetilde{\gamma},\widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u),\widetilde{\nu}(u))$ be Legendre curves. If (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$ then*

$$(\ell,\beta)(t_0) = (\widetilde{\ell},\widetilde{\beta})(u_0), \ \frac{d}{dt}(\ell,\beta)(t_0) = \frac{d}{du}(\widetilde{\ell},\widetilde{\beta})(u_0),$$
$$\cdots, \ \frac{d^{k-1}}{dt^{k-1}}(\ell,\beta)(t_0) = \frac{d^{k-1}}{du^{k-1}}(\widetilde{\ell},\widetilde{\beta})(u_0). \tag{3.1}$$

Conversely, if the condition (3.1) *holds, then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$, up to congruence as Legendre curves.*

Proof. Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least second order contact at $t = t_0, u = u_0$. Since $\nu(t_0) = \widetilde{\nu}(u_0)$, we have $\boldsymbol{\mu}(t_0) = \widetilde{\boldsymbol{\mu}}(u_0)$. By (1.1) and (1.2), $(d/dt)(\gamma,\nu)(t) = (\beta(t)\boldsymbol{\mu}(t), \ell(t)\boldsymbol{\mu}(t))$ and $(d/du)(\widetilde{\gamma}\ ,\widetilde{\nu})(u)=(\widetilde{\beta}(u)\widetilde{\boldsymbol{\mu}}(u), \widetilde{\ell}(u)\widetilde{\boldsymbol{\mu}}(u))$. It follows that $\ell(t_0) = \widetilde{\ell}(u_0), \beta(t_0) = \widetilde{\beta}(u_0)$. Hence, the first assertion of Theorem 3.1 holds in the case of $k = 1$.

Suppose that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t=t_0, u=u_0$ and

$$
\begin{aligned}
(\ell,\beta)(t_0)=(\widetilde{\ell},\widetilde{\beta})(u_0),\ \frac{d}{dt}(\ell,\beta)(t_0)&=\frac{d}{du}(\widetilde{\ell},\widetilde{\beta})(u_0),\\
\cdots,\ \frac{d^{k-2}}{dt^{k-2}}(\ell,\beta)(t_0)&=\frac{d^{k-2}}{du^{k-2}}(\widetilde{\ell},\widetilde{\beta})(u_0)
\end{aligned}
$$

hold. It follows that $(d^k/dt^k)\gamma(t)$ and $(d^k/dt^k)\nu(t)$ are given by the form

$$
\begin{aligned}
&\frac{d^{k-1}}{dt^{k-1}}\beta(t)\boldsymbol{\mu}(t)+f_1\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\nu(t)\\
&\qquad+f_2\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\boldsymbol{\mu}(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{d^{k-1}}{dt^{k-1}}\ell(t)\boldsymbol{\mu}(t)+g_1\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\nu(t)\\
&\qquad+g_2\left(\beta(t),\ell(t),\dots,\frac{d^{k-2}}{dt^{k-2}}\beta(t),\frac{d^{k-2}}{dt^{k-2}}\ell(t)\right)\boldsymbol{\mu}(t)
\end{aligned}
$$

for some smooth functions f_1, f_2, g_1 and g_2. By the same calculations,

$$
\begin{aligned}
\frac{d^k}{du^k}\widetilde{\gamma}(u) &= \frac{d^{k-1}}{du^{k-1}}\widetilde{\beta}(u)\widetilde{\boldsymbol{\mu}}(u)\\
&\quad+f_1\left(\widetilde{\beta}(u),\widetilde{\ell}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u),\frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\nu}(u)\\
&\quad+f_2\left(\widetilde{\beta}(u),\widetilde{\ell}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u),\frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\boldsymbol{\mu}}(u),\\
\frac{d^k}{du^k}\widetilde{\nu}(u) &= \frac{d^{k-1}}{du^{k-1}}\widetilde{\ell}(u)\widetilde{\boldsymbol{\mu}}(u)\\
&\quad+g_1\left(\widetilde{\beta}(u),\widetilde{\ell}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u),\frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\nu}(u)\\
&\quad+g_2\left(\widetilde{\beta}(u),\widetilde{\ell}(u),\dots,\frac{d^{k-2}}{du^{k-2}}\widetilde{\beta}(u),\frac{d^{k-2}}{du^{k-2}}\widetilde{\ell}(u)\right)\widetilde{\boldsymbol{\mu}}(u).
\end{aligned}
$$

It follows that $(d^{k-1}/dt^{k-1})(\ell,\beta)(t_0)=(d^{k-1}/du^{k-1})(\widetilde{\ell},\widetilde{\beta})(u_0)$. By the induction, we have the first assertion.

Suppose that the condition (3.1) holds. By the above calculations, we have $(d^i/dt^i)(\gamma,\nu)(t_0)=(d^i/du^i)(\widetilde{\gamma},\widetilde{\nu})(u_0)$ for $i=1,\dots,k$. Therefore, (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t=t_0, u=u_0$, up to congruence as Legendre curves. □

Note that if γ is a regular curve, then we also consider a contact between curves (cf. [4]). Let $\gamma: I\to\mathbb{R}^2; t\mapsto\gamma(t)$ and $\widetilde{\gamma}:\widetilde{I}\to\mathbb{R}^2; u\mapsto\widetilde{\gamma}(u)$ be regular plane curves, respectively. We say that γ *and* $\widetilde{\gamma}$ *have at least* k*-th order contact* *at* $t = t_0, u = u_0$ if

$$\gamma(t_0) = \widetilde{\gamma}(u_0),\ \frac{d\gamma}{dt}(t_0) = \frac{d\widetilde{\gamma}}{du}(u_0),\ \cdots,\ \frac{d^k\gamma}{dt^k}(t_0) = \frac{d^k\widetilde{\gamma}}{du^k}(u_0).$$

By Example 1, we take $\nu : I \to S^1, \nu(t) = \boldsymbol{n}(t)$ and $\widetilde{\nu} : \widetilde{I} \to S^1, \widetilde{\nu}(u) = \widetilde{\boldsymbol{n}}(u)$.

If s be the arc-length parameter of γ, then $\ell(t) = \kappa(t)$ and $|\beta(t)| = 1$ by Proposition 2.5. Therefore, we have following result as a corollary of Theorem 3.1.

Corollary 3.2. *Let $\gamma : I \to \mathbb{R}^2$ and $\widetilde{\gamma} : \widetilde{I} \to \mathbb{R}^2$ be regular curves with the arc-length parameters. Under the above notations, (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are Legendre immersions. Then (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ have at least $(k+1)$-order contact at $t = t_0, u = u_0$ if and only if γ and $\widetilde{\gamma}$ have at least $(k+1)$-order contact at $t = t_0, u = u_0$.*

4. Legendre immersions

In this section, we consider Legendre immersions in the unit tangent bundle. Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion. Then the curvature of the Legendre immersion $(\ell(t), \beta(t)) \neq (0, 0)$. In this case, we define *the normalized curvature for the Legendre immersion* by

$$\left(\overline{\ell}(t), \overline{\beta}(t)\right) = \left(\frac{\ell(t)}{\sqrt{\ell(t)^2 + \beta^2(t)}}, \frac{\beta(t)}{\sqrt{\ell(t)^2 + \beta(t)^2}}\right).$$

By a direct calculation, the normalized curvature $\left(\overline{\ell}(t), \overline{\beta}(t)\right)$ is independent on the choice of a parametrization, see §2. Moreover, since $\overline{\ell}(t)^2 + \overline{\beta}(t)^2 = 1$, there exists a smooth function $\theta(t)$ such that

$$\overline{\ell}(t) = \cos\theta(t),\ \overline{\beta}(t) = \sin\theta(t).$$

It is helpful to introduce the notion of the arc-length parameter of Legendre immersions. In general, we can not consider the arc-length parameter of the front γ, since γ may have singularities. However, (γ, ν) is an immersion, we introduce the arc-length parameter for the Legendre immersion (γ, ν). The *speed* $s(t)$ of the Legendre immersion at the parameter t is defined to be the length of the tangent vector at t, namely,

$$s(t) = |(\dot{\gamma}(t), \dot{\nu}(t))| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t) + \dot{\nu}(t) \cdot \dot{\nu}(t)}.$$

Given scalars $a, b \in I$, we define the arc-length from $t = a$ to $t = b$ to be the integral of the speed,

$$L(\gamma, \nu) = \int_a^b s(t)\ dt.$$

By the same method for the are-length parameter of a regular plane curve, one can prove the following:

Proposition 4.1. *Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t),\nu(t))$ be a Legendre immersion, and let $t_0 \in I$. Then (γ,ν) is parametrically equivalent to a unit speed curve*

$$(\overline{\gamma},\overline{\nu}) : \overline{I} \to \mathbb{R}^2 \times S^1; s \mapsto (\overline{\gamma}(s),\overline{\nu}(s)) = (\gamma \circ u(s), \nu \circ u(s)),$$

under a change of parameter $u : \overline{I} \to I$ with $u(0) = t_0$ and with $u'(s) > 0$.

We call the above parameter s in Proposition 4.1 *the arc-length parameter for the Legendre immersion.* Let s be the are-length parameter for (γ,ν). By definition, we have $\gamma'(s)\cdot\gamma'(s) + \nu'(s)\cdot\nu'(s) = 1$, where $'$ is the derivation with respect to s. It follows that $\ell(s)^2 + \beta(s)^2 = 1$. Then there exists a smooth function $\theta(s)$ such that

$$\ell(s) = \cos\theta(s),\ \beta(s) = \sin\theta(s).$$

Also, as a corollary of Theorem 3.1, we have the following corollary:

Corollary 4.2. *Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t),\nu(t))$ and $(\widetilde{\gamma},\widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u),\widetilde{\nu}(u))$ be Legendre immersions with the arc-length parameters. Suppose that $\theta : I \to \mathbb{R}$ and $\widetilde{\theta} : \widetilde{I} \to \mathbb{R}$ are smooth functions with the conditions*

$$\ell(t) = \cos\theta(t),\ \beta(t) = \sin\theta(t),\ \widetilde{\ell}(u) = \cos\widetilde{\theta}(u),\ \widetilde{\beta}(u) = \sin\widetilde{\theta}(u).$$

If (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$, then there exists a integer $n \in \mathbb{Z}$ such that

$$\theta(t_0) = \widetilde{\theta}(u_0) + 2n\pi,\ \frac{d\theta}{dt}(t_0) = \frac{d\widetilde{\theta}}{du}(u_0),\ \cdots,\ \frac{d^{k-1}\theta}{dt^{k-1}}(t_0) = \frac{d^{k-1}\widetilde{\theta}}{du^{k-1}}(u_0). \tag{4.1}$$

Conversely, if the condition (4.1) *holds, then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$, up to congruence as Legendre immersions.*

Finally, we consider a relation between the curvature of the Legendre immersion and the zigzag number (or Maslov index) (cf. [8]).

Proposition 4.3. *Let $(\gamma,\nu) : [a,b] \to \mathbb{R}^2 \times S^1$ be a closed Legendre immersion with the curvature of the Legendre immersion (ℓ,β). Suppose that (γ,ν) is parametrized by the arc-length parameter t and θ is a smooth function which satisfies $\ell(t) = \cos\theta(t)$ and $\beta(t) = \sin\theta(t)$. Then $(1/2\pi)|\theta(b) - \theta(a)|$ is equal to the zigzag number of the front γ.*

Proof. Let $z(\gamma)$ be the zigzag number of γ, By the definition of the zigzag number (see [8] for example), $z(\gamma)=|\deg([-\dot{\nu}(t),\dot{\gamma}(t)])|=|\deg([-\ell(t),\beta(t)])|=|\deg([-\cos\theta(t),\sin\theta(t)])|$, where $[-\dot{\nu}(t),\dot{\gamma}(t)]$ is a ratio as the proportional constant of two vectors, $[-\ell(t),\beta(t)]$ and $[-\cos\theta(t),\sin\theta(t)]$ are ratios of two real numbers, in other words, elements of the real projective line. We consider the real projective line as S^1, then $\deg([-\cos\theta(t),\sin\theta(t)])$ means a rotation number of the map $t \mapsto (-\cos\theta(t),\sin\theta(t)) \in S^1$. It follows that $|\theta(b) - \theta(a)| = 2\pi|\deg([-\cos\theta(t),\sin\theta(t)])|$. Therefore, we obtain $(1/2\pi)|\theta(b)-\theta(a)| = z(\gamma)$. $\square$

Remark 4.4. Let $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ be a closed Legendre immersion with the curvature (ℓ,β). Suppose that (γ,ν) is parametrized by the arc-length parameter t and θ is a smooth function which satisfies $\ell(t)=\cos\theta(t)$ and $\beta(t)=\sin\theta(t)$. Then the curvature of the Legendre immersion $(\gamma,-\nu)$ is equal to $(\ell,-\beta)$ (Remark 2.4). We denote $(\ell(t),-\beta(t))=(\cos\widetilde{\theta}(t),\sin\widetilde{\theta}(t))$ for a smooth function $\widetilde{\theta}$. Then we obtain simultaneous equations $\cos\theta(t)=\cos\widetilde{\theta}(t)$ and $\sin\theta(t)=-\sin\widetilde{\theta}(t)$. It follows that there exists an integer n such that $\theta(t)=-\widetilde{\theta}(t)+2n\pi$. Thus $\theta(b)-\theta(a)=-\widetilde{\theta}(b)+\widetilde{\theta}(a)$.

Acknowledgment

The authors would like to thank Professor Toshizumi Fukui for valuable comments, and the referee for helpful comments to improve the original manuscript. The second author was supported by a Grant-in-Aid for Young Scientists (B) No. 23740041.

References

[1] Arnol'd, V.I. : *Singularities of Caustics and Wave Fronts.* Mathematics and Its Applications **62**, Kluwer Academic Publishers (1990).

[2] Arnol'd, V.I., Gusein-Zade, S.M., Varchenko, A.N. : *Singularities of Differentiable Maps vol.* I. Birkhäuser (1986).

[3] Bruce, J.W., Giblin, P.J. : *Curves and singularities. A geometrical introduction to singularity theory. Second edition.* Cambridge University Press, Cambridge (1992).

[4] Fukunaga, T., Takahashi, M. : *Evolutes of fronts in the Euclidean plane.* Preprint, Hokkaido University Preprint Series, No.1026 (2012).

[5] Gibson, C.G. : *Elementary geometry of differentiable curves. An undergraduate introduction.* Cambridge University Press, Cambridge (2001).

[6] Gray, A., Abbena, E., Salamon, S. : *Modern differential geometry of curves and surfaces with Mathematica. Third edition.* Studies in Advanced Mathematics. Chapman and Hall/CRC, Boca Raton, FL (2006)

[7] Ishikawa, G. : *Classifying singular Legendre curves by contactomorphisms.* J. Geom. Phys. **52**, 113–126 (2004).

[8] Saji, K., Umehara, M., Yamada, K. : *The geometry of fronts.* Ann. Math. **169**, 491–529 (2009).

[9] Sasai, T. : *Geometry of analytic space curves with singularities and regular singularities of differential equations.* Funkcial. Ekvac. **30**, 283–303 (1987).

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