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CONGRUENCES TO IKEDA-MIYAWAKI LIFTS AND TRIPLE L-VALUES OF ELLIPTIC MODULAR FORMS

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Abstract. In this paper, we consider congruences between the Ikeda-Miyawaki lift and other Siegel modular forms, relating these congruences to critical values of L functions by using Ikeda's conjecture on periods. We also give general formulas for critical values of triple L functions and prove results, both in theory and in examples, on the relation between such congruences and critical values.

1. Introduction

Congruences between modular forms are important in the arithmetic theory of modular forms. In particular, congruences between lifts and non-lifts sometimes produce non-trivial elements of the Bloch-Kato Selmer group (cf. [Br], [BDS],[DIK]). In [Kat5], the second named author considered the congruence between the Duke-Imamoḡlu-Ikeda lift $I_{2n}(h)$ of a Hecke eigenform h of half-integral weight and non-Duke-Imamoḡlu-Ikeda lifts, and proved that a prime ideal dividing a certain L-value of f gives such a congruence, where f is the primitive form of integral weight corresponding to h under the Shimura correspondence. This result is based on the relation between the periods of $I_{2n}(h)$ and h proved by the second-named author and Kawamura [KK], which forms a part of the relations conjectured by Ikeda [Ik2]. A similar result concerning congruences between Yoshida lifts and non-Yoshida lifts was proved by Böcherer, Dummigan, and Schulze-Pillot [BDS]; this proof is also based on period relations, this time for the Yoshida lift. In general, the algebraic part of critical values of the standard L function sometimes gives congruence primes between Siegel modular forms, see [Kat3]. In view of the above results, we can expect that if there is a formula to describe the period of a lift F from some form G by that of G, then the critical values of some L function of G are related to congruences between the lift F and non-lifts.

In this article, we consider congruences between Ikeda-Miyawaki lifts and other Siegel modular Hecke eigenforms. Let k and n be positive integers such that $k+n+1$ is even. For a Hecke eigenform h of weight $k+1/2$ for $\Gamma_0(4)$ and a primitive form g of weight $k+n+1$ for $\mathrm{SL}_2(\mathbf{Z})$, let $\mathcal{F}_{h,g}$ be the cusp form of weight $k+n+1$ for $\mathrm{Sp}_{2n+1}(\mathbf{Z})$ constructed by Ikeda [Ik2]. For the precise definition of $\mathcal{F}_{h,g}$ see Section 3. This type of lift was conjectured by Miyawaki [Miy] in the case $n=1$, therefore we call $\mathcal{F}_{h,g}$ the Ikeda-Miyawaki lift of h and g. We also denote by f the primitive form of weight $2k$ for $\mathrm{SL}_2(\mathbf{Z})$ corresponding to h under the Shimura correspondence. Then, roughly speaking, our conjecture can be stated as follows, (more precisely, see Conjecture B and Problem B'):

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Let $\mathfrak{P}$ be a "big prime ideal" in the composite $\mathbf{Q}(f)\mathbf{Q}(g)$ of the Hecke fields of f and g. Then $\mathfrak{P}$ divides the algebraic part $L_{alg}(2k+2n, f\otimes g\otimes g)$ of the triple product L-function at $2k+2n$ if and only if there exists a congruence modulo $\mathfrak{P}$ between $\mathcal{F}_{h,g}$ and a cuspidal Hecke eigenform G of the same weight, where G is a non-Ikeda-Miyawaki lift.

This type of conjecture has already been proposed in the case of the Saito-Kurokawa lift and the Duke-Imamoḡlu-Ikeda lift (cf. [Kat2]). Our conjecture is based on the conjecture concerning the period of the Ikeda-Miyawaki lift proposed by Ikeda [Ik2] (cf. Conjecture A, Theorem 3.1 and its corollary). We note that Bergström, Faber, and van der Geer [BFG] have proposed a conjecture on the congruence of (not necessarily scalar valued) modular forms of degree three from a different point of view. We discuss the relation between their conjecture and ours in Section 3. Since certain types of triple L functions appear in the description of the congruence primes, we give a concrete general formula for the special values of any triple L function in the *balanced* case, and execute this calculation to give values in several cases, including the cases which appear as examples in the conjecture of [BFG]. Finally we construct examples of non-Ikeda Miyawaki lifts of degree 3 by using a pullback of Eisenstein series of degree 6 and prove the appropriate congruences for these examples.

The content of this paper is as follows. In Section 2, we review the arithmetic properties of various L-functions. In Section 3, we prove a weaker version of our conjecture assuming Ikeda's conjecture about the period of the Ikeda-Miyawaki lift (cf. [Ik2]). In Section 4, we give a formula for triple L-values in the balanced case that is a modification of Boecherer and Schulze-Pillot's formula [BS]. Our method relies on Ibukiyama and Zagier's differential operators [IZ] and our formula enables us to compute any critical values of a triple product L-function (in the balanced case) more easily than was done in Mizumoto [Miz3] or Lanphier [La]. By using this, it is also possible to give critical values of the symmetric cube L functions and, as an example, we tabulate them for an elliptic cusp form of weight 16 in Table 4. We also give norms of critical values of triple L functions in several cases, see Tables 1, 2, 3. In particular, we show that the numerator of $L_{alg}(30, f\otimes g\otimes g)$ lies in a prime ideal of $\mathbf{Q}(f)$ above 107 in the case where f and g are primitive forms of weights 28 and 16, respectively; this is the case $k=14$ in Table 2. In Section 5, we show this prime ideal gives a congruence between $\mathcal{F}_{h,g}$ and a non-Ikeda-Miyawaki lift. This example shows the validity of our conjecture in this case, and does so without the assumption of any open conjectures. We verify our conjecture in other cases as well. These computations of spaces of Siegel modular cusp forms in degree three may be viewed as a continuation of the work of Miyawaki [Miy], whose computations motivated his lifting conjectures in the first place. The computational method used here is an application of [Kat2], where the Siegel series for general degree n are given recursively.

Notation. We denote by $\mathbf{Z}$, the ring of rational integers, and by $\mathbf{Q}, \mathbf{R}$, and $\mathbf{C}$ the fields of rational numbers, real numbers, and complex numbers, respectively. Moreover for a prime number p, we denote by $\mathbf{Q}_p$ and $\mathbf{Z}_p$ the field of p-adic numbers and the ring of p-adic integers, respectively. We also denote by ord_p the normalized additive valuation on $\mathbf{Q}_p$ and write $e(z)=e^{2\pi i z}$ for $z\in\mathbf{C}$.

For a commutative ring R, we denote by $M_{mn}(R)$ the set of (m,n)-matrices with entries in R. In particular put $M_n(R) = M_{nn}(R)$. For an (m,n)-matrix X and an (m,m)-matrix A, we write $A[X] = {}^tXAX$, where ${}^tX$ denotes the transpose of X. Put $\mathrm{GL}_m(R) = \{A \in M_m(R) \mid \det A \in R^*\}$, where $\det A$ denotes the determinant of the square matrix A, and R^* denotes the unit group of R. Let $S_n(R)$ denote the set of symmetric matrices of degree n with entries in R. Furthermore, for an integral domain R of characteristic different from 2, let $\mathcal{L}_n(R)$ denote the set of half-integral matrices of degree n over R, that is, $\mathcal{L}_n(R)$ is the set of symmetric matrices of degree n whose (i,j)-component belongs to R or $\frac{1}{2}R$ according as $i = j$ or not. In particular we put $\mathcal{L}_n = \mathcal{L}_n(\mathbf{Z})$ and, for each prime p, put $\mathcal{L}_{n,p} = \mathcal{L}_n(\mathbf{Z}_p)$. For a subset S of $M_n(R)$ we denote by $S^\times$ the subset of S consisting of non-degenerate matrices. In particular, if S is a subset of $S_n(\mathbf{R})$ with $\mathbf{R}$ the field of real numbers, we denote by $S_{>0}$ (resp. $S_{\geq 0}$) the subset of S consisting of positive definite (resp. semi-positive definite) matrices. Let R' be a subring of R. Two symmetric matrices A and A' with entries in R are called equivalent over R' and we write $A \underset{R'}{\sim} A'$ if there is an element X of $\mathrm{GL}_n(R')$ such that $A' = A[X]$. We also write $A \sim A'$ if there is no fear of confusion. For square matrices X and Y we write $X \perp Y = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$.

2. Several L-values

Put $J_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$, where 1_n denotes the unit matrix of degree n. For a subring K of $\mathbf{R}$ put

$$\mathrm{GSp}_n^+(K) = \{M \in \mathrm{GL}_{2n}(K) \mid J_n[M] = \mu(M)J_n \text{ for some } \mu(M) > 0\},$$

and

$$\mathrm{Sp}_n(K) = \{M \in \mathrm{GL}_{2n}(K) \mid J_n[M] = J_n\}.$$

Define the standard subgroup $\Gamma_0^{(n)}(N)$ by

$$\Gamma_0^{(n)}(N) = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbf{Z}) \mid C \equiv 0 \bmod N\}.$$

If $n = 1$, we drop the superscript (n). Let $\mathbf{H}_n$ be Siegel's upper half-space. For a function F on $\mathbf{H}_n$, an integer k and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GSp}_n^+(\mathbf{R})$, put $j(g,Z) = \det(CZ+D)$, and

$$(F|_k g)(Z) = (\det(g))^{k/2} j(g,Z)^{-k} F\left((AZ+B)(CZ+D)^{-1}\right).$$

Let κ be an integer or a half-integer. We denote by $M_\kappa(\Gamma_0^{(n)}(N))$ or $M_\kappa^\infty(\Gamma_0^{(n)}(N))$ the space of holomorphic or C^∞-modular forms, respectively, of weight κ with respect to $\Gamma_0^{(n)}(N)$. We denote by $S_\kappa(\Gamma_0^{(n)}(N))$ the vector subspace of $M_\kappa(\Gamma_0^{(n)}(N))$ consisting of cusp forms. For two C^∞-modular forms F and G of weight κ with respect to $\Gamma_0^{(n)}(N)$ we define the Petersson scalar product $\langle F, G\rangle$ by

$$\langle F, G\rangle = [\mathrm{Sp}_n(\mathbf{Z}) : \Gamma_0^{(n)}(N)]^{-1} \int_{\Phi_{\Gamma_0^{(n)}(N)}} F(Z)\overline{G}(Z)\,(\det(\mathrm{Im}(Z)))^{\kappa-n-1}\,dXdY,$$

provided the integral converges, where $\Phi_{\Gamma_0^{(n)}(N)}$ is a fundamental domain in $\mathbf{H}_n$ for $\Gamma_0^{(n)}(N)$. Let $\tilde{\mathbf{L}}_n = \mathcal{R}_{\mathbf{Q}}(\mathrm{GSp}_n^+(\mathbf{Q}), \mathrm{Sp}_n(\mathbf{Z}))$, $\mathbf{L}_n^\circ = \mathcal{R}_{\mathbf{Q}}(\mathrm{Sp}_n(\mathbf{Q}), \mathrm{Sp}_n(\mathbf{Z}))$ be the Hecke rings over $\mathbf{Q}$ for the Hecke pairs $(\mathrm{GSp}_n^+(\mathbf{Q}), \mathrm{Sp}_n(\mathbf{Z}))$, $(\mathrm{Sp}_n(\mathbf{Q}), \mathrm{Sp}_n(\mathbf{Z}))$,

respectively. The Hecke ring $\mathbf{L}_n^\circ$ is a subring of $\tilde{\mathbf{L}}_n$. Let $T = \mathrm{Sp}_n(\mathbf{Z})M\,\mathrm{Sp}_n(\mathbf{Z}) \in \mathrm{GSp}_n^+(\mathbf{Q}))$ be an element of $\tilde{\mathbf{L}}_n$. Write T as a disjoint union $T = \bigcup_g \mathrm{Sp}_n(\mathbf{Z})g$. For $k \in \mathbf{N}$ and for $F \in M_k(\mathrm{Sp}_n(\mathbf{Z}))$ we define the Hecke operator $F|_kT$ as

$$F|_kT = (\det M)^{k/2-(n+1)/2} \sum_g F|_k g.$$

We define the elements $T(p)$ and $T_j(p^2)$ of $\tilde{\mathbf{L}}_n$ in a standard way (cf. [An]): $T(p) = \mathrm{Sp}_n(\mathbf{Z})(1_n \perp p1_n)\,\mathrm{Sp}_n(\mathbf{Z})$ and $T_j(p^2) = \mathrm{Sp}_n(\mathbf{Z})(1_{n-j} \perp p1_j \perp p^2 1_{n-j} \perp p1_j)\,\mathrm{Sp}_n(\mathbf{Z})$. For any subring of $R \subseteq \tilde{\mathbf{L}}_n$, we say that a modular form $F \in M_k(\mathrm{Sp}_n(\mathbf{Z}))$ is a Hecke eigenform with respect to R if F is a common eigenfunction of all $T \in R$. If $R = \tilde{\mathbf{L}}_n$, we simply say that F is a Hecke eigenform. In this case, we denote by $\mathbf{Q}(F)$ the field generated over $\mathbf{Q}$ by all the Hecke eigenvalues of $T \in \tilde{\mathbf{L}}_n$, and call it the Hecke field of F. We remark that $\mathbf{Q}(F)$ is a totally real algebraic number field of finite degree over $\mathbf{Q}$ (cf. [Miz2]).

In this section we review several L-values of modular forms that appear in this article. Let

$$f(z) = \sum_{m=1}^{\infty} c_f(m)\mathbf{e}(mz)$$

be a primitive form in $S_k(\mathrm{SL}_2(\mathbf{Z}))$. For each prime p let $\alpha_p = \alpha_{f,p}$ be a complex number such that $\alpha_p + \alpha_p^{-1} = p^{-k/2+1/2}c_f(p)$. We define

$$L(s,f) = \prod_p \Big((1-\alpha_p p^{k/2-1/2-s})(1-\alpha_p^{-1}p^{k/2-1/2-s})\Big)^{-1}.$$

We write $\Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$ and write $\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ as usual. For $l \in \mathbf{N}$ satisfying $1 \le l \le k-1$, put

$$\Lambda(l,f) = \Gamma_C(l)L(l,f).$$

Then there exist two (positive) real numbers $\Omega_+(f)$ and $\Omega_-(f)$ such that for the sign $j = (-1)^l$, we have (cf. [Sh1])

$$\frac{\Lambda(l,f)}{\Omega_j(f)} \in \mathbf{Q}(f). \tag{1}$$

We note that $\Omega_+(f)$ and $\Omega_-(f)$ are determined by f only up to constant multiples of $\mathbf{Q}(f)^\times$. Fixing $\Omega_\pm(f)$, we define an algebraic part of $L(l,f)$ by

$$L_{alg}(l,f) = \frac{\Lambda(l,f)}{\Omega_j(f)}.$$

Another rationality result (cf. [R], [Sh2]) is that

$$\langle f,f\rangle/\Omega_+(f)\Omega_-(f) \in \mathbf{Q}(f).$$

So if we change $\Omega_\pm(f)$ by a multiple in $\mathbf{Q}(f)^\times$, we can make $\Omega_+(f)\Omega_-(f) = \langle f,f\rangle$, though we do not assume this in this article. For two positive integers $l_1, l_2 \le k-1$ such that $l_1 + l_2 \equiv 1 \bmod 2$, the value

$$\frac{\Gamma_{\mathbf{C}}(l_1)\Gamma_{\mathbf{C}}(l_2)L(l_1,f)L(l_2,f)}{\langle f,f\rangle}$$

belongs to $\mathbf{Q}(f)$; we will denote this value by $L_{alg}(l_1,l_2;f)$. This value does not depend upon the choice of $\Omega_\pm(f)$.

Let F be a Hecke eigenform in $S_k(\mathrm{Sp}_n(\mathbf{Z}))$ with respect to $\mathbf{L}_n^\circ$, and let the p-Satake parameters of F be $\beta_1(p), \cdots, \beta_n(p)$. We then define the standard L-function by

$$L(s, F, \mathrm{St}) = \prod_p \{(1-p^{-s}) \prod_{i=1}^n (1-\beta_i(p)p^{-s})(1-\beta_i(p)^{-1}p^{-s})\}^{-1}.$$

We put

$$\widetilde{\Lambda}(s, F, \mathrm{St}) = \Gamma_{\mathbf{C}}(s) \left(\prod_{i=1}^n \Gamma_{\mathbf{C}}(s+k-i) \right) L(s, F, \mathrm{St}).$$

Here the gamma factor is different from the one which appears in the functional equation, but this gamma factor is suitable for special values. Let $\rho(n) = 3$ or 1 according as $n \equiv 1 \bmod 4$ and $n \geq 5$, or not. For a positive integer m such that $\rho(n) \leq m \leq k-n$ and $m \equiv n \bmod 2$ put

$$L_{alg}(m, F, \mathrm{St}) = \frac{\widetilde{\Lambda}(m, F, \mathrm{St})}{\langle F, F \rangle}.$$

It is known that $L_{alg}(m, F, \mathrm{St})$ belongs to $\mathbf{Q}(F)$ if all the Fourier coefficients of F belong to $\mathbf{Q}(F)$, compare [Bo1], [Miz1]. When F is a Ikeda-Miyawaki lift, which we shall treat later, the algebraicity holds also for $m = 1$. For general F, it is expected that $L_{alg}(1, F, \mathrm{St}) \in \mathbf{Q}(F)$ even when $n \equiv 1 \bmod 4$ and $n \geq 5$, but this has not been proved in general.

Let $f_i(z)$ be a primitive Hecke eigenform in $S_{k_i}(\mathrm{SL}_2(\mathbf{Z}))$ for $i = 1, 2, 3$. Then we define the triple product L-function $L(s, f_1 \otimes f_2 \otimes f_3)$ as

$$L(s, f_1 \otimes f_2 \otimes f_3) = \prod_p \prod_{i,j,l \in \{1,-1\}} \left(1 - \alpha_{f_1,p}^i \alpha_{f_2,p}^j \alpha_{f_3,p}^l \, p^{(k_1+k_2+k_3-3)/2-s}\right)^{-1}.$$

This satisfies a functional equation for $s \to k_1+k_2+k_3-2-s$. Note that $L(s, f_1 \otimes f_2 \otimes f_3)$ is symmetric in the f_i, so that we may assume that $k_1 \geq k_2 \geq k_3$ without loss of generality. In addition, we always assume that $k_2 + k_3 > k_1$ hereafter in this article; this case is called the *balanced case*. Then there is an integer l satisfying $(k_1+k_2+k_3)/2 - 1 \leq l \leq k_2 + k_3 - 2$, and for such l, we put

$$L_{alg}(l, f_1 \otimes f_2 \otimes f_3) = \\ \frac{L(l, f_1 \otimes f_2 \otimes f_3)\Gamma_{\mathbf{C}}(l)\Gamma_{\mathbf{C}}(l-k_1+1)\Gamma_{\mathbf{C}}(l-k_2+1)\Gamma_{\mathbf{C}}(l-k_3+1)}{\langle f_1, f_1 \rangle \langle f_2, f_2 \rangle \langle f_3, f_3 \rangle}.$$

Then $L_{alg}(l, f_1 \otimes f_2 \otimes f_3)$ belongs to $\mathbf{Q}(f_1)\mathbf{Q}(f_2)\mathbf{Q}(f_3)$ ([Ga], [Or], [Sat]) and is also symmetric in the f_i. An algorithm to compute $L_{alg}(l, f_1 \otimes f_2 \otimes f_3)$ will be given in Section 4.

Finally, we define $L(s, \mathrm{St}(f_2) \otimes f_1)$ as

$$L(s, \mathrm{St}(f_2) \otimes f_1) = \prod_p \left(\prod_{i \in \{1,-1\}} \prod_{j \in \{1,-1,0\}} (1 - \alpha_{f_1,p}^i \alpha_{f_2,p}^{2j} p^{k_1/2-1/2-s}) \right)^{-1},$$

and put

$$\Lambda(s, \mathrm{St}(f_2) \otimes f_1) = \Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s-k_1+k_2)\Gamma_{\mathbf{C}}(s+k_2-1)L(s, \mathrm{St}(f_2) \otimes f_1).$$

Moreover we fix the period $\Omega_\pm(f_1)$ satisfying (1) and for a positive integer l with $k_1/2 \le l \le k_2 - 1$, put

$$L_{alg}(l, \mathrm{St}(f_2) \otimes f_1) = \frac{\Lambda(l, \mathrm{St}(f_2) \otimes f_1)}{\langle f_2, f_2\rangle^2 \Omega_{j'}(f_1)},$$

where $j' = +$ or $-$ according as l is odd or even. (We note that $L_{alg}(l, \mathrm{St}(f_2) \otimes f_1)$ would more properly be denoted as $L_{alg}(l, \mathrm{St}(f_2) \otimes f_1; \Omega_{j'}(f_1))$ since it does depend upon the choice of $\Omega_\pm(f_1)$, but we use the above short notation.) From the Euler product, we note that

$$L(s + k_2 - 1, f_2 \otimes f_2 \otimes f_1) = L(s, \mathrm{St}(f_2) \otimes f_1) L(s, f_1)$$

and we have

$$L_{alg}(l + k_2 - 1, f_2 \otimes f_2 \otimes f_1) = L_{alg}(l, \mathrm{St}(f_2) \otimes f_1) L_{alg}(l, f_1) \times \frac{\Omega_+(f_1)\Omega_-(f_1)}{\langle f_1, f_1\rangle}.$$

We also note that $\Lambda(l, f_1)/\Omega_j(f_1)$ belongs to $\mathbf{Q}(f_1)^\times$ for $k_1/2+1 \le l \le k_2-1$, since we assumed $k_1/2 + 1 \le k_2 \le k_1$ and the Euler product of $L(s, f_1)$ converges in this range. Hence $L_{alg}(l, \mathrm{St}(f_2) \otimes f_1)$ belongs to $\mathbf{Q}(f_1)\mathbf{Q}(f_2)$ for $k_1/2+1 \le l \le k_2-1$. We note that $L(k_1/2+k_2-1, f_2 \otimes f_2 \otimes f_1)$ is always zero (cf. [Sat]), and that $L(k_1/2, f)$ may be zero. Therefore we cannot prove the algebraicity of $L_{alg}(k_1/2, \mathrm{St}(f_2) \otimes f_1)$ by using the above argument. However, in the case $k_1/2$ is odd, Ichino [Ich] proved that $L_{alg}(k_1/2, \mathrm{St}(f_2) \otimes f_1)$ is algebraic by using a different method.

3. Ikeda-Miyawaki lift

Throughout this section, fix positive integers k, n such that $k \equiv n + 1 \bmod 2$ and $k > n$. Let h be a Hecke eigenform in the Kohnen plus subspace $S^+_{k+1/2}(\Gamma_0(4))$ with Fourier expansion

$$h(z) = \sum_{m \in \mathbf{N}:\, (-1)^k m \equiv 0,1 \bmod 4} c_h(m) \mathbf{e}(mz)$$

and let g be a primitive form in $S_{k+n+1}(\mathrm{SL}_2(\mathbf{Z}))$ with Fourier expansion

$$g(z) = \sum_{m=1}^{\infty} c_g(m) \mathbf{e}(mz).$$

Moreover let

$$f(z) = \sum_{m=1}^{\infty} c_f(m) \mathbf{e}(mz)$$

be the primitive form in $S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ corresponding to h under the Shimura correspondence. Let $\alpha_p \in \mathbf{C}$ be taken such that $\alpha_p + \alpha_p^{-1} = p^{-k+1/2} c_f(p)$. For $T \in \mathcal{L}_{2n+2>0}$, define $c_{I_{2n+2}(h)}(T)$ as

$$c_{I_{2n+2}(h)}(T) = c_h(|\mathfrak{d}_T|)\, \mathfrak{f}_T^{k-1/2} \prod_p \alpha_p^{-\mathrm{ord}_p(\mathfrak{f}_T)} F_p(T, p^{-n-3/2}\alpha_p),$$

where $\mathfrak{d}_T$ is the discriminant of $\mathbf{Q}(\sqrt{(-1)^{n+1}\det(2T)})$, $\mathfrak{f}_T = \sqrt{(-1)^{n+1}\det(2T)/\mathfrak{d}_T}$ so that $\mathfrak{d}_T\, \mathfrak{f}_T^2 = (-1)^{n+1}\det(2T)$, and $F_p(T, X)$ is a polynomial in X with coefficients in $\mathbf{Q}$, which will be defined in Section 4.3. We then define a Fourier series

$I_{2n+2}(h)(Z)$ for $Z \in \mathbf{H}_{2n+2}$:

$$I_{2n+2}(h)(Z) = \sum_{T \in \mathcal{L}_{2n+2>0}} c_{I_{2n+2}(h)}(T)\mathbf{e}(\mathrm{tr}(TZ)).$$

Then $I_{2n+2}(h)$ is a Hecke eigenform in $S_{k+n+1}(\mathrm{Sp}_{2n+2}(\mathbf{Z}))$, see [Ik1]. A proof that $I_{2n+2}(h)$ is a Hecke eigenform for the entire Hecke algebra $\tilde{\mathbf{L}}_{2n+2}$, not just the even part $\mathbf{L}^{\circ}_{2n+2}$, may be found in [Kat5]. We call $I_{2n+2}(h)$ the Duke-Imamoḡlu-Ikeda lift of h (or of f) to $S_{k+n+1}(\mathrm{Sp}_{2n+2}(\mathbf{Z}))$. For $z \in \mathbf{H}_{2n+1}$ and $w = x + iy \in \mathbf{H}_1$, set

$$\mathcal{F}_{h,g}(z) = \int_{\mathrm{SL}_2(\mathbf{Z})\backslash \mathbf{H}_1} I_{2n+2}(h)(\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix})\overline{g(w)}y^{k+n-1}dw.$$

Ikeda [Ik2] showed the following:

If $\mathcal{F}_{h,g}(z)$ is not identically zero in $S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbf{Z}))$, then it is a Hecke eigenform with respect to $\mathbf{L}^{\circ}_{2n+1}$ and its standard L-function is given by

$$L(s, \mathcal{F}_{h,g}, \mathrm{St}) = L(s, g, \mathrm{St}) \prod_{i=1}^{2n} L(s + k + n - i, f).$$

This is a part of Ikeda's results: the case $r = 1$ in [Ik2]. The existence of this type of Hecke eigenform was conjectured by Miyawaki [Miy]; therefore, we call $\mathcal{F}_{h,g}$ the Ikeda-Miyawaki lift of h and g when $\mathcal{F}_{h,g}$ is not identically zero. It is known that the Ikeda-Miyawaki lift $\mathcal{F}_{h,g}$ is a Hecke eigenform also with respect to $\tilde{\mathbf{L}}_{2n+1}$. (See Hayashida [Haya], also Heim [He] for a special case.)

Ikeda proposed the following conjecture relating the Petersson norm of $\mathcal{F}_{h,g}$ to those of f and g. Thus, if correct, we may relate the algebraic parts of L-values for $\mathcal{F}_{h,g}$ to those of f and g as we will see in the theorems to follow.

Conjecture A. (Ikeda [Ik2]) *Let $k > n$ be positive integers with $k + n$ odd. Let $h \in S^{+}_{k+1/2}(\Gamma_0(4))$ be a Hecke eigenform corresponding to the primitive eigenform $f \in S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ under the Shimura correspondence. Let $g \in S_{k+n+1}(\mathrm{SL}_2(\mathbf{Z}))$ be a primitve eigenform and assume that the Ikeda-Miyawaki lift $\mathcal{F}_{h,g}$ is not identically zero. Put $\widetilde{\xi}(s) = \Gamma_{\mathbf{C}}(s)\zeta(s)$. Then*

$$\frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g} \rangle}{\langle g, g \rangle \langle h, h \rangle} = 2^{\beta_{n,k}} \Lambda(k + n, \mathrm{St}(g) \otimes f) \prod_{i=1}^{n} \tilde{\xi}(2i) \prod_{i=1}^{n-1} \tilde{\Lambda}(2i + 1, f, \mathrm{St}),$$

where $\beta_{n,k}$ is an integer depending only on n and k.

The expression in the above conjecture appears to be different from the one in [Ik2] but it is the same since we have $\widetilde{\Lambda}(1, f, St) = 2^{2k}\langle f, f \rangle$ (see [Za]).

Now we modify $\mathcal{F}_{h,g}$ and put

$$\widetilde{\mathcal{F}}_{h,g} = \langle g, g \rangle^{-1} \mathcal{F}_{h,g}.$$

Let $\{g_i\}_{i=1}^{d_2}$ be a basis of $S_{k+n+1}(\mathrm{SL}_2(\mathbf{Z}))$ consisting of Hecke eigenforms, then we have easily from the definition of the Ikeda-Miyawaki lift:

$$I_{2n+2}(h)(\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}) = \sum_{i=1}^{d_2} \widetilde{\mathcal{F}}_{h,g_i}(z) g_i(w).$$

We have

$$I_{2n+2}(h)(\begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}) = \sum_{A>0, m>0} c_{2n+1,h}(A,m)\mathbf{e}(\mathrm{tr}(Az))\mathbf{e}(mw),$$

where

$$c_{2n+1,h}(A,m) = \sum_r c_{I_{2n+2}(h)}(\begin{pmatrix} A & r/2 \\ {}^t r/2 & m \end{pmatrix}).$$

By replacing the Hecke eigenform h by a constant multiple of h, we can assume that every Fourier coefficient $c_h(m)$ belongs to $\mathbf{Q}(f)$ for any $m > 0$. If we assume so, then by definition, $c_{I_{2n+2}(h)}(T)$ belongs to $\mathbf{Q}(f)$ for any $T \in \mathcal{L}_{2n+2>0}$ and, therefore, so does $c_{2n+1,h}(A,m)$ for any $A \in \mathcal{L}_{2n+1>0}$ and $m > 0$. Hence, the A-th Fourier coefficient $c_{\widetilde{\mathcal{F}}_{h,g}}(A)$ of $\widetilde{\mathcal{F}}_{h,g}$ belongs to $\mathbf{Q}(f)\mathbf{Q}(g)$ for any $A \in \mathcal{L}_{2n+1>0}$. Actually, in this section we do not assume any normalization of h since every complete expression below is invariant if we change h by a constant multiple.

We consider congruences of Siegel modular Hecke eigenforms to Ikeda-Miyawaki lifts. For any integer $r \geq 1$, let $\mathbf{L}'_r = \mathbf{L}_{\mathbf{Z}}(M_{2r}(\mathbf{Z}) \cap \mathrm{GSp}_r(\mathbf{Q}), \mathrm{Sp}_r(\mathbf{Z}))$ be the Hecke ring over $\mathbf{Z}$ associated with the Hecke pair $(M_{2r}(\mathbf{Z}) \cap \mathrm{GSp}_r(\mathbf{Q}), \mathrm{Sp}_r(\mathbf{Z}))$. Let F and G be Hecke eigenforms in $S_l(\mathrm{Sp}_r(\mathbf{Z}))$ and let K be an algebraic number field of finite degree containing $\mathbf{Q}(F)\mathbf{Q}(G)$. For a prime ideal $\mathfrak{P}$ of $\mathfrak{O}_K$ we write

$$G \equiv_{e.v.} F \bmod \mathfrak{P}$$

if

$$\lambda_F(T) \equiv \lambda_G(T) \bmod \mathfrak{P}$$

for all $T \in \mathbf{L}'_r$, where $\lambda_F(T)$ and $\lambda_G(T)$ are the eigenvalues of T with respect to F and G, respectively. We denote by $K(S_l(\mathrm{Sp}_r(\mathbf{Z})))$ the composite of all the Hecke fields of Hecke eigenforms in $S_l\,(\mathrm{Sp}_r(\mathbf{Z}))$.

Conjecture B. Let $\mathfrak{P}$ be a "big prime ideal" of $K(S_{k+n+1}(\mathrm{Sp}_{2n+1})(\mathbf{Z}))$. Then $\mathfrak{P}$ divides $L_{alg}(k+n, \mathrm{St}(g) \otimes f) \prod_{i=1}^{n-1} L_{alg}(2i+1, f, \mathrm{St})$ if and only if there exists a Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbf{Z}))$, not coming from the Ikeda-Miyawaki lift, such that

$$G \equiv_{e.v.} \widetilde{\mathcal{F}}_{h,g} \bmod \mathfrak{P}.$$

The above conjecture is rather ambiguous because there are many choices for the pair $\Omega_+(f)$, $\Omega_-(f)$. Moreover, the word "big prime ideal" is not defined rigorously. To avoid this ambiguity, we usually use the so called "canonical periods" in Vatsal [Vat]. However, we do not know how to rigorously compute the algebraic part $L_{alg}(k+n, \mathrm{St}(g) \otimes f)$ if we use these canonical periods. A serious attempt to address this issue locally is due to Harder in [Ha3]. For practical purposes, we often normalize periods as follows. For $f \in S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ and a prime ideal $\mathfrak{P}$ of a number field K containing $\mathbf{Q}(f)$ we take the period $\Omega_\pm(f) = \Omega_\pm(\mathfrak{P}; f)$ so that

$$\frac{\Lambda(j,f)}{\Omega_{(-1)^j}(f)} \in \mathbf{Q}(f)$$

for any $1 \leq j \leq 2k-1$, and so that

$$\min_{1 \leq i \leq k} \mathrm{ord}_{\mathfrak{P}} \left(\frac{\Lambda(2i-1,f)}{\Omega_-(f)} \right) = 0, \text{ and } \min_{1 \leq i \leq k-1} \mathrm{ord}_{\mathfrak{P}} \left(\frac{\Lambda(2i,f)}{\Omega_+(f)} \right) = 0,$$

where $\mathrm{ord}_{\mathfrak{P}}(a)$ denotes the $\mathfrak{P}$-adic order of $a \in K$. We do not know whether these periods coincide with those in [Vat] or not. In fact, some answers by Harder (still slightly conjectural) for ordinary primes seem to suggest that the value of $\mathrm{ord}_{\mathfrak{P}}$ on these algebraic values would differ slightly from the true ones. Subsequently in Problem B$'$, however, we will exclude certain small primes to ensure that this difference does not occur. To summarize, although there does exist a theoretical definition of the periods uniquely determined up to $\mathfrak{P}$-unit, we will use the above practical definition of the period and use the L-values associated with those; namely:

$$L_{alg}(l, \mathrm{St}(f_2) \otimes f_1) = \frac{\Lambda(l, \mathrm{St}(f_2) \otimes f_1)}{\langle f_2, f_2 \rangle^2 \Omega_{(-1)^{l+1}}(f_1)}$$

and

$$L_{alg}(i, f) = \frac{\Lambda(i, f)}{\Omega_{(-1)^i}(f)}.$$

We note that

$$L_{alg}(2k+2n, g \otimes g \otimes f) = \frac{\Lambda(k+n, \mathrm{St}(g) \otimes f)\Lambda(k+n, f)}{\langle f, f \rangle \langle g, g \rangle^2} \tag{2}$$

$$= L_{alg}(k+n, \mathrm{St}(g) \otimes f) L_{alg}(k+n, f) \times \frac{\Omega_+(f)\Omega_-(f)}{\langle f, f \rangle}.$$

We also believe that a "big prime ideal" should mean a prime ideal which does not divide $(2k+2n-3)!$ in a suitable setting. Hence we give the following guess, which is a variant of Conjecture B, as a problem:

Problem B$'$. Let $\mathfrak{P}$ be a prime ideal of $K(S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbf{Z})))$ not dividing $(2k+2n-3)!$. Then can we show the following claim? The prime ideal $\mathfrak{P}$ divides

$$\frac{L_{alg}(2k+2n, g \otimes g \otimes f)}{L_{alg}(k+n, f)} \prod_{i=1}^{n-1} L_{alg}(2i+1, f, \mathrm{St})$$

if and only if there exists a Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbf{Z}))$, not coming from the Ikeda-Miyawaki lift, such that

$$G \equiv_{e.v.} \widetilde{\mathcal{F}_{h,g}} \bmod \mathfrak{P}.$$

Here we did not include the term $\langle f, f \rangle / \Omega_+ \Omega_-$ because the primes dividing this seem to be taken care by the denominator of the triple L value $L_{alg}(2k+2n, g \otimes g \otimes f)$.

To explain why the above conjecture and problem are reasonable, we here prove a weaker version of them assuming Conjecture A. To do this, we rewrite Conjecture A in terms of the algebraic parts of L-functions. Thus by testing Problem B', we also test Conjecture A; see section 5 for examples.

Theorem 3.1. *We assume that $k > n$ and n is odd. We fix a real number $\Omega_+(f)$ which satisfies* (1) *of section 2.*
(1) Assume that Conjecture A holds. Then

$$|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 L_{alg}(l, \widetilde{\mathcal{F}}_{h,g}, \mathrm{St}) =$$

$$2^{\gamma_{n,k}} \frac{|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 \langle f, f \rangle}{\langle h, h \rangle \Omega_+(f)} \frac{L_{alg}(l, g, \mathrm{St}) \prod_{i=1}^{n} L_{alg}(l+k+n-2i+1, l+k+n-2i; f)}{L_{alg}(k+n, \mathrm{St}(g) \otimes f) \prod_{i=1}^{n-1} L_{alg}(2i+1, f, \mathrm{St}) \prod_{i=1}^{n} \widetilde{\xi}(2i)}$$

for any positive definite half-integral matrix A of degree $2n+1$ and an odd integer l with $1 \le l \le k-n$, where $\gamma_{n,k}$ is a certain integer depending only on n and k.
(2) Conversely, if the above equality holds for some positive definite half-integral matrix A of degree $2n+1$ and an odd integer l with $n+1 \le l \le k-n$ such that $c_{\widetilde{\mathcal{F}}_{h,g}}(A) \neq 0$, then Conjecture A holds.

Here we have multiplied both sides by $|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2$ since this appears naturally in the pullback formula that we use to prove congruences. In (2), we assume $n+1 \le l$ since $L(l+k+n-i, f)$ might vanish for some i with $1 \le i \le 2n$ without this assumption. Using equation (2) we may rewrite this Theorem.

Corollary. *Assume that Conjecture A holds. Then*

$$|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 L_{alg}(l, \widetilde{\mathcal{F}}_{h,g}, \mathrm{St}) = 2^{\gamma_{n,k}} \frac{|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2\, \Omega_-(f)}{\langle h, h\rangle}$$
$$\times \frac{L_{alg}(k+n, f) L_{alg}(l, g, \mathrm{St}) \prod_{i=1}^{n} L_{alg}(l+k+n-2i+1, l+k+n-2i; f)}{L_{alg}(2k+2n, g\otimes g\otimes f) \prod_{i=1}^{n-1} L_{alg}(2i+1, f, \mathrm{St}) \prod_{i=1}^{n} \widetilde{\xi}(2i)}$$

for any positive definite half-integral matrix A of degree $2n+1$ and an odd integer l with $1 \le l \le k-n$. Conversely, if the above equality holds for some positive definite half-integral matrix A of degree $2n+1$ and an odd integer l with $n+1 \le l \le k-n$ such that $c_{\widetilde{\mathcal{F}}_{h,g}}(A) \neq 0$, then Conjecture A holds.

It would not be useless to review here algebraicity properties that follow easily from known results (for example, see [KZ]).

Proposition 3.2. *Let h, g and f be as in Conjecture A and assume that the Ikeda-Miyawaki lift $\mathcal{F}_{h,g}$ is not identically zero.*

(1) *Let $\Omega_+(f)$ and $\Omega_-(f)$ be fixed real numbers satisfying equation (1) of section 2. For any $A \in \mathcal{L}_{2n+1>0}$, the number $\dfrac{|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 \langle f, f\rangle}{\langle h, h\rangle \Omega_+(f)}$ belongs to $\mathbf{Q}(f)$.*

(2) *The value $|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 L_{alg}(l, \widetilde{\mathcal{F}}_{h,g}, \mathrm{St})$ belongs to $\mathbf{Q}(f)\mathbf{Q}(g)$ for any odd integer l with $1 \le l \le k-n$ and $A \in \mathcal{L}_{2n+1>0}$.*

We can prove a congruence theorem for Ikeda-Miyawaki lifts if we assume Ikeda's Conjecture A.

Theorem 3.3. *Assume Conjecture A. Fix a real number $\Omega_+(f)$ satisfying equation (1) of section 2. Let $\mathfrak{P}$ be a prime ideal of $K(M_{k+n+1}(\mathrm{Sp}_{2n+1}))$ satisfying the following conditions:*

(1) *$\mathfrak{P}$ divides $L_{alg}(k+n, \mathrm{St}(g)\otimes f) \prod_{i=1}^{n-1} L_{alg}(2i+1, f, \mathrm{St}) \prod_{i=1}^{n} \widetilde{\xi}(2i)$.*

(2) *$\mathfrak{P}$ is odd (i.e. $\mathfrak{P} \nmid 2$) and does not divide*

$$(2k+2n-3)! \frac{|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 \langle f, f\rangle}{\langle h, h\rangle \Omega_+(f)} L_{alg}(l, g, \mathrm{St}) \prod_{i=1}^{n} L_{alg}(l+k-2i+1, l+k-2i; f)$$

for some positive definite half-integral matrix A of degree $2n+1$ and some odd integer l with $1 \le l \le k-n-2$.

Then there exists a Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbf{Z}))$, which is not a constant multiple of $\widetilde{\mathcal{F}}_{h,g}$, such that

$$G \equiv_{e.v.} \widetilde{\mathcal{F}}_{f,g} \bmod \mathfrak{P}.$$

Proof. Take an integer l and a matrix A satisfying condition (2) of the Theorem. Then by Theorem 3.1 we have

$$|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 L_{alg}(l, \widetilde{\mathcal{F}}_{h,g}, \mathrm{St})$$

$$= 2^{\gamma_{n,k}} \frac{|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 \langle f, f\rangle}{\langle h, h\rangle \Omega_+(f)} \frac{L_{alg}(l, g, \mathrm{St}) \prod_{i=1}^{n} L_{alg}(l+k-2i+1, l+k-2i; f)}{L_{alg}(k+n, \mathrm{St}(g) \otimes f) \prod_{i=1}^{n-1} L_{alg}(2i+1, f, \mathrm{St}) \prod_{i=1}^{n} \widetilde{\xi}(2i)},$$

where $\gamma_{n,k}$ is a certain integer depending only on n and k. Then, by the assumption, $\mathfrak{P}$ divides the denominator of the right hand side of the equation for $|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2 L_{alg}(l, \widetilde{\mathcal{F}}_{h,g}, \mathrm{St})$, and by Theorem 5.2 of [Kat3], we have proved the assertion. (We note that there is an error in Theorem 5.2 of [Kat3]. The integer "$(2l-1)!$" on page 108, line 15 should be "$(2l+2n-1)!$".) □

We rewrite the above theorem in terms of triple L-values.

Theorem 3.4. *Assume Conjecture A. Let $\mathfrak{P}$ be a prime ideal in the ring of integers of $K(M_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbf{Z})))$ satisfying the following conditions:*

(1) *$\mathfrak{P}$ divides $\frac{L_{alg}(2k+2n, g\otimes g\otimes f)}{L_{alg}(k+n, f)} \prod_{i=1}^{n-1} L_{alg}(2i+1, f, \mathrm{St}) \prod_{i=1}^{n} \widetilde{\xi}(2i)$.*

(2) *$\mathfrak{P}$ does not divide*

$$(2k+2n-3)! \frac{|c_{\widetilde{\mathcal{F}}_{h,g}}(A)|^2\, \Omega_-(f)}{\langle h, h\rangle} L_{alg}(l, g, \mathrm{St}) \prod_{i=1}^{n} L_{alg}(l+k-2i+1, l+k-2i; f)$$

for some positive definite half-integral matrix A of degree $2n+1$ and some odd integer l with $1 \le l \le k-n-2$.

Then there exists a Hecke eigenform $G \in S_{k+n+1}(\mathrm{Sp}_{2n+1}(\mathbf{Z}))$ which is not a constant multiple of $\widetilde{F}_{h,g}$ such that

$$G \equiv_{e.v.} \widetilde{\mathcal{F}}_{h,g} \bmod \mathfrak{P}.$$

In Section 5, we will rigorously verify that the answer to Problem B' is affirmative without assuming Conjecture A in the two particular cases where $(n, k) = (1, 14)$ and $(1, 18)$.

Finally, we refer to a conjecture due to Bergström, Faber, and van der Geer. Our conjecture is on a congruence between an Ikeda-Miyawaki lift and another Hecke eigenform in the scalar valued case. In the vector valued case, it is likely that there are no Ikeda-Miyawaki lifts in general but still we can consider an Ikeda-Miyawaki type L-function $L^{IK}(s, f, g)$ defined by a pair of elliptic modular forms f and g. They predict the existence of a vector valued Siegel modular form F of degree 3 such that its $T^{(3)}(p)$-eigenvalue is congruent to the p^{-s} coefficient of $L^{IK}(s, f, g)$. To explain this, let U be the standard representation of $\mathrm{GL}_3(\mathbf{C})$, and for a triple (n_1, n_2, n_3) of non-negative integers, let U_{n_1,n_2,n_3} be the irreducible representation of $\mathrm{GL}_3(\mathbf{C})$ of with signature $(n_1+n_2+n_3, n_2+n_3, n_3)$ in the sense of Weyl [We]. We then denote by $\mathcal{S}_{n_1,n_2,n_3} = \mathcal{S}_{n_1,n_2,n_3}(\mathrm{Sp}_3(\mathbf{Z}))$ the space of cusp forms of weight U_{n_1,n_2,n_3} for $\mathrm{Sp}_3(\mathbf{Z})$. We note that $\mathcal{S}_{0,0,n_3} = S_{n_3}(\mathrm{Sp}_3(\mathbf{Z}))$. Now we assume

$a \geq b \geq c \geq 0$. For primitive forms $g \in S_{a+4}(\mathrm{SL}_2(\mathbf{Z}))$ and $f \in S_{b+c+4}(\mathrm{SL}_2(\mathbf{Z}))$, define $L(s, Sym^2(g) \otimes f)$ as

$$L(s, Sym^2(g) \otimes f) = L(s-a-3, \mathrm{St}(g) \otimes f),$$

and put

$$L_{alg}(a+b+6, Sym^2(g) \otimes f) = L_{alg}(b+3, \mathrm{St}(g) \otimes f),$$

fixing $\Omega_{\pm}(f)$.

Bergström, Faber, and van der Geer proposed the following conjecture:

Conjecture C. ([[BFG] Conjecture 10.8]) *Let $\mathfrak{P}$ be a prime ideal of $\mathbf{Q}(f)\mathbf{Q}(g)$. Assume that, for some $s \in \mathbf{N}$, $\mathfrak{P}^s$ divides the algebraic part $L_{alg}(a+b+6, Sym^2(g) \otimes f)$ of $L(a+b+6, Sym^2(g) \otimes f)$. Then there exists a Hecke eigenform F in $\mathcal{S}_{a-b,b-c,c+4}$ such that*

$$\lambda_F(T^{(3)}(q)) \equiv \lambda_g(T^{(1)}(q))(\lambda_f(T^{(1)}(q)) + q^{b+2} + q^{c+1}) \bmod \widetilde{\mathfrak{P}}^s$$

for all prime numbers q, where $T^{(r)}(q)$ is the element of $\mathbf{L}'_r$ defined by

$$T^{(r)}(q) = \mathrm{Sp}_r(\mathbf{Z})(1_r \perp q 1_r)\,\mathrm{Sp}_r(\mathbf{Z}),$$

and $\widetilde{\mathfrak{P}}$ is some prime ideal of $\mathbf{Q}(f)\mathbf{Q}(g)\mathbf{Q}(F)$.

Now let us consider the case $a = b = c$. In this case, if we take $F = \mathcal{F}_{h,g}$, where $h \in S^+_{a+5/2}(\Gamma_0(4))$ corresponds to f under the Shimura correspondence, then Conjecture C is obviously satisfied, not just with a congruence but with equality. In contrast, Conjecture B predicts the existence of a cusp form F, not a constant multiple of $\mathcal{F}_{h,g}$, such that $F \equiv_{e.v.} \mathcal{F}_{h,g} \bmod \widetilde{\mathfrak{P}}$. Later, in Table 3, we give exact special values (in contrast to approximate values) for all (a, b, c) which appear in their conjecture as examples, and prove that these values are divisible by the primes that they were able to guess by approximation.

4. Triple L-values

4.1. **Böcherer and Schulze-Pillot's formula for the triple L-values.** We review a formula for the triple L-values of elliptic modular forms due to Böcherer and Schulze-Pillot [BS]. We define two types of imbeddings, ι_{12}, and ι_{111}, of products of upper half spaces into $\mathbf{H}_3$ via:

$$\iota_{12} : \mathbf{H}_1 \times \mathbf{H}_2 \ni (z, Z) \mapsto \begin{pmatrix} z & O \\ O & Z \end{pmatrix} \in \mathbf{H}_3,$$

and

$$\iota_{111} : \mathbf{H}_1^3 \ni (z_1, z_2, z_3) \mapsto \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_2 & 0 \\ 0 & 0 & z_3 \end{pmatrix} \in \mathbf{H}_3.$$

We use the same symbol i_{111} to denote the corresponding diagonal imbeddings of $\mathrm{SL}_2(\mathbf{Z})^3$ into $\mathrm{Sp}_3(\mathbf{Z})$. For an integer α we define the Maaß operator $\mathcal{M}_\alpha$ on $C^\infty(\mathbf{H}_3)$ as

$$\mathcal{M}_\alpha = \det(Z - \bar{Z})^{2-\alpha} \det(\partial_{ij}) \det(Z - \bar{Z})^{\alpha-1},$$

and for a non-negative integer ν put

$$\mathcal{M}_\alpha^{[\nu]} = \mathcal{M}_{\alpha+\nu-1} \circ \cdots \circ \mathcal{M}_{\alpha+1} \circ \mathcal{M}_\alpha.$$

Here $Z = (z_{ij}) \in \mathbf{H}_3$, and $\partial_{ij} = \frac{\partial}{\partial z_{ii}}$ or $\frac{1}{2}\frac{\partial}{\partial z_{ij}}$ according as $i = j$ or not. The operator $\mathcal{M}_\alpha^{[\nu]}$ maps $C^\infty(\mathbf{H}_3)$ to itself.

For a non-negative integer b, let $V_3^{(b)}$ be the $\mathbf{C}$-vector space of homogeneous polynomials in X_2 and X_3 of degree b with coefficients in $\mathbf{C}$, and define the map $\mathbf{L}_\alpha^{[b]}$ from $C^\infty(\mathbf{H}_3)$ to $C^\infty(\mathbf{H}_1 \times \mathbf{H}_2, V_3^{(b)})$ as

$$\mathbf{L}_\alpha^{[b]}(F) =$$
$$\frac{1}{(\alpha)_b} \sum_{0 \le 2\nu \le b} \frac{1}{\nu!(b-2\nu)!(2-\alpha-b)_\nu} \cdot \left((D_\uparrow D_\downarrow)^\nu \cdot (D - D_\uparrow - D_\downarrow)^{b-2\nu} F\right)(\iota_{12}(z, Z))$$

$((z, Z) \in \mathbf{H}_1 \times \mathbf{H}_2)$ for $F \in C^\infty(\mathbf{H}_3)$, where

$$D_\uparrow = \partial_{11},$$
$$D_\downarrow = \sum_{2 \le i,j \le 3} \partial_{ij} X_i X_j,$$
$$D - D_\uparrow - D_\downarrow = 2(\partial_{12} X_2 + \partial_{13} X_3),$$

and $(\alpha)_\nu$ is the Pochhammer symbol:

$$(\alpha)_\nu = \frac{\Gamma(\alpha+\nu)}{\Gamma(\alpha)} = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+\nu-1) & \text{if } \nu \neq 0, \\ 1 & \text{if } \nu = 0. \end{cases}$$

Composing the above two maps and restricting to $\mathbf{H}_1^3$, for an even positive integer a we define the map $\mathcal{D}_\alpha^{*(a,b)}$ from $C^\infty(\mathbf{H}_3)$ to $C^\infty(\mathbf{H}_1^3, V_3^{(b)})$ as

$$(\mathcal{D}_\alpha^{*(a,b)}(F))(z_1, z_2, z_3) = \left(\mathbf{L}_{\alpha+a/2}^{[b]} \circ \mathcal{M}_\alpha^{[a/2]}(F)\right)(\iota_{111}(z_1, z_2, z_3))$$

for $(z_1, z_2, z_3) \in \mathbf{H}_1^3$ and $F \in C^\infty(\mathbf{H}_3)$. Then $(\mathcal{D}_\alpha^{*(a,b)}(F))$ can be decomposed as

$$\mathcal{D}_\alpha^{*(a,b)}(F) = \sum_{\nu_2+\nu_3=b, \nu_2 \ge 0} (\mathcal{D}_\alpha^{*(a,\nu_2,\nu_3)}(F)) X_2^{\nu_2} X_3^{\nu_3},$$

making $\mathcal{D}_\alpha^{*(a,\nu_2,\nu_3)}$ a map from $C^\infty(\mathbf{H}_3)$ to $C^\infty(\mathbf{H}_1^3)$. The map $\mathcal{D}_\alpha^{*(a,\nu_2,\nu_3)}$ preserves automorphy but does not preserve holomorphy. To construct a holomorphic differential operator which preserves automorphy, let

$$\delta_\alpha = \frac{1}{2\pi\sqrt{-1}} \left(\frac{\alpha}{2\sqrt{-1}y} + \frac{\partial}{\partial z}\right),$$

and

$$\delta_\alpha^\mu(z) = \delta_{\alpha+2\mu-2} \circ \cdots \circ \delta_\alpha.$$

Then we have

$$(y_1 y_2 y_3)^{-a/2} \cdot \mathcal{D}_\alpha^{*(a,\nu_2,\nu_3)} =$$
$$\sum \delta_{\alpha+a+\nu_2+\nu_3-2\mu_1}^{\mu_1}(z_1) \delta_{\alpha+a+\nu_2-2\mu_2}^{\mu_2}(z_2) \delta_{\alpha+a+\nu_3-2\mu_3}^{\mu_3}(z_3) \mathbf{D}_\alpha(a, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3),$$

where the summation is over $0 \le \mu_1, \mu_2, \mu_3 \le a/2$ and where $\mathbf{D}_\alpha(a, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3)$ is a holomorphic differential operator, that is, it maps from $C^\infty(\mathbf{H}_3)$ to $C^\infty(\mathbf{H}_1^3)$ and sends $\mathrm{Hol}(\mathbf{H}_3)$ to $\mathrm{Hol}(\mathbf{H}_1^3)$. Moreover it preserves the holomorphy at cusps, and we have

$$\mathbf{D}_\alpha(a, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3)(F|_\alpha \iota_{111}(g_1, g_2, g_3))$$
$$= \mathbf{D}_\alpha(a, \nu_2, \nu_3, \mu_1, \mu_2, \mu_3)(F)|_{\alpha+a+\nu_2+\nu_3-2\mu_2}^{z_1} g_1|_{\alpha+a+\nu_2-2\mu_2}^{z_2} g_2|_{\alpha+a+\nu_3-2\mu_3}^{z_3} g_3,$$

where the upper indices z_1, z_2 and z_3 indicate which variable is relevant at the moment. In particular, for a positive integer r and non-negative integers ν_1, ν_2, ν_3 such that $\nu_2 \ge \nu_1, \nu_3 \ge \nu_1$ $\mathbf{D}_r(2\nu_1, \nu_2 - \nu_1, \nu_3 - \nu_1, 0, 0, 0)$ maps $M_r(\mathrm{Sp}_3(\mathbf{Z}))$ to

$M_{k_1}(\mathrm{SL}_2(\mathbf{Z}))\otimes M_{k_2}(\mathrm{SL}_2(\mathbf{Z}))\otimes M_{k_3}(\mathrm{SL}_2(\mathbf{Z}))$, where $k_1=r+\nu_2+\nu_3, k_2=r+\nu_1+\nu_3$, and $k_3=r+\nu_2+\nu_1$. From now on we put $\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3}=\mathbf{D}_r(2\nu_1,\nu_2-\nu_1,\nu_3-\nu_1,0,0,0)$.

Let

$$E_{n,r}(Z)=\zeta(1-r)\prod_{i=1}^{[n/2]}\zeta(2i+1-2r)\sum_{\gamma\in\Gamma_\infty^{(n)}\backslash\mathrm{Sp}_n(\mathbf{Z})}j(\gamma,Z)^{-r}$$

be the Eisenstein series of weight r and degree n, where

$$\Gamma_\infty^{(n)}=\{\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in\mathrm{Sp}_n(\mathbf{Z})\mid C=0\}.$$

By a careful examination of the proof of Theorem 4.2 in [BS], we obtain:

Theorem 4.1. *For a positive integer r and non-negative integers ν_1,ν_2,ν_3 such that $\nu_2\geq\nu_1,\nu_3\geq\nu_1$, put $k_1=r+\nu_2+\nu_3, k_2=r+\nu_1+\nu_3$ and $k_3=r+\nu_2+\nu_1$. Let f_1,f_2, and f_3 be primitive forms in $S_{k_1}(\mathrm{SL}_2(\mathbf{Z})), S_{k_2}(\mathrm{SL}_2(\mathbf{Z}))$ and $S_{k_3}(\mathrm{SL}_2(\mathbf{Z}))$, respectively. For an even positive integer r such that $2\leq r\leq k_2+k_3-k_1$ put*

$$\widetilde{\mathbf{M}}(r;f_1,f_2,f_3)=$$
$$-\frac{(\sqrt{-1})^{(k_1-k_2-k_3+r)/2}}{2^{3k_1/2+k_2/2+k_3/2+r/2-3}((k_2+k_3-k_1+r)/2)_{2k_1-k_2-k_3}(k_1-k_3)!(k_1-k_2)!}$$
$$\times L_{alg}(\frac{k_1+k_2+k_3+r}{2}-2,f_1\otimes f_2\otimes f_3).$$

Then we have

$$\langle\langle\langle(2\pi\sqrt{-1})^{3(-k_1+k_2+k_3-r)/2}\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3}(E_{3,r}),f_1\rangle,f_2\rangle,f_3\rangle$$
$$=\langle f_1,f_1\rangle\langle f_2,f_2\rangle\langle f_3,f_3\rangle\widetilde{\mathbf{M}}(r;f_1,f_2,f_3).$$

Proof. We can prove this by using (3.1) (or equivalently (2.41)) of [BS] and replacing $T_p(s)$ by (3.22) there. This is the same sort of calculation that was done for $r=2$ but not for general r in Theorem 4.1 of [BS]. Although the calculation follows [BS], we give some details here since it is fairly complicated. We must calculate

$$\frac{(2\pi i)^{-(k_1+k_2+k_3-3r)/2}\zeta(1-r)\zeta(3-2r)A(f_1,f_2,f_3,0)}{c_r(0)L_\infty(\frac{k_1+k_2+k_3+r}{2}-2)}\tag{3}$$

in the notation of [BS]. By definition (1.19) in p.7 of [BS], we have $c_r(0)=1$. By (3.1) loc. cit., we have

$$A(f_1,f_2,f_3,0)=2\binom{b}{\tilde{\nu}_2}A(r,b)\mu(r+a+b,-a')T_\infty(-a')\prod_p T_p(0).$$

Here we have

$$a=k_2+k_3-k_1-r,\quad a'=a/2,\quad b=2k_1-k_2-k_3,\quad \tilde{\nu}_2=k_1-k_3,\quad b-\tilde{\nu}_2=k_1-k_2.$$

By (3.22) in [BS], we have

$$\prod_p T_p(0)=\frac{L(\frac{k_1+k_2+k_3+r}{2}-2)}{\zeta(r)\zeta(2r-2)}.$$

By the functional equation, we have

$$\frac{\zeta(1-r)\zeta(3-2r)}{\zeta(r)\zeta(2r-2)} = -(-1)^{r/2}2^{4-3r}\pi^{2-3r}.$$

By the definition in [BS], we have

$$\binom{b}{\tilde{\nu}_2}A(r,b) = \frac{2^{k_1-k_2-k_3+r}}{(k_1-k_2)!(k_1-k_3)!}\frac{\Gamma(k_1+r-2)\Gamma(\frac{k_2+k_3-k_1+r}{2})\Gamma(\frac{k_2+k_3-k_1+r}{2}-1)}{\Gamma(r)\Gamma(2r-2)\Gamma(\frac{3k_1-k_2-k_3+r}{2}-1)},$$

$$\mu(r+a+b,-a^{'}) = (-1)^{k_1/2}\,2^{3-2k_1+k_2+k_3-r}\frac{\pi}{\frac{3k_1-k_2-k_3+r}{2}-1}.$$

On the other hand, we have

$$T_\infty(-a^{'}) = (4\pi)^{-(k_1+k_2+k_3+r)/2+2} \times \frac{\Gamma(\frac{k_1+k_2+k_3+r}{2}-2)\Gamma(\frac{k_1+k_2-k_3+r}{2}-1)\Gamma(\frac{k_1-k_2+k_3+r}{2}-1)}{\Gamma(k_1+r-2)},$$

and

$$\frac{1}{L_\infty(\frac{k_1+k_2+k_3+r}{2}-2)} = \frac{2^{-4}(2\pi)^{(k_1+k_2+k_3+2r)-5}}{\Gamma(\frac{k_1+k_2+k_3+r}{2}-2)\Gamma(\frac{-k_1+k_2+k_3+r}{2}-1)\Gamma(\frac{k_1-k_2+k_3+r}{2}-1)\Gamma(\frac{k_1+k_2-k_3+r}{2}-1)}.$$

Here we used the fact that k_i and b are even integers and that $(-1)^b = 1$. Using $((3k_1-k_2-k_3+r)/2-1)\Gamma((3k_1-k_2-k_3+r)/2-1) = \Gamma((3k_1-k_2-k_3+r)/2)$, we see that the part involving the Γ function in (3) is given by

$$\frac{\Gamma(\frac{k_2+k_3-k_1+r}{2})}{\Gamma(\frac{3k_1-k_2-k_3+r}{2})} = \frac{1}{((k_2+k_3-k_1+r)/2)_{(2k_1-k_2-k_3)}}.$$

The other factors consisting of the power of i, 2 or π are calculated from the above data and the theorem is proved. □

Lemma 4.2. *The map* $\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3} : C^\infty(\mathbf{H}_3) \to C^\infty(\mathbf{H}_1^3)$ *satisfies*

$$\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3}(z_{12}^{\nu_3}z_{13}^{\nu_2}z_{23}^{\nu_1}) = \frac{2^{\nu_1}(\sqrt{-1})^{3\nu_1}\nu_1!\nu_2!\nu_3!}{(\nu_3-\nu_1)!(\nu_2-\nu_1)!(r+\nu_1)_{(\nu_2+\nu_3-2\nu_1)}}.$$

Proof. Put $\widetilde{\nu}_2 = \nu_2-\nu_1, \widetilde{\nu}_3 = \nu_3-\nu_1,$. Then in the notation of [BS], we have $b = \widetilde{\nu}_2+\widetilde{\nu}_3 = \nu_2+\nu_3-2\nu_1$. Then it follows from (1.11) of [BS] that $(y_1y_2y_3)^{-\nu_1}\mathcal{D}_r^{\nu_1,b}$ can be expressed as

$$(y_1y_2y_3)^{-\nu_1}\mathcal{D}_r^{\nu_1,b} = (\partial_{12}\partial_{13}\partial_{23})^{\nu_1}\frac{1}{(r+\nu_1)_b(b)!}2^{b+4\nu_1}\sqrt{-1}^{3\nu_1}(\partial_{12}X_2+\partial_{13}X_3)^b + Q,$$

where Q is a polynomial in $\partial_{12},\partial_{13},\partial_{23}$ of total degree smaller than $3\nu_1+b$ with coefficients in $\mathbf{C}[\partial_{11},\partial_{22},\partial_{33},y_1^{-1},y_2^{-1},y_3^{-1},X_2,X_3]$. (There is a misprint in (1.11) of [BS]. For the correction, see the Appendix of [BSS].) Hence $\mathcal{D}_r^{*(2\nu_1,\widetilde{\nu}_2,\widetilde{\nu}_3)}$ can be expressed as

$$\mathcal{D}_r^{*(2\nu_1,\widetilde{\nu}_2,\widetilde{\nu}_3)} = \partial_{12}^{\nu_1}\partial_{13}^{\nu_2}\partial_{23}^{\nu_3}\frac{1}{(r+\nu_1)_b\widetilde{\nu}_2!\widetilde{\nu}_3!}2^{b+4\nu_1}\sqrt{-1}^{3\nu_1} + R,$$

where R is a polynomial in $\partial_{12}, \partial_{13}, \partial_{23}$ of total degree smaller than $3\nu_1 + b$ with coefficients in $\mathbf{C}[\partial_{11}, \partial_{22}, \partial_{33}, y_1^{-1}, y_2^{-1}, y_3^{-1}]$. Hence $\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3} = \mathbf{D}_r(2\nu_1, \widetilde{\nu}_2, \widetilde{\nu}_3, 0, 0, 0)$ can be expressed as

$$\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3} = \partial_{12}^{\nu_1}\partial_{13}^{\nu_2}\partial_{23}^{\nu_3}\frac{1}{(r+\nu_1)_b\widetilde{\nu}_2!\widetilde{\nu}_3!}2^{b+4\nu_1}\sqrt{-1}^{3\nu_1} + \widetilde{R},$$

where $\widetilde{R}$ is a polynomial in $\partial_{12}, \partial_{13}, \partial_{23}$ of total degree smaller than $3\nu_1 + b$ with coefficients in $\mathbf{C}[\partial_{11}, \partial_{22}, \partial_{33}]$. Since $b + 3\nu_1 = \nu_1 + \nu_2 + \nu_3$, we also have

$$2^{b+3\nu_1}\partial_{12}^{\nu_1}\partial_{13}^{\nu_2}\partial_{23}^{\nu_1}(z_{12}^{\nu_3}z_{13}^{\nu_2}z_{23}^{\nu_1}) = \nu_1!\nu_2!\nu_3!.$$

So the assertion is proved. □

4.2. **Alternative simple differential operators from** [IZ] **applied to triple L-values.** The above formula from Theorem 4.1 due to Böcherer and Schulze-Pillot is useful for investigating the qualitative nature of triple L-values. However, these operators are fairly complicated. There exists another formulation on similar differential operators in a quite general setting slightly different from theirs (cf. [Ib]) and these are easier to handle since they come from harmonic polynomials that possess some invariance properties. Indeed, in the case treated in this article, there are simply described differential operators $\mathbf{D}_{r,\nu_1,\nu_2,\nu_3}$ such that $Res_\Delta(\mathbf{D}_{r,\nu_1,\nu_2,\nu_3}(F))$ is equal to $Res_\Delta(\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3}(F))$ for any holomorphic function F on H_3, up to a common constant (depending on r, ν_i), where Res_Δ is the restriction to the diagonal $\Delta = H_1^3 \subset H_3$. We can give a formula for $\mathbf{D}_{\nu_1,\nu_2,\nu_3}$ (cf. [IZ]) as a polynomial of partial derivatives. These operators are well suited to give special values of triple L functions, so we will adjust the constant in the above formula by Böcherer and Schulze-Pillot into the version using these new differential operators from [IZ]. In order to use the differential operators from [IZ], we introduce the following notation. Let $\{x_i\}_{1\le i\le 3}$ and $\{t_{ij}\}_{1\le i,j\le 3}$ (where $t_{ij} = t_{ji}$) be variables and put $X = \begin{pmatrix} 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 & x_1 & 0 \end{pmatrix}$ and $T = (t_{ij})_{1\le i,j\le 3}$. We write $P = P(T)$ if P is a polynomial in t_{ij} $(1 \le i \le j \le 3)$ and say that P is a polynomial in T. We denote by $\sigma_i = \sigma_i(TX)$ the i-th symmetric function of eigenvalues of TX. We note that σ_i is a polynomial in x_1, x_2, x_3 and T. Then for a non-negative integer s we define the formal power series $G^{(3)}(s; \sigma_1, \sigma_2, \sigma_3)$ by

$$G^{(3)}(s; \sigma_1, \sigma_2, \sigma_3) = \sum_{a,b,c\ge 0} 2^{c-a}\binom{a+2b+4c+2s+1}{a}\binom{b+2c+s}{b}\binom{2c+s}{c}\sigma_1^a\sigma_2^b\sigma_3^c,$$

where $\binom{*}{*}$ is the binomial coefficient. Write $G^{(3)}(s; \sigma_1, \sigma_2, \sigma_3)$ as

$$G^{(3)}(s; \sigma_1, \sigma_2, \sigma_3) = \sum_{\nu_1,\nu_2,\nu_3\ge 0} P_{s,\nu_1,\nu_2,\nu_3}(T)x_1^{\nu_1}x_2^{\nu_2}x_3^{\nu_3}$$

and say that $P_{s,\nu_1,\nu_2,\nu_3}(T)$ is a polynomial in T and s. For an integer $r \ge 2$ and non-negative integers ν_1, ν_2, ν_3 we define the operator $\mathbf{D}_{r,\nu_1,\nu_2,\nu_3}$ on $C^\infty(\mathbf{H}_3)$ by

$$\mathbf{D}_{r,\nu_1,\nu_2,\nu_3} = P_{r-2,\nu_1,\nu_2,\nu_3}((2\pi i)^{-1}(\partial_{ij})).$$

For any holomorphic function F on H_3, we define $\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}$ by

$$\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}(F) = Res_\Delta(\mathbf{D}_{r,\nu_1,\nu_2,\nu_3}(F))$$

where Res_Δ means the restriction of the function to $\Delta = \mathbf{H}_1^3 \subset \mathbf{H}_3$. We skip the theoretical details here but we know the following fact as a special case of results from [Ib] and [IZ].

Theorem 4.3 ([IZ]). *Let $k_1 = r + \nu_2 + \nu_3, k_2 = r + \nu_1 + \nu_3, k_3 = r + \nu_1 + \nu_2$ with $r \in \mathbb{Z}_{\geq 2}$ and ν_1, ν_2, ν_3 non-negative integers. The operator $\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}$ maps $M_r(\mathrm{Sp}_3(\mathbf{Z}))$ to $M_{k_1}(\mathrm{SL}_2(\mathbf{Z})) \otimes M_{k_2}(\mathrm{SL}_2(\mathbf{Z})) \otimes M_{k_3}(\mathrm{SL}_2(\mathbf{Z}))$. In particular if $\nu_1 > 0$ and $\nu_2 = \nu_3 = 0$, then the image of $\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}$ is included in $M_{k_1}(\mathrm{SL}_2(\mathbf{Z})) \otimes S_{k_2}(\mathrm{SL}_2(\mathbf{Z})) \otimes S_{k_3}(\mathrm{SL}_2(\mathbf{Z}))$; and if at least two of ν_1, ν_2, ν_3 are positive, then the image is included in $S_{k_1}(\mathrm{SL}_2(\mathbf{Z})) \otimes S_{k_2}(\mathrm{SL}_2(\mathbf{Z})) \otimes S_{k_3}(\mathrm{SL}_2(\mathbf{Z}))$.*

Lemma 4.4 ([IZ]). *The map $\mathbf{D}_{r,\nu_1,\nu_2,\nu_3} : C^\infty(\mathbf{H}_3) \to C^\infty(\mathbf{H}_1^3)$ satisfies*

$$\mathbf{D}_{r,\nu_1,\nu_2,\nu_3}(z_{12}^{\nu_3} z_{13}^{\nu_2} z_{23}^{\nu_1}) = \frac{(\nu_2+\nu_1+r-2)!(\nu_1+\nu_3+r-2)!(\nu_1+\nu_2+r-2)!}{(\nu_1+r-2)!(\nu_2+r-2)!(\nu_3+r-2)!}.$$

For any holomorphic function F, we define an operator $\widetilde{\mathbb{D}}_{r,\nu_1,\nu_2,\nu_3}$ by

$$\widetilde{\mathbb{D}}_{r,\nu_1,\nu_2,\nu_3}(F) = Res_\Delta(\widetilde{\mathbf{D}}_{r,\nu_1,\nu_2,\nu_3} F).$$

Proposition 4.5. *We have*

$$\mathbb{D}_{r,\nu_1,\nu_2,\nu_3} = \frac{(r+\nu_1)_{(\nu_2+\nu_3-2\nu_1)}(\nu_3-\nu_1)!(\nu_2-\nu_1)!}{2^{\nu_1}(\sqrt{-1})^{3\nu_1}}$$
$$\times \binom{\nu_2+\nu_3+r-2}{\nu_2+r-2}\binom{\nu_1+\nu_3+r-2}{\nu_3+r-2}\binom{\nu_1+\nu_2+r-2}{\nu_1+r-2}\widetilde{\mathbb{D}}_{r,\nu_1,\nu_2,\nu_3}.$$

In particular, if $k_1 = r+\nu_2+\nu_3, k_2 = r+\nu_1+\nu_3$ and $k_3 = r+\nu_2+\nu_1$, then

$$\mathbb{D}_{r,\nu_1,\nu_2,\nu_3} = \frac{((k_2+k_3-k_1+r)/2)_{(2k_1-k_2-k_3)}(k_1-k_2)!(k_1-k_3)!}{2^{(k_2+k_3-k_1-r)/2}(\sqrt{-1})^{3(k_2+k_3-k_1-r)/2}}$$
$$\times \binom{k_1-2}{(k_1+k_3-k_2+r)/2-2}\binom{k_2-2}{(k_1+k_2-k_3+r)/2-2}$$
$$\times \binom{k_3-2}{(k_2+k_3-k_1+r)/2-2}\widetilde{\mathbb{D}}_{r,\nu_1,\nu_2,\nu_3}.$$

Proof. We remark that $\nu_1 = (-k_1+k_2+k_3-r)/2, \nu_2 = (k_1-k_2+k_3-r)/2$ and $\nu_3 = (k_1+k_2-k_3-r)/2$. This proves the latter half of the assertions. □

By definition we have

$$\text{(A)} \qquad P_{s,\nu_1,\nu_2,\nu_3}(T) = \frac{1}{\nu_1!\nu_2!\nu_3!}\frac{\partial^{\nu_1+\nu_2+\nu_3}}{\partial x_1^{\nu_1}\partial x_2^{\nu_2}\partial x_3^{\nu_3}}(G^{(3)}(s;\sigma_1,\sigma_2,\sigma_3))|_{x_1=x_2=x_3=0}.$$

When we apply this to $L(s, f\otimes g\otimes g)$ where two modular forms are the same, we need only coefficients at $(\nu_1,\nu_2,\nu_3) = (0,\nu,\nu)$, and we have a simpler formula in this case, which is useful sometimes.

Theorem 4.6. *We have $P_{s,0,\nu,\nu}(T) =$*

$$\sum_{a_0=0}^{\nu}\sum_{j,l}\frac{2^{\nu-l-j-a_0}(2\nu+2s+1)!(\nu-a_0+s)!P(l,j,a_0)}{s!(2\nu+2s+1-2a_0)!(a_0+l-j)!(a_0-l+j)!(\nu-a_0-l-j)!j!l!},$$

where j, l run over all non-negative integers such that $\max(j,l) \le \nu/2$ *and* $|j-l| \le a_0 \le \nu - l - j$, *and*

$$P(l,j,a_0) = t_{12}^{a_0-l+j} t_{13}^{a_0+l-j} (t_{12}^2 - t_{11}t_{22})^l (t_{13}^2 - t_{11}t_{33})^j (t_{12}t_{13} - t_{11}t_{23})^{\nu-a_0-l-j}.$$

Proof. Since $\nu_1 = 0$, we put $x_1 = 0$ in $G^{(3)}(s, \sigma_1, \sigma_2, \sigma_3)$. By definition, under the restriction to $x_1 = 0$, we have

$$\begin{aligned}
\sigma_1 &= 2(t_{13}x_2 + t_{12}x_3) \\
\sigma_2 &= (t_{13}^2 - t_{11}t_{33})x_2^2 + 2(t_{12}t_{13} - t_{11}t_{23})x_2x_3 + (t_{12}^2 - t_{11}t_{22})x_3^2, \\
\sigma_3 &= 0.
\end{aligned}$$

So we have

$$\begin{aligned}
&G^{(3)}(s, \sigma_1, \sigma_2, \sigma_3)\big|_{x_1=0} \\
&= \sum_{a,b\ge 0} 2^{-a} \binom{a+2b+2s+1}{a} \binom{b+s}{b} (\sigma_1^a \sigma_2^b)|_{x_1=0} \\
&= \sum_{a,b\ge 0} \binom{a+2b+2s+1}{a} \binom{b+s}{b} \sum_{i=0}^{a} \binom{a}{i} t_{13}^i t_{12}^{a-i} \\
&\times \sum_{\substack{l+m+j=b \\ l,m,j\ge 0}} \frac{2^m b!}{l!m!j!} (t_{12}^2 - t_{11}t_{22})^l (t_{12}t_{13} - t_{11}t_{23})^m (t_{13}^2 - t_{11}t_{33})^j x_2^{i+2j+m} x_3^{a-i+2l+m}.
\end{aligned}$$

Now we assume that $i + 2j + m = a - i + 2l + m = \nu$. Then $a = 2(i + j - l)$ is even, so we write $a = 2a_0$ and we have $a_0 = i + j - l$. We also have $2\nu = (2a_0 - i + 2l + m) + (i + 2j + m) = 2a_0 + 2j + 2l + 2m$, so $\nu = a_0 + j + l + m$. For each fixed $a_0 \in \mathbf{Z}$ with $0 \le a_0 \le \nu$, we erase i, m and $b = l + m + j$ by these relations. The condition $0 \le i \le 2a_0$ means $|j - l| \le a_0$ and the condition $0 \le m$ means $0 \le a_0 \le \nu - (j + l)$. (As a consequence, we have j, $l \le \nu/2$.) Since $a + 2b = 2a_0 + 2(\nu - a_0) = 2\nu$, we have

$$\begin{aligned}
&2^m \binom{a+2b+2s+1}{a} \binom{b+s}{b} \binom{2a_0}{i} \frac{b!}{j!l!m!} \\
&= 2^m \binom{2\nu+2s+1}{2a_0} \binom{\nu-a_0+s}{\nu-a_0} \binom{2a_0}{a_0-j+l} \frac{(\nu-a_0)!}{j!l!(\nu-a_0-j-l)!} = \\
&2^m \frac{(2\nu+2s+1)!(\nu-a_0+s)!(\nu-a_0)!(2a_0)!}{(2\nu+2s-2a_0+1)!(2a_0)!s!(a_0-j+l)!(a_0+j-l)!(\nu-a_0)!j!l!(\nu-a_0-j-l)!} \\
&= 2^{\nu-a_0-j-l} \frac{(2\nu+2s+1)!(\nu-a_0+s)!}{(2\nu+2s-2a_0+1)!s!(a_0-j+l)!(a_0+j-l)!j!l!(\nu-a_0-j-l)!}
\end{aligned}$$

This gives the assertion of Theorem 4.6. □

The following theorem is a reformulation of Theorem 4.1

Theorem 4.7. *Let k_1, k_2, k_3 be positive even integers such that $k_1 \ge k_2 \ge k_3$ and $k_1 < k_2 + k_3$. Let f_1, f_2 and f_3 be primitive forms in $S_{k_1}(\mathrm{SL}_2(\mathbf{Z})), S_{k_2}(\mathrm{SL}_2(\mathbf{Z}))$,*

and $S_{k_3}(\mathrm{SL}_2(\mathbf{Z}))$, *respectively. For an even positive integer* r *such that* $2 \le r \le k_2 + k_3 - k_1$ *put*

$$\mathbf{M}(r; f_1, f_2, f_3) = -\frac{1}{2^{k_1+k_2+k_3-3}} \times \binom{k_1-2}{\frac{k_1+k_3-k_2+r}{2}-2}\binom{k_2-2}{\frac{k_1+k_2-k_3+r}{2}-2}\binom{k_3-2}{\frac{k_2+k_3-k_1+r}{2}-2} \times L_{alg}(\frac{k_1+k_2+k_3+r}{2}-2, f_1 \otimes f_2 \otimes f_3).$$

Then we have

$$\langle\langle\langle(2\pi\sqrt{-1})^{-(k_1+k_2+k_3-3r)/2}\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}(E_{3,r}), f_1\rangle, f_2\rangle, f_3\rangle = \langle f_1, f_1\rangle\langle f_2, f_2\rangle\langle f_3, f_3\rangle\mathbf{M}(r; f_1, f_2, f_3),$$

where $\nu_1 = (k_2+k_3-k_1-r)/2, \nu_2 = (k_1+k_3-k_2-r)/2$ *and* $\nu_3 = (k_1+k_2-k_3-r)/2$.

In order to calculate $L_{alg}(\frac{k_1+k_2+k_3+r}{2}-2, f_1\otimes f_2\otimes f_3)$ explicitly enough to write a computer program, we need an expression for $\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}E_{3,r}$ as a linear combination of triple tensors of modular forms of one variable. Now let $d_i = \dim S_{k_i}(\mathrm{SL}_2(\mathbf{Z}))$ for $i = 1, 2, 3$, and f_i $(i = 1, ..., d_1), g_j$ $(j = 1, 2, .., d_2), h_l$ $(l = 1, ..., d_3)$ be a basis of $S_{k_1}(\mathrm{SL}_2(\mathbf{Z})), S_{k_2}(\mathrm{SL}_2(\mathbf{Z}))$, and $S_{k_3}(\mathrm{SL}_2(\mathbf{Z}))$, respectively, consisting of primitive forms. Moreover let $f_0 = E_{1,k_1}, g_0 = E_{1,k_2}$, and $h_0 = E_{1,k_3}$.

Then we obtain:

Theorem 4.8. *Under the above notation,*

$$(2\pi\sqrt{-1})^{-(k_1+k_2+k_3-3r)/2}(\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}E_{3,r})(z_1, z_2, z_3) = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\sum_{l=0}^{d_3} m(r; f_i, g_j, h_l)f_i(z_1)g_j(z_2)h_l(z_3).$$

Here $m(r; f_i, g_j, h_l)$ *is a certain algebraic number such that*
(1) $m(r; f_i, g_j, h_l) = \mathbf{M}(r; f_i, g_j, h_l)$ *for* $i, j, l \ge 1$.
(2) If $k_1 > r$, *then* $m(f_0, g_j, h_l) = 0$, *if* $k_2 > r$ *then* $m(f_i, g_0, h_l) = 0$, *and if* $k_3 > 0$, *then* $m(f_i, g_j, h_0) = 0$ *for any* i, j, l.

In particular, if at least two of ν_1, ν_2, ν_3 *are positive, then*

$$(2\pi\sqrt{-1})^{-(k_1+k_2+k_3-3r)/2}(\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}E_{3,r})(z_1, z_2, z_3) = \sum_{i=1}^{d_1}\sum_{j=1}^{d_2}\sum_{l=1}^{d_3} \mathbf{M}(r; f_i, g_j, h_l)f_i(z_1)g_j(z_2)h_l(z_3).$$

In order to obtain the coefficients of the above linear combination, we need the Fourier coefficients of $E_{3,r}$ and f_i, g_j, h_l. Let

$$E_{n,r}(Z) = \sum_{A\ge 0} c_{n,r}(A)\mathbf{e}(\mathrm{tr}(AZ))$$

and

$$(\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}E_{3,r})(z_1,z_2,z_3) = \sum_{m_1,m_2,m_3=0}^{\infty} c_{3,r,\nu_1,\nu_2,\nu_3}(m_1,m_2,m_3)\mathbf{e}(m_1z_1+m_2z_2+m_3z_3)$$

be the Fourier expansion of $E_{n,r}(Z)$ and $(\mathbb{D}_{r,\nu_1,\nu_2,\nu_3}E_{3,r})(z_1,z_2,z_3)$, respectively. Then it is easily seen that

$$c_{3,r,\nu_1,\nu_2,\nu_3}(m_1,m_2,m_3) = \sum_{r_{12},r_{13},r_{23}\in\mathbf{Z}} c_{3,r}\begin{pmatrix} m_1 & r_{12}/2 & r_{13}/2 \\ r_{12}/2 & m_2 & r_{23}/2 \\ r_{13}/2 & r_{23}/2 & m_3 \end{pmatrix} P_{r,\nu_1,\nu_2,\nu_3}\begin{pmatrix} m_1 & r_{12}/2 & r_{13}/2 \\ r_{12}/2 & m_2 & r_{23}/2 \\ r_{13}/2 & r_{23}/2 & m_3 \end{pmatrix},$$

where P_{r,ν_1,ν_2,ν_3} is the polynomial used to define $\mathbf{D}_{r,\nu_1,\nu_2,\nu_3}$. If we omit zero summands then this is a finite sum. Let

$$f_i(z) = \sum_{m=0}^{\infty} a_i(m)\mathbf{e}(mz),\ g_j(z) = \sum_{m=0}^{\infty} b_j(m)\mathbf{e}(mz),\ h_l(z) = \sum_{m=0}^{\infty} c_l(m)\mathbf{e}(mz).$$

By Theorem 4.8, we obviously have:

Corollary. *For any non-negative integers m_1, m_2, m_3, we have*

$$c_{3,r,\nu_1,\nu_2,\nu_3}(m_1,m_2,m_3) = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\sum_{l=0}^{d_3} m(r;f_i,g_j,h_l)a_i(m_1)b_j(m_2)c_l(m_3).$$

We would like to calculate $m(r,f_i,g_j,h_l)$ as solutions of the above relations, regarded as simultaneous linear equations with known data $c_{3,r,\nu_1,\nu_2,\nu_3}(m_1,m_2,m_3)$ and $a_i(m_1)$, $b_j(m_2)$ and $c_l(m_3)$. Since the latter three numbers are determined by the values when m_i are primes, it would be convenient to write a formula in terms of data from primes. As given in [Go] and easily shown by the usual Hecke theory, we have

$$a_i(p)^n = \sum_{0\le r\le[n/2]} \mathrm{Co}(n,r)p^{r(k_1-1)}a_i(p^{n-2r})$$

where we put

$$\mathrm{Co}(n,r) = \begin{cases} \binom{n}{r} - \binom{n}{r-1} & \text{if } r>0, \\ 1 & \text{if } r=0. \end{cases}$$

The same relations hold for $b_j(m_2)$ and $c_l(m_3)$. For integers i_1, i_2, i_3, we put

$$b(i_1,i_2,i_3,r,k_1,k_2,k_3;q) = \sum_{j_1=0}^{[(i_1-1)/2]}\sum_{j_2=0}^{[(i_2-1)/2]}\sum_{j_3=0}^{[(i_3-1)/2]} \mathrm{Co}(i_1-1,j_1)\mathrm{Co}(i_2-1,j_2)\mathrm{Co}(i_3-1,j_3) \times q^{j_1(k_1-1)}q^{j_2(k_2-1)}q^{j_3(k_3-1)}c_{3,r,k_1,k_2,k_3}(q^{i_1-1-2j_1},q^{i_2-1-2j_2},q^{i_3-1-2j_3}).$$

Then by the above relations, we obviously have

$$b(i_1,i_2,i_3,r,k_1,k_2,k_3;q) = \sum_{i=0}^{d_1}\sum_{j=0}^{d_2}\sum_{l=0}^{d_3} m(r;f_i,g_j,h_l)a_i(q)^{i_1}b_j(q)^{i_2}c_l(q)^{i_3}$$

for any $0 \le i_1 \le d_1$, $0 \le i_2 \le d_2$, $0 \le i_3 \le d_3$. If we regard these relations as simultaneous linear equations for the unknowns $m(r; f_i, g_j, h_l)$, then this can be solved by using a Vandermonde determinant as in [Go]. To write down the formula for the solution, we prepare the following notation. For a prime number q and a positive integer k, let $\Phi_{q,k,0}(X) = \sum_{i=0}^{d+1} A_{k,0,i} X^{d+1-i}$ and $\Phi_{q,k,1}(X) = \sum_{i=0}^{d} A_{k,1,i} X^{d-i}$ be the characteristic polynomials of the Hecke operators $T(q)$ on $M_k(\mathrm{SL}_2(\mathbf{Z}))$ and on $S_k(\mathrm{SL}_2(\mathbf{Z}))$, respectively. We note that $\Phi_{q,k,0}(X) = (X - q^{k-1} - 1)\Phi_{q,k,1}(X)$. Now we put $e_i = 0$ or 1 for $k_i = r$ or $k_i > r$ for each $i = 1, 2, 3$, respectively. Then using Lemma 2.2 from [Go], we have

Theorem 4.9. *Define e_i for $i = 1, 2, 3$ as above. For a fixed i, j, l with $e_1 \le i \le d_1$, $e_2 \le j \le d_2$, $e_3 \le l \le d_3$, assume that $\Phi'_{q,k_1,e_1}(a_i(q))\Phi'_{q,k_2,e_2}(b_j(q))\Phi'_{q,k_3,e_3}(c_l(q)) \neq 0$ for some prime number q. Then we have*

$$m(r, f_i, g_j, h_l) = \left(\Phi'_{q,k_1,e_1}(a_i(q))\Phi'_{q,k_2,e_2}(b_j(q))\Phi'_{q,k_3,e_3}(c_l(q))\right)^{-1}$$

$$\times \sum_{i_1=0}^{d_1-e_1} \sum_{i_2=0}^{d_2-e_2} \sum_{i_3=0}^{d_3-e_3} \sum_{j_1=0}^{d_1-e_1-i_1} \sum_{j_2=0}^{d_2-e_2-i_2} \sum_{j_3=0}^{d_3-e_3-i_3} A_{k_1,e_1,j_1} A_{k_2,e_2,j_2} A_{k_3,e_3,j_3}$$

$$\times a_i(q)^{d_1-e_1-i_1-j_1} b_j(q)^{d_2-e_2-i_2-j_2} c_l(q)^{d_3-e_3-i_3-j_3} b(i_1, i_2, i_3, r, k_1, k_2, k_3; q).$$

In particular if $k_1 - r$, $k_2 - r$ and $k_3 - r$ are all positive, then

$$\mathbf{M}(r, f_i, g_j, h_l) = \left(\Phi'_{q,k_1,1}(a_i(q))\Phi'_{q,k_2,1}(b_j(q))\Phi'_{q,k_3,1}(c_l(q))\right)^{-1}$$

$$\times \sum_{i_1=0}^{d_1-1} \sum_{i_2=0}^{d_2-1} \sum_{i_3=0}^{d_3-1} \sum_{j_1=0}^{d_1-1-i_1} \sum_{j_2=0}^{d_2-1-i_2} \sum_{j_3=0}^{d_3-1-i_3} A_{k_1,1,j_1} A_{k_2,1,j_2} A_{k_3,1,j_3}$$

$$\times a_i(q)^{d_1-1-i_1-j_1} b_j(q)^{d_2-1-i_2-j_2} c_l(q)^{d_3-1-i_3-j_3} b(i_1, i_2, i_3, r, k_1, k_2, k_3; q).$$

Proof. This is easily proved by the same argument as in [Go] Lemma 2.2. □

Remark. The right-hand sides of the formulas in the above Theorem do not depend on the choice of q.

4.3. **Fourier coefficients of Siegel Eisenstein series of degree 3.** Now we give a formula for the Fourier coefficients of $E_{3,k}$. From now on, for a while, we fix a prime number p. We define $\chi_p(a)$ for $a \in \mathbf{Q}_p \backslash \{0\}$ as follows;

$$\chi_p(a) = \begin{cases} +1 & \text{if } \mathbf{Q}_p(\sqrt{a}) = \mathbf{Q}_p \\ -1 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic unramified} \\ 0 & \text{if } \mathbf{Q}_p(\sqrt{a})/\mathbf{Q}_p \text{ is quadratic ramified.} \end{cases}$$

For each $T \in \mathcal{L}_{r,p}^{\times}$ we define the local Siegel series $b_p(T, s)$ by

$$b_p(T, s) = \sum_{R \in S_r(\mathbf{Q}_p)/S_r(\mathbf{Z}_p)} \mathbf{e}_p(\mathrm{tr}(TR)) p^{-\nu_p(\mu_p(R))s},$$

where $\mathbf{e}_p$ is the character of the additive group $\mathbf{Q}_p$ such that $\mathbf{e}_p(a) = \mathbf{e}(a)$ for $a \in \mathbf{Q}$, and $\mu_p(R) = [R\mathbf{Z}_p^r + \mathbf{Z}_p^r : \mathbf{Z}_p^r]$. There exists a unique polynomial $F_p(T, X)$ in X such that

$$b_p(T,s) = F_p(T,p^{-s})(1-p^{-s}) \prod_{i=1}^{[r/2]} (1-p^{2i-2s}) \times \begin{cases} (1-\chi_p((-1)^{r/2}\det(2T))p^{r/2-s})^{-1}, & \text{if } r \text{ is even,} \\ 1, & \text{if } r \text{ is odd.} \end{cases}$$

(cf. Kitaoka [Ki]). We give a formula for $F_p(T, X)$ for $T \in \mathcal{L}_{r,p}^{\times}$ with $r \leq 3$.

Proposition 4.10. (1) *Let $T = (a) \in \mathcal{L}_{1,p}^{\times}$. Then we have*

$$F_p(T,X) = \sum_{i=0}^{\mathrm{ord}_p(a)} (pX)^i.$$

(2) *Let $T = (a_{ij})_{1\leq i,j\leq 2} \in \mathcal{L}_{2,p}^{\times}$. Put $\xi_p(T) = \chi_p(-4\det T)$, and set $e_1 = \mathrm{Min}_{1\leq i,j\leq 2}(\mathrm{ord}_p(2^{1-\delta_{ij}}a_{ij}))$, and $e_2 = [(\mathrm{ord}_p(4\det T) + 1 - \delta_{2p})/2] + \xi_p(T)^2 - 1$. Then we have*

$$F_p(T,X) = \sum_{i=0}^{e_1}(p^2X)^i \sum_{j=0}^{e_2-i}(p^3X^2)^j - \xi_p(T)pX\sum_{i=0}^{e_1}(p^2X)^i \sum_{j=0}^{e_2-i-1}(p^3X^2)^j.$$

Let $T = (t_{ij}) \in \mathcal{L}_{3,p}^{\times}$. Let $m_1 = m_1(T) := \mathrm{Min}_{1\leq i,j\leq 3}(\mathrm{ord}_p(2^{1-\delta_{ij}}t_{ij})), m_2 = m_2(T) := \mathrm{Min}_{1\leq i,j\leq 3}(\mathrm{ord}_p(2^{3-\delta_{ij}}T_{ij}))$, and $m_3 = m_3(T) = \mathrm{ord}(4\det T)$, where T_{ij} denotes the (i,j)-th cofactor of T. For a prime number p, let $\eta_p(T) := (-1)^{\delta_{2p}}h_p(T)$, where $h_p(T)$ denotes the Hasse invariant defined on $S_3(\mathbf{Q}_p)$. Let $p \neq 2$. Then T is $\mathrm{GL}_3(\mathbf{Z}_p)$-equivalent to

$$p^{r_1}u_1 \perp p^{r_2}u_2 \perp p^{r_3}u_3$$

with $r_1 \geq r_2 \geq r_3$ and $u_1, u_2, u_3 \in \mathbf{Z}_p^{\times}$. We note that r_1, r_2, r_3 are uniquely determined by T. Then put $\tilde{\xi}_p(T) := \chi_p(-p^{r_2+r_3}u_2u_3)$ or $(\chi_p(-p^{r_2+r_3}u_2u_3))^2$ according as $r_1 > r_2$ or $r_1 = r_2$. This $\tilde{\xi}_p(T)$ does not depend on the choices of u_1, u_2, u_3. For the case $p = 2$, then T is $\mathrm{GL}_3(\mathbf{Z}_2)$-equivalent to one of the following forms:

$(C1)$ $2^{r_1}u_1 \perp 2^{r_2}K$

with $r_1 \geq r_2, K = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, and $u_1 \in \mathbf{Z}_2^{\times}$,

$(C2)$ $2^{r_1}K \perp 2^{r_3}u_3$

with $r_1 \geq r_3 + 2, K = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}$, and $u_3 \in \mathbf{Z}_2^{\times}$,

$(C3)$ $2^{r_1}u_1 \perp 2^{r_2}u_2 \perp 2^{r_3}u_3$

with $r_1 \geq r_2 \geq r_3$ and $u_1, u_2, u_3 \in \mathbf{Z}_2^{\times}$.

Then define $\tilde{\xi}_2(T)$ by

$$\tilde{\xi}_2(T) := \begin{cases} \chi_2(-\det K) & \text{if } T \text{ is type } C1 \text{ and } r_1 \geq r_3 + 1, \\ \chi_2(-2^{r_2-r_3}u_2u_3) & \text{if } T \text{ is type } C3 \text{ and } r_1 \geq r_2 + 3, \\ \chi_2(-2^{r_2-r_3}u_2u_3)^2 & \text{if } T \text{ is type } C3 \text{ and } r_1 = r_2 + 2, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore put

$$n'_p(T) := \begin{cases} 1 & \text{if } p \neq 2 \text{ and } m_2 \equiv 0 \bmod 2 \\ & \text{or if } p = 2, m_3 - 2m_2 + m_1 = -4, \text{ and } m_2 \equiv 0 \bmod 2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have a formula for $F_p(T, X)$ for a nondegenerate semi-integral matrix T of size not greater than three (cf. [Kat1],[Kat2]).

Proposition 4.11. *For $T = (t_{ij}) \in \mathcal{L}_{3,p}^{\times}$ we have*

$$F_p(T, X) = \sum_{i=0}^{m_1} \left(\sum_{j=0}^{[(m_2-\delta_{2p}-1)/2]-i} (p^5X^2)^j \right) (p^3X)^i$$

$$+ \eta_p(T)(p^2X)^{m_3}(p^3X^2)^{-[m_2/2]+\delta_{2p}} \sum_{i=0}^{m_1} \left(\sum_{j=n'}^{[m_2/2]-\delta_{2p}-i} (p^3X^2)^j \right) (p^2X)^i$$

$$+ (p^5X^2)^{[m_2/2]}(p^2X)^{-m_1} \sum_{i=0}^{m_3-2m_2+m_1} (p^2X)^i \tilde{\xi}_p(T)^{i+2} \sum_{j=0}^{m_1} (p^2X)^i.$$

For an element $B \in \mathcal{L}_{n,p}$ of rank $m \geq 0$, there exists an element $\tilde{B} \in \mathcal{H}_{m,p} \cap GL_m(\mathbf{Q}_p)$ such that $B \sim \tilde{B} \perp O_{n-m}$. We note that $F_p(\tilde{B}, X)$ does not depend on the choice of $\tilde{B}$. Then we put $F_p^{(m)}(B, X) = F_p(\tilde{B}, X)$. For an element $B \in \mathcal{L}_{n\geq 0}$ of rank $m \geq 0$, there exists an element $\tilde{B} \in \mathcal{L}_{m>0}$ such that $B \sim \tilde{B} \perp O_{n-m}$. Then, $\det \tilde{B}$ does not depend on the choice of B. Thus we put $\det^{(m)} B = \det \tilde{B}$. Similarly, we write $\chi_B^{(m)} = \chi_{\tilde{B}}$ if m is even. The following is essentially due to [Bo2] (see also [Kat4].)

Proposition 4.12. *Let T be a semi-positive definite half-integral matrix of degree n and of rank m. Then the T-th Fourier coefficient $c_{n,l}(T)$ of $E_{n,l}$ is given as follows:*

$$c_{n,l}(T) = 2^{[(m+1)/2]} \prod_p F_p^{(m)}(T, p^{l-m-1})$$

$$\times \begin{cases} L(1 + m/2 - l, \chi_T^{(m)}) \prod_{i=m/2+1}^{[n/2]} \zeta(1 + 2i - 2l) & \text{if } m \text{ is even} \\ \prod_{i=(m+1)/2}^{[n/2]} \zeta(1 + 2i - 2l) & \text{if } m \text{ is odd} \end{cases}.$$

Here we make the convention $F_p^{(m)}(T, p^{l-m-1}) = 1$ and $L(1 + m/2 - l, \chi_T^{(m)}) = \zeta(1 - l)$ if $m = 0$.

Remark. The factor "$(-1)^{(m^2-1)/8}$" on page 101, line 9 of [Kat3] should be deleted.

Corollary. *Let $T \in \mathcal{L}_{3\geq 0}$.*

(1) *Assume that* $\mathrm{rank}(T) = 3$. *Then*

$$c_{3,k}(T) = 4 \prod_p F_p(T, p^{k-4}).$$

(2) *Assume that* $\mathrm{rank}(T) = 2$, *and* $T \sim_{\mathbf{Z}} T_2 \perp O_1$ *with* $T_2 \in \mathcal{L}_{2>0}$. *Then*

$$c_{3,k}(T) = 2L(2 - k, \left(\frac{\mathfrak{d}_{T_2}}{\cdot}\right)) \prod_p F_p(T_2, p^{k-3}),$$

where $\left(\frac{\mathfrak{d}_{T_2}}{\cdot}\right)$ *is the Kronecker character associated with the fundamantal discriminant* $\mathfrak{d}_{T_2}$ *(cf. Section 3.)*

(3) *Assume that* $\mathrm{rank}(T) = 1$, *and* $T \sim_{\mathbf{Z}} T_1 \perp O_2$ *with* $T_1 \in \mathcal{L}_{1>0}$. *Then*

$$c_{3,k}(T) = 2\zeta(3-2k)\prod_p F_p(T_1, p^{k-2}).$$

(4) *Assume that* $T = O_3$. *Then*

$$c_{3,k}(T) = \zeta(1-k)\zeta(3-2k).$$

4.4. **Numerical examples.** By Proposition 4.5, Theorem 4.6, Theorem 4.9, and the corollary of Proposition 4.12, we can obtain the value of $\mathbf{M}(l; f_1, f_2, f_3)$ for any primitive forms f_1, f_2 and f_3 of weight k_1, k_2 and k_3, respectively, and any integer l such that $k_1 \geq k_2 \geq k_3, k_2 + k_3 > k_1$ and $(k_1 + k_2 + k_3)/2 - 1 \leq l \leq k_2 + k_3 - 2$. For an even integer $k = 12, 16$ let Δ_k be the unique primitive form in $S_k(\mathrm{SL}_2(\mathbf{Z}))$. First we treat the case $\Delta_{12} \in S_{12}(\mathrm{SL}_2(\mathbf{Z}))$.

Table 1. Triple L-values.

l	$L_{alg}(l, \Delta_{12} \otimes \Delta_{12} \otimes \Delta_{12})$
18	$\dfrac{2^{39} \cdot 3^2}{7^2}$
19	$2^{35} \cdot 3 \cdot 5$
20	$\dfrac{2^{42} \cdot 7}{3^2}$
21	$\dfrac{2^{46} \cdot 3^4}{5^2 \cdot 7}$
22	$\dfrac{2^{48} \cdot 3^9 \cdot 5^3 \cdot 7}{23 \cdot 691^2}$

We also compute that we have

$$L_{alg}(26, \Delta_{12} \otimes \Delta_{16} \otimes \Delta_{16}) = \frac{2^{53} \cdot 3^5 \cdot 5^2 \cdot 7^2}{13 \cdot 691}.$$

We note that

$$\frac{L(l, \Delta_{12} \otimes \Delta_{12} \otimes \Delta_{12})}{\pi^{4l-33}\langle \Delta_{12}, \Delta_{12}\rangle^3} = \frac{L_{alg}(l, \Delta_{12} \otimes \Delta_{12} \otimes \Delta_{12}) 2^{4l-37}}{\Gamma(l)\Gamma(l-11)^3}$$

for $18 \leq l \leq 22$, and that

$$\frac{L(26, \Delta_{12} \otimes \Delta_{16} \otimes \Delta_{16})}{\pi^{63}\langle \Delta_{12}, \Delta_{12}\rangle\langle \Delta_{16}, \Delta_{16}\rangle^2} = \frac{L_{alg}(l, \Delta_{12} \otimes \Delta_{16} \otimes \Delta_{16}) 2^{59}}{\Gamma(26)\Gamma(15)\Gamma(11)^2}.$$

Mizumoto's table and the result on page 208 of [Miz2] give the right-hand sides of the above equalities, and we see that Table 1 above is consistent with Proposition 3.4 in Mizumoto [Miz3]. Mizumoto used nearly holomorphic modular forms to compute the triple values of holomorphic modular forms. On the other hand, here everything is treated within the framework of holomorphic modular forms.

Let $f \in S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ and $g \in S_{k+2}(\mathrm{SL}_2(\mathbf{Z}))$ be primitive forms. We assume that the Hecke polynomial $\Phi_{2,2k,1}$ of $T(2)$ on $S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ (resp. $\Phi_{2,k+2,1}$ of $T(2)$ on $S_{k+2}(\mathrm{SL}_2(\mathbf{Z}))$) is irreducible over $\mathbf{Q}$. Put $K = \mathbf{Q}(f)\mathbf{Q}(g)$, and put

$$I_k = N_{K/\mathbf{Q}}\left(L_{alg}(2k+2, f \otimes g \otimes g)\right).$$

By Theorem 4.9, I_k does not depend on the choice of f and g (cf. [Sat].)

Table 2. Norms of triple L-values.

k	k+2	2k	I_k
10	12	20	$\frac{2^{51} \cdot 5}{11 \cdot 17}$
14	16	28	$\frac{2^{129} \cdot 3^2 \cdot 7^3 \cdot 11 \cdot 107}{5^2 \cdot 13 \cdot 19 \cdot 23 \cdot 131 \cdot 139}$
16	18	32	$\frac{2^{153} \cdot 5^3 \cdot 7 \cdot 11^4 \cdot 13}{3 \cdot 17 \cdot 23 \cdot 29^2 \cdot 67 \cdot 273067}$
18	20	36	$\frac{2^{241} \cdot 3^{12} \cdot 5 \cdot 7^4 \cdot 11^5 \cdot 13^5 \cdot 157}{17^2 \cdot 19 \cdot 23^2 \cdot 29^2 \cdot 31^2 \cdot 1259 \cdot 269461929553}$

For a triple (a, b, c) of integers with $a \geq b \geq c \geq 0$, let $f \in S_{b+c+4}(\mathrm{SL}_2(\mathbf{Z}))$ and $g \in S_{a+4}(\mathrm{SL}_2(\mathbf{Z}))$ be primitive forms. We assume that the Hecke polynomial $\Phi_{2,b+c+4,1}$ of $T(2)$ on $S_{b+c+4}(\mathrm{SL}_2(\mathbf{Z}))$ (resp. $\Phi_{2,a+4,1}$ of $T(2)$ on $S_{a+4}(\mathrm{SL}_2(\mathbf{Z}))$) is irreducible over $\mathbf{Q}$. Put $K = \mathbf{Q}(f)\mathbf{Q}(g)$, and

$$J_{a,b,c} = N_{K/\mathbf{Q}}(L_{alg}(a+b+6, g \otimes g \otimes f)).$$

We note that $J_{a,a,a} = I_{a+2}$. We now compute $J_{a,b,c}$ for all (a, b, c) in Table 3 in [BFG]. They assumed that these are divisible by certain primes by using approximation. Here we give the true values by using the finite sum expression of algebraic numbers in Theorem 4.9 for $r = b - c + 4$, $\nu_1 = \nu_2 = c$, $\nu_3 = a - b$, and prove the above divisibility as a result, though we consider the triple L function instead of $L(s, Sym^2(g) \otimes f)$ and ignore values $L_{alg}(b+3, f)$ because of the ambiguity of the periods of f.

Table 3. More norms of triple L-values.

(a,b,c)	$J_{a,b,c}$	(a,b,c)	$J_{a,b,c}$
$(12,6,2)$	$\frac{2^{49}\cdot 7\cdot 101}{3\cdot 13^2}$	$(14,7,1)$	$\frac{2^{50}\cdot 3^3\cdot 17^2}{5\cdot 7}$
$(16,8,0)$	$\frac{2^{61}\cdot 3^5\cdot 5^2\cdot 7\cdot 43}{17\cdot 691}$	$(16,7,1)$	$\frac{2^{60}\cdot 3\cdot 263}{5\cdot 17}$
$(16,5,3)$	$\frac{2^{53}\cdot 5\cdot 11\cdot 127}{3^2\cdot 17^2}$	$(16,4,4)$	$\frac{2^{55}\cdot 11\cdot 29}{3\cdot 7\cdot 17^2}$
$(12,12,0)$	$\frac{2^{60}\cdot 3^9\cdot 5^3\cdot 7^3\cdot 11\cdot 13\cdot 37}{31\cdot 3617^2}$	$(12,9,3)$	$\frac{2^{50}\cdot 5\cdot 7^2\cdot 137}{13^2}$
$(12,6,6)$	$\frac{2^{51}\cdot 229}{3\cdot 13^3}$	$(14,11,1)$	$\frac{2^{55}\cdot 3^5\cdot 11\cdot 13\cdot 37}{7}$
$(14,7,5)$	$\frac{2^{49}\cdot 3\cdot 71}{13}$	$(12,8,6)$	$\frac{2^{51}\cdot 3\cdot 5\cdot 73}{11\cdot 13^2}$
$(14,14,0)$	$\frac{2^{71}\cdot 3^8\cdot 5^8\cdot 7\cdot 11\cdot 13\cdot 59}{43867^2}$	$(12,8,8)$	$\frac{2^{55}\cdot 7^2\cdot 61}{3\cdot 11\cdot 13^2\cdot 17}$
$(12,12,12)$	$\frac{2^{129}\cdot 3^2\cdot 7^3\cdot 11\cdot 107}{5^2\cdot 13\cdot 19\cdot 23\cdot 131\cdot 139}$	$(14,13,11)$	$\frac{2^{133}\cdot 5^5\cdot 11\cdot 41}{19\cdot 23\cdot 131\cdot 139}$
$(16,16,16)$	$\frac{2^{241}\cdot 3^{12}\cdot 5\cdot 7^4\cdot 11^5\cdot 13^5\cdot 157}{17^2\cdot 19\cdot 23^2\cdot 29^2\cdot 31^2\cdot 1259\cdot 269461929553}$		

Here the cases $(a,b,c)=(12,12,12)$ and $(16,16,16)$ are the same as I_{14} and I_{18} in Table 2, respectively.

Finally we note that the special values of the symmetric cube L function of $f\in S_{2k}(SL_2(\mathbf{Z}))$ can be calculated from the triple L function since $L(s,f\otimes f\otimes f)=L(s-k+1,f)^2L(s,f,Sym(3))$. The sign of the functional equation of $L(s,f,Sym(3))$ is -1 and the central critical value $L(3k/2-1,f,Sym(3))$ vanishes always (see [Sat]). The other critical values can be calculated from those of the triple L functions by Theorem 4.9 since $L(l-k+1,f)\neq 0$ for $3k/2\leq l\leq 2k-2$. We give one example here when f is a primitive form of $S_{16}(\mathrm{SL}_2(\mathbf{Z}))$. Here we define periods $\Omega_\pm$ by $\Lambda(15,f)=2^3\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 3617^{-1}\,\Omega_-$ and $\Lambda(12,f)=5\cdot 7^2\,\Omega_+$. The ratios of $L(l,f)$ for even or odd l with $1\leq l\leq 15$ are calculated by period relations (See [Lang] Chapter VI). The results are given in the following table. By using [R] Theorem 4, we see

$$\Lambda(f,12)\Lambda(f,15)=\frac{2^{16}\cdot 3\cdot 5\cdot 7}{3617}\ \langle f,f\rangle,$$

and this gives the relation

$$\langle f,f\rangle=\frac{3^2\cdot 5\cdot 7^2\cdot 11\cdot 13}{2^{13}}\Omega_+\Omega_-.$$

We put $\Lambda(s,f,Sym(3))=\Gamma_{\mathbf{C}}(s)\Gamma_{\mathbf{C}}(s-15)L(s,f,Sym(3))$ and for the above periods, we define $L_{alg}(l,f,Sym(3))=\Lambda(s,f,Sym(3))/\Omega^3_{(-1)^l}\Omega_{(-1)^{l+1}}$. We also put $L_{alg}(l-15,f)=\Lambda(l-15,f)/\Omega_{(-1)^{l+1}}$. In the following table, l runs over the right half of the critical values.

Table 4. Critical values of the symmetric cube L function for weight 16.

l	$L_{alg}(l-15,f)$	$L_{alg}(l,f\otimes f\otimes f)$	$L_{alg}(l,f,Sym(3))$
23	$2\cdot 5\cdot 7$	0	0
24	11	$\dfrac{2^{51}\cdot 229}{3\cdot 13^3}$	$2^{12}\cdot 3^5\cdot 5^3\cdot 7^6\cdot 11\cdot 229$
25	$2\cdot 7^2$	$\dfrac{2^{51}\cdot 3\cdot 5\cdot 7^2}{13^3}$	$2^{10}\cdot 3^7\cdot 5^4\cdot 7^4\cdot 11^3$
26	$3\cdot 7$	$\dfrac{2^{51}\cdot 3^3\cdot 7^3\cdot 137}{11^2\cdot 13^3}$	$2^{12}\cdot 3^7\cdot 5^3\cdot 7^7\cdot 11\cdot 137$
27	$5\cdot 7^2$	$\dfrac{2^{50}\cdot 5\cdot 7^2\cdot 137}{13^2}$	$2^{11}\cdot 3^6\cdot 5^2\cdot 7^4\cdot 11^3\cdot 13\cdot 137$
28	$2\cdot 3\cdot 11$	$\dfrac{2^{54}\cdot 3^2\cdot 11\cdot 613}{13^2}$	$2^{13}\cdot 3^6\cdot 5^3\cdot 7^6\cdot 11^4\cdot 13\cdot 613$
29	$2^3\cdot 3^2\cdot 13$	$\dfrac{2^{57}\cdot 3^8\cdot 13}{7^2}$	$2^{12}\cdot 3^{10}\cdot 5^3\cdot 7^4\cdot 11^3\cdot 13^2$
30	$\dfrac{2^3\cdot 3^3\cdot 5\cdot 7\cdot 11\cdot 13}{3617}$	$\dfrac{2^{60}\cdot 3^9\cdot 5^3\cdot 7^3\cdot 11\cdot 13\cdot 37}{31\cdot 3617^2}$	$\dfrac{2^{15}\cdot 3^9\cdot 5^4\cdot 7^7\cdot 11^2\cdot 13^2\cdot 37}{31}$

Here the cases $l = 24,\ 27,\ 30$ are the same as $(a,b,c) = (12,6,6)$, $(12,9,3)$ and $(12,12,0)$ in Table 3, respectively.

5. Congruences between Ikeda-Miyawaki lifts and non-Ikeda-Miyawaki lifts

This section presents examples of congruences between Siegel modular forms that confirm that the answer to Problem B' is affirmative in the cases where $n = 1$ and $k \in \{14, 16, 18\}$. When $n = 1$, Problem B' reads as follows:

Problem B' for $n = 1$. *Let $k > 2$ be an even integer. Assume that there exist primitive Hecke eigenforms $f \in S_{2k}(\mathrm{SL}_2(\mathbf{Z}))$ and $g \in S_{k+2}(\mathrm{SL}_2(\mathbf{Z}))$ with nontrivial Ikeda-Miyawaki lift $\mathcal{F}_{f,g}$. Let $\mathfrak{P}$ be a prime ideal of $K(S_{k+2}(\mathrm{Sp}_3(\mathbf{Z})))$ not dividing $(2k-1)!$. Can we show that $\mathfrak{P}$ divides $L_{alg}(2k+2, g\otimes g\otimes f)/L_{alg}(k+1,f)$ if and only if there is a Hecke eigenform $G \in S_{k+2}(\mathrm{Sp}_3(\mathbf{Z}))$, not coming from the Ikeda-Miyawaki lift, such that $G \equiv_{e.v.} \widetilde{\mathcal{F}}_{f,g} \mod \mathfrak{P}$?*

Our examples verify that the answer to the Problem B' is yes for $(n,k) = (1,14), (1,16)$ and $(1,18)$. Let's describe the situation for $(n,k) = (1,14)$ in some detail. For $k = 14$, $S_{28}(\mathrm{SL}_2(\mathbf{Z}))$ is two dimensional with Hecke field $K = \mathbf{Q}\left(\sqrt{18209}\right)$ and $S_{16}(\mathrm{SL}_2(\mathbf{Z}))$ is one dimensional with Hecke field $\mathbf{Q}$. Let f and its conjugate $\bar{f}$ be a basis of primitive eigenforms for $S_{28}(\mathrm{SL}_2(\mathbf{Z}))$ and g for $S_{16}(\mathrm{SL}_2(\mathbf{Z}))$.

$$f(\tau) = q + \lambda q^2 + (151740 + 192\lambda)q^3 + \dots, \text{ where } \lambda = -4140 + 108\sqrt{18209}.$$

$$g(\tau) = q + 216q^2 - 334q^3 + \dots.$$

The space $S_{16}(\mathrm{Sp}_3(\mathbf{Z}))$ is three dimensional with a basis of Hecke eigenforms given by two Ikeda-Miyawaki lifts, f_1 a multiple of $\mathcal{F}_{f,g}$, f_2 a conjugate multiple of $\mathcal{F}_{\bar{f},g}$ and one non-Ikeda-Miyawaki lift f_3. The Fourier coefficients of f_1, f_2 are algebraic numbers in K and f_3 has its Fourier coefficients in $\mathbf{Q}$. The eigenvalues of these three cusp forms under the Hecke operators $T(2)$ and $T_j(4)$ for $j = 0, 1, 2, 3$ are given in the following Table. Note that $T(2)$ has distinct eigenvalues and hence by itself separates the space $S_{16}(\mathrm{Sp}_3(\mathbf{Z}))$ into one dimensional eigenspaces.

Table 5. Eigenvalues $\lambda_{f_i}(T)$ for $f_i \in S_{16}(\mathrm{Sp}_3(\mathbf{Z}))$.

T	f_1	f_3
$T(2)$	$4414176 + 23328\sqrt{18209}$	-115200
$T_0(4)$	$55296(-17632637 + 1160109\sqrt{18209})$	-784548495360
$T_1(4)$	$-4718592(-1757519 + 1503\sqrt{18209})$	-1062815662080
$T_2(4)$	$1207959552(-209 + 9\sqrt{18209})$	-352724189184
$T_3(4)$	68719476736	68719476736

Let $\mathfrak{P} = \langle 107, 33 + \sqrt{18209}\rangle$ be a prime ideal in $\mathfrak{O}_K$ over 107. Conveniently, but not essential to any argument, this ideal is actually principal, being $\mathfrak{P} = \varpi\mathfrak{O}_K$ for

$$\varpi = 4727989352201351990560778 06 + 35037526710991883522259 41\sqrt{18209}.$$

The ideal $\mathfrak{P}$ is relatively prime to $(2k-1)! = 27!$. Furthermore, $\mathfrak{P}$ (as well as $\bar{\mathfrak{P}}$) divides $L_{\mathrm{alg}}(30, f \otimes g \otimes g)$ because, according to Table 2, the norm I_{14} equals a rational fraction with 107 occurring only in the numerator. We can also calculate the ratio of the critical values $L(l, f_1)$ for odd l by solving the usual rational linear relations and the action of Hecke operators $T(2)$ and $T(4)$ on the periods (see [Lang] and [Ma]). Normalizing these as in the explanation between Conjecture B and Problem B', we see that 107 does not divide the norm of $L_{alg}(15, f)$. So $\mathfrak{P}$ should be the congruence prime if the answer to Problem B' is affirmative. One may check directly that $\lambda_{f_1}(T) \equiv \lambda_{f_3}(T) \mod \mathfrak{P}$ for the Hecke operators T in Table 5, and that there is no other prime ideal of K giving such a congruence except over 2, 3 and 5. Thus, the only if part of Problem B' is true in this case $(n, k) = (1, 14)$. It remains to show that $\lambda_{f_1}(T) \equiv \lambda_{f_3}(T) \mod \mathfrak{P}$ for *all* Hecke operators $T \in \mathbf{L}'_n$. We see this from a pullback of an Eisenstein series, which gives an apparently stronger result: the congruence modulo $\mathfrak{P}$ of all the Fourier coefficients of f_1 and f_3.

Let $W_{33} : \mathbf{H}_3 \times \mathbf{H}_3 \to \mathbf{H}_6$ be defined by sending $(Z_1, Z_2) \mapsto Z_1 \perp Z_2$ so that $W_{33}^* : M_k(\mathrm{Sp}_6(\mathbf{Z})) \to M_k(\mathrm{Sp}_3(\mathbf{Z})) \otimes M_k(\mathrm{Sp}_3(\mathbf{Z}))$ is the Witt map. Recall the definition of the Eisenstein series $E_{n,r}$ from section 4.1 and the formula for its Fourier coefficients in Proposition 4.12. According to Tsuyumine [Tsu], $\dim M_{16}(\mathrm{Sp}_3(\mathbf{Z})) = 7$ and so if $f_1, f_2, \ldots, f_7$ is a basis of Hecke eigenforms extending our previous basis of cusp forms, then we have $W_{33}^* E_{6,16} = \sum_{i=1}^{7} a_i f_i \otimes f_i$ for some $a_i \in \mathbf{C}$ as the pullback of the Eisenstein series of degree 6 and weight 16. We have the following Fourier

expansion of $W_{33}^* E_{6,16}$:

$$W_{33}^* E_{6,16}(Z_1, Z_2) = \sum_{T_1, T_2 \in \mathcal{L}_3} c(T_1, T_2)\mathbf{e}(\mathrm{tr}(T_1 Z_1 + T_2 Z_2),$$

where

$$c(T_1, T_2) = \sum_{R \in M_3(\mathbf{Z})} c_{6,16}(\begin{pmatrix} T_1 & R/2 \\ {}^t R/2 & T_2 \end{pmatrix}).$$

The Eisenstein series $E_{6,16}$ has bounded denominators in its Fourier coefficients and the primes which may occur are bounded by twice the weight minus one, in this case 31. This follows from the work of Böcherer, see Satz 5.1 of [Bo2]. The f_i almost share this property because their Fourier coefficients are linear combinations of the Fourier coefficients of this Eisenstein series; however, the coefficients we used in the linear combinations also had the large primes 97, 373, 1721, 3617, 9349, 362903, 657931, 1001259881 in some denominators. To determine the coefficients a_i, and to allow our work to be reproduced, we must specify the f_i: Put

$$T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ T_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ T_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}, \ T_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$T_5 = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}, \ T_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}, \ T_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the cusp forms f_1, f_2, f_3 the Fourier coefficients $c_{f_i}(T)$ at $T = T_5, T_6$ and T_7 are given in the following Table:

Table 6. Fourier Coefficients of f_1 and f_3 for $S_{16}(\mathrm{Sp}_3(\mathbf{Z}))$.

T_i	f_1	f_3
T_5	1	1
T_6	$(3(29+\sqrt{18209}))/26$	16
T_7	$(2(2293+19\sqrt{18209}))/13$	40

We have $c_{f_2}(T) = c_{f_1}(T)^\sigma$, where σ is the nontrivial element of $\mathrm{Gal}(K/\mathbf{Q})$. Furthermore, Fourier coefficients of f_4, f_5 and f_6 at $T = T_i$ $(i = 1, \cdots, 7)$ are given in the following Table:

Table 7. Fourier Coefficients of f_4, f_5, f_6 and f_7 for $M_{16}(\mathrm{Sp}_3(\mathbf{Z}))$.

T_i	f_4	f_5	f_6
T_1	1	0	0
T_2	$\frac{16320}{3617}$	1	0
T_3	$\frac{2157050618257920}{6232699579062017}$	$\frac{124}{539}$	1
T_4	$\frac{139792940605422720}{6232699579062017}$	$\frac{5394}{539}$	$\frac{-103-\sqrt{51349}}{21}$
T_5	$\frac{45951373923840}{6232699579062017}$	$\frac{17979408396}{1828708499233}$	$\frac{35208358-20711\sqrt{51349}}{437762325}$
T_6	$\frac{13415366601584640}{6232699579062017}$	$\frac{3523449472896}{1828708499233}$	$\frac{2053241914-6201578\sqrt{51349}}{437762325}$
T_7	$\frac{752913261742118400}{6232699579062017}$	$\frac{148770817399800}{1828708499233}$	$\frac{-2(1553343619+15439717\sqrt{51349})}{39796575}$

We have $c_{f_7}(T_i) = c_{f_6}(T_i)^\tau$ for $i = 1, \cdots, 7$, where τ is the nontrivial element of $\mathrm{Gal}\left(\mathbf{Q}(\sqrt{51349})/\mathbf{Q}\right)$. With these normalizations, we have

$$b_j = \sum_{i=1}^{7} a_i c_{ij} \ (j = 1, \cdots, 7)$$

where $b_j = c(T_j, T_j)$, and $c_{ij} = c_{f_i}(T_j)$. By a simple calculation we see that we have $\det(c_{ij})_{1\le i,j\le 7} \neq 0$, and we may compute the a_i and obtain the following results: The coefficient a_1 is the quotient of two algebraic integers where ϖ divides the denominator but not the numerator. Actually, if we use the argument of Lemma 5.1 in [Kat3], this fact alone suffices to prove that f_1 has eigenvalues congruent to one of the other f_i, which must be f_3 if we note the $T(2)$ eigenvalues. We continue the description of the coefficients a_i, however, since we can extend the congruence to the Fourier coefficients of f_1 and f_3. The coefficient a_2 is the quotient of two algebraic integers where $\bar{\varpi}$ divides the denominator but not the numerator. The coefficient a_3 is a rational number where $\varpi\bar{\varpi} = 107$ divides the denominator but neither ϖ nor $\bar{\varpi}$ divide the numerator. The coefficients a_4, a_5, a_6, a_7 are the quotients of algebraic integers where neither ϖ nor $\bar{\varpi}$ divide the denominator. All the Fourier coefficients of the f_i and of $W_{33}^* E_{6,16}$ have well defined reductions modulo the ideal $\mathfrak{P}$, so that if we define $A_1 = \varpi a_1$ and $A_3 = \varpi a_3$, we obtain

$$A_1 f_1 \otimes f_1 + A_3 f_3 \otimes f_3 \equiv 0 \mod \mathfrak{P}.$$

As a relation between formal series, this implies that $f_1 \equiv \alpha\, f_3 \mod \mathfrak{P}$, where α is an element of $\mathfrak{O}_K$ such that $\alpha^2 \equiv -\frac{A_3}{A_1} \mod \mathfrak{P}$. Since we have $c_{f_1}(T_5) = c_{f_3}(T_5) = 1$, we can determine that this congruence is nontrivial and that $\alpha \equiv 1 \mod \mathfrak{P}$. Thus we have the apparently stronger result: over $\mathfrak{O}_K/\mathfrak{P}$ the Fourier expansions of f_1

and f_3 agree and, hence, all their Hecke eigenvalues are congruent modulo $\mathfrak{P}$, which is the conclusion of Problem B' in the case $(n,k)=(1,14)$. Of course, conjugation gives $f_2 \equiv_{\text{e.v.}} f_3 \mod \bar{\mathfrak{P}}$ as well.

Twenty years ago, Miyawaki computed the cases of weight 12 and 14 in degree 3. No further examples have been published since that time, so it is appropriate say a few words about how the Fourier coefficients of the basis $f_1,\dots,f_7$ of $M_{16}(\mathrm{Sp}_3(\mathbf{Z}))$ were computed; the details that made the computation tractable will be published elsewhere [PY13]. For each $T_1 \in \mathcal{L}_3$,

$$Z \mapsto \sum_T a\left(T_1 \otimes T; W_{33}^* E_{6,16}\right) e\left(\langle T, Z\rangle\right)$$

defines an element of $M_{16}(\mathrm{Sp}_3(\mathbf{Z}))$ with rational Fourier coefficients, and this is how this space was spanned, relying on Tsuyumine [Tsu] for $\dim M_{16}(\mathrm{Sp}_3(\mathbf{Z}))=7$. The computation of the Fourier coefficients of the Eisenstein series as given by Proposition 4.12 required the recursion for the F_p polynomials given by the second author in [Kat2]. The computation of the F_p polynomials was implemented by modifying a LISP program written by O. King, compare [King]. Beyond spanning this space, enough Fourier coefficients need to be computed to apply $T(2)$. The action of $T(2)$ and the Hecke operators $T_j(4)$ on the Fourier expansions may be found in Miyawaki [Miy]; however, we used the recursive formulae of Breulmann and Küss, see [BK] or [PRY]. We computed some Fourier coefficients for the cusp forms as linear combinations of theta series with pluriharmonic coefficients as a check. The computational point here is that it would not have been feasible for us to compute enough Fourier coefficients of the theta series to apply $T(2)$ directly to them. Control of the primes occurring in the denominators of the Fourier coefficients of the Eisenstein series is crucial for the above reduction to prime ideals. The primes occurring in the denominators of the Fourier coefficients of the Eisenstein series $E_{n,r}$ are bounded by $2r-1$. This bound is a rather weak corollary of the work of Böcherer [Bo2]. Here one must use the von Staudt-Clausen Theorem controlling which primes may occur in the denominator of a Bernoulli number as well as the corresponding results of Leopoldt [Leo] and Carlitz [Car] on generalized Bernoulli numbers.

For $(n,k)=(1,18)$, the space $S_{36}(\mathrm{SL}_2(\mathbf{Z}))$ is three dimensional with a totally real Hecke filed $K=\mathbf{Q}(\lambda)$ for

$$\lambda^3 - 139656\lambda^2 - 59208339456\lambda - 1467625047588864 = 0$$

and a basis of primitive Hecke eigenforms f, f', f'' with

$$f(\tau) = q + \lambda q^2 + \left(478011548 + 2697\lambda - \lambda^2/72\right) q^3 + \dots.$$

For the sake of definiteness, we associate f to the root $\lambda \approx -165109.167$; f' to the root $\lambda \approx -26808.007$; and f'' to $\lambda \approx 331573.175$. The space $S_{20}(\mathrm{SL}_2(\mathbf{Z}))$ is one dimensional, spanned by

$$g(\tau) = q + 456q^2 + 50652q^3 + \dots.$$

In the following, we set $z = 179306496 + 456\lambda$. There are three Ikeda-Miyawaki lifts f_1 a multiple of $\mathcal{F}_{f,g}$, f_2 a corresponding multiple of $\mathcal{F}_{f',g}$ and f_3 a corresponding multiple of $\mathcal{F}_{f'',g}$ in $S_{20}(\mathrm{SL}_2(\mathbf{Z}))$. There are two eigenfunctions f_4, f_5 that appear to be lifts of a second type, also conjectured by Miyawaki [Miy], and one rational non-Ikeda-Miyawaki lift f_6. By Tsuyumine's work, $\dim S_{20}(\mathrm{Sp}_3(\mathbf{Z}))=6$.

Table 8. Eigenvalues $\lambda_{f_i}(T)$ for $1 \leq i \leq 3$ for $f_i \in S_{20}(\mathrm{Sp}_3(\mathbf{Z}))$.

T	f_1
$T(2)$	z
$T_0(4)$	$-125204137833922560 + (20659568640/19)z - (3013/1083)z^2$
$T_1(4)$	$87934816820920320 - (12255232000/19)z + (4096/3249)z^2$
$T_2(4)$	$-11773295632318464 + (1073741824/19)z$
$T_3(4)$	2^{48}

The $T(2)$ eigenvalues of f_1, f_2, f_3 are just the real algebraic integers z given above; the other 2-eigenvalues are given in Tables 8 and 9. We have a congruence $f_1 \equiv_{\text{e.v.}} f_6 \mod \mathfrak{P}$, where $\mathfrak{P} =< 157, 77 + z >$ is a prime ideal in $\mathfrak{O}_K$ above 157, and corresponding congruences for f_2 and f_3 follow from the Galois action. Since I_{18} from Table 2 has a factor of 157 in the numerator only, and we can show that any prime dividing $L_{alg}(19, f_1)$ does not cancel this, this validates Problem B' for $(n, k) = (1, 18)$. Incidentally, by checking the eigenvalues for $T(2)$ and $T_j(4)$, one may check that there are no congruences between f_4 and f_6, at least for prime ideals in $\mathfrak{O}_K$ above rational primes greater than 19.

Table 9. Eigenvalues $\lambda_{f_i}(T)$ for $4 \leq i \leq 6$ for $f_i \in S_{20}(\mathrm{Sp}_3(\mathbf{Z}))$.

T	f_4	f_6
$T(2)$	$-25344(14359 + \sqrt{63737521})$	47162880
$T_0(4)$	$3538944(8268884107 + 1270173\sqrt{63737521})$	1360866622832640
$T_1(4)$	$2113929216(14357863 + 737\sqrt{63737521})$	7864145852497920
$T_2(4)$	$618475290624(1623 + \sqrt{63737521})$	-4048539252424704
$T_3(4)$	2^{48}	2^{48}

For $(n, k) = (1, 16)$, we have $\dim S_{18}(\mathrm{Sp}_3(\mathbf{Z})) = 4$ and $S_{18}(\mathrm{Sp}_3(\mathbf{Z}))$ is spanned by two Ikeda-Miyawaki lifts and two other conjugate eigenforms, apparently of Miyawaki's second conjectural type. By computing the eigenvalues for $T(2)$ and $T_j(4)$, one may check that there are no congruences between any of these forms, except, perhaps, for prime ideals above 2, 3, 5, 7 or 11. In this case $(2k-1)! = 31!$ and from Table 2 all the prime factors of the numerator of I_{16} are less than 31; thus, Problem B' holds true in the case of $(n, k) = (1, 16)$. For $(n, k) = (1, 10)$, we simply comment that $S_{12}(\mathrm{Sp}_3(\mathbf{Z}))$ is spanned by one Ikeda-Miyawaki lift and that, as $(2k-1)! = 19!$ and $I_{10} = 2^{51}5/(11 \cdot 17)$, there are no eligible congruence primes. In the case of $(n, k) = (1, 12)$, there are no Ikeda-Miyawaki lifts in $S_{14}(\mathrm{Sp}_3(\mathbf{Z}))$.

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