On Operational Equations

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Abstract

This paper deals with regular solutions of the operational equations: (A) ⋯ $\boldsymbol{P}(u)=f(x,y,z)$, (B) ⋯ $\boldsymbol{P}(u)=\phi(x,y,z,u)$ where $\boldsymbol{P}$ is supposed as a certain partial differential compound operator of rational integral form. For (A), some important formal solutions are given with some examples and especially for the equation $\boldsymbol{P}(u)=\boldsymbol{D}_t^n(u)$ ($\boldsymbol{P}$ and $\boldsymbol{D}$ are independent) the Initial-value-Problem is studied on an important theorem.

For (B) $\boldsymbol{P}$ is shown its composition by means of function-theoretical calculus and is characterized by the parameters λ, μ, ν and a function $\psi(\xi,\eta,\zeta;\ x,y,z)$, which corresponds to a solution in the sense of one-to-one as far as the solutions are regular.

It has been my attempt to reach after some new points of view on partial differential equations, which might look over them more systematically than the classic methods, and there have been found out two ways on the whole. In this paper, giving each chapter to each of them I will show some important results. The above-mentioned title has been chosen to explain the methods to state, but in this paper I do not mean to expand the field beyond the differential equation's.

Chapter I Formal Calculus

1. Definition of the Operator $\boldsymbol{P}$. In this chapter equations of the form

$$\boldsymbol{P}(u)=f(x,y,z) \qquad \text{(A)}$$

will be principally investigated, while in the next chapter equations of the form

$$\boldsymbol{P}(u)=\phi(x,y,z\,;\,u) \qquad \text{(B)}$$

will be discussed, giving another definition for $\boldsymbol{P}$, which is different because of calculating method but not essentially. In this chapter the operator $\boldsymbol{P}$ is difined by the following six assumptions: (i) If $f(x,y,z)$ is a function which is continuous and has every partial derivative as continuous in a certain domain, $\boldsymbol{P}(f)$, $\boldsymbol{P}^2(f)$, $\boldsymbol{P}^3(f), \cdots$; $\boldsymbol{P}^{k+1}(f)=\boldsymbol{P}\{\boldsymbol{P}^k(f)\}$, are all continuous in the same domain. (In this case f will be called "endlessly operatable" or "operatable for the operator $\boldsymbol{P}$".) (ii) $\boldsymbol{P}(0)=0$. (iii) If h is a function of certain variables independent of x,y,z, $\boldsymbol{P}(hf)=h\boldsymbol{P}(f)$. (iv) If Q is another operator of the same ensemble as $\boldsymbol{P}$, $(\boldsymbol{P}\pm Q)(f)=\boldsymbol{P}(f)\pm$

$Q(f)$. (v) $\boldsymbol{PQ}(f)=\boldsymbol{QP}(f)$; $\boldsymbol{PQ}(f)=\boldsymbol{P}\{\boldsymbol{Q}(f)\}$. (vi) If $g(x,y,z)$ is another operatable function, $\boldsymbol{Q}(f\pm g)=\boldsymbol{P}(f)\pm\boldsymbol{P}(g)$.

The assumption (ii) makes our operator impossible to include any integration's or general inversion's process, for which the operator must be difined under another system of restrictions. And we must pay attention to the ensemble to which our operator belogs, for if we take two operators

$$\boldsymbol{P}=x^2\frac{\partial}{\partial x} \quad \text{and} \quad \boldsymbol{Q}=\frac{\partial}{\partial x}$$

then $\boldsymbol{QP}=\boldsymbol{PQ}+2x\frac{\partial}{\partial x}$ i.e. $\boldsymbol{PQ}\neq\boldsymbol{QP}$. In this case we see the assumption (v) is not satisfied, and $\boldsymbol{P}$ and $\boldsymbol{Q}$ belong to different ensembles. But if two operators are written in the forms

$$\boldsymbol{R}_1=a_0+a_1\boldsymbol{P}+\cdots+a_\lambda\boldsymbol{P}^\lambda$$
$$\boldsymbol{R}_2=b_0+b_1\boldsymbol{P}+\cdots+b_\nu\boldsymbol{P}^\nu$$

where a_k $(k=0,1,2,\cdots,\lambda)$ and $b_k(k=0,1,2,\cdots,\nu)$ are all constant coefficients, they conform to the condition (v), viz.

$$\boldsymbol{R}_1\boldsymbol{R}_2=\boldsymbol{R}_2\boldsymbol{R}_1$$

and both $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ belong to the same ensemble as $\boldsymbol{P}$.

2. **Reiteration Principle.** If $f(x,y,z)$ is an endlessly operatable function for the operator $\boldsymbol{P}$, a function defined in the form

$$u_0(x,y,z)=f-\boldsymbol{P}(f)+\boldsymbol{P}^2(f)-\boldsymbol{P}(f)+\cdots \tag{2, 1}$$

satisfies the formal relation

$$\boldsymbol{P}(u_0)=\boldsymbol{P}(f)-\boldsymbol{P}^2(f)+\boldsymbol{P}^3(f)-\boldsymbol{P}^4(f)+\cdots$$

and therefore makes a solution of the operational equation

$$(1+\boldsymbol{P})(u_0)\equiv u_0+\boldsymbol{P}(u_0)=f(x,y,z).$$

Here we take the operator $(\boldsymbol{P}-1)$ in place of $\boldsymbol{P}$ and find (A) is solved formally by $u(x,y,z)$ which is defined as

$$u(x,y,z)=\sum_{\kappa=0}^{\infty}(1-\boldsymbol{P})^k(f)\ . \tag{2, 2}$$

This solution is, in point of fact, a special one of the following cases.

$g(x,y,z)$ be an arbitrary operatable function, then the function $v(x,y,z)$ befined as follows is found to be a formal solution of (A):

$$v(x,y,z)=g\,(f_0-f_1+f_2-f_3+\cdots) \tag{2, 3}$$

where $f_{k+1}=\boldsymbol{P}(f_kg)-f$ $(k=0,1,2,\cdots)$, $f_0=f$. Here naturally comes another yet similar way solves (A), i.e.……if $h(t)$ is supposed as an arbitrary continuous function of t inependent of x, y, z for which naturally

$$\boldsymbol{P}(h)=h\boldsymbol{P}(1)$$

and is taken in place of g in (2, 3), it is found that

$$w(x,y,z;\, t) = h\sum_{k=0}^{\infty}(1-h\boldsymbol{P})^k(f) \tag{2, 4}$$

solves (A) too: Morever $u = \partial w/\partial t$ may solves $\boldsymbol{P}(u) = 0$, but by some computation this solution v is proved to be but a trivial one which vanishes identically and therefore the series of type (2, 4) can converge and give a solution of (A) only when $w\ (x,\, y,\, z;\, t)$ is shown to be independent of t, while the solution (2, 2) remains to be tested in this respect.

There is another method to be stated about, which is particular yet important. Let us suppose the following conditions: (a) There exists an operator $\boldsymbol{S}$ for which the equation $\boldsymbol{S}(v) = 0$ is solved by $v = h(t)(\equiv h_0 \mathrel{\ne} 0)$, and the equations $\boldsymbol{S}(h_{j+1}) = h_j$ have their solutions h_{j+1} for every $j = 0, 1, 2, \cdots$ (b) t is independent of x, y, z. Let $g(x,y,z)$ be taken as an arbitrary operatable function for $\boldsymbol{P}$ to make a funktion of the definition

$$u\,(x,y,z;\, t) = \sum_{j=0}^{\infty} h_j\, g_j \tag{2, 5}$$

where $g_0 = g$, $\boldsymbol{P}^j(g) = g_j\ \ (j = 1, 2, \cdots)$, then

$$\boldsymbol{P}(u) = \sum_{j=0}^{\infty}\boldsymbol{P}(h_j\, g_j) = \sum_{j=0}^{\infty} h_j\,\boldsymbol{P}(g_j) = \sum_{j=0}^{\infty} h_j g_{j+1}.$$

$$\boldsymbol{S}(u) = \sum_{j=0}^{\infty}\boldsymbol{S}(h_j g_j) = \sum_{j=0}^{\infty} g_j \boldsymbol{S}(h_j) = g_0\boldsymbol{S}(h_0) + \sum_{j=0}^{\infty} g_j h_{j-1} = g\boldsymbol{S}(h) + \sum_{j=0}^{\infty} h_j g_{j+1}$$

accordingly $\qquad (\boldsymbol{P}-\boldsymbol{S})(u) = -g\boldsymbol{S}(h) = 0$

i. e. the equation $\qquad (\boldsymbol{P}-\boldsymbol{S})(u) = 0 \qquad$ (2, 6)

is solved by (2, 5).

But it must be noted that this last course involves the inverse process by the condition (a) viz.

$$h_{j+1} = \boldsymbol{S}^{-1}(h_j)$$

which means a general inversion and belongs not to the ensemble restricted by the primarily shown assumptions. because $\boldsymbol{S}^{-1}(0) = h \mathrel{\ne} 0$ i. e. (ii) is not satisfied.

3. Examples.

Ex. 1) Put $\boldsymbol{P} = a - Q$ and suppose $Q^n\ (f) = 0$ in (2, 4), then we have

$$(1-h\boldsymbol{P})^\nu(f) = \{C_0^\nu A^\nu + C_1^\nu A^{\nu-1}hQ + \cdots + C_{n-1}^\nu A^{\nu-n+1}(hQ)^{n-1}\}\cdot(f) \qquad (\nu = 0, 1, 2, \cdots)$$

where $A = 1 - ah$. Hence we gain

$$w(x,y,z;\, t) = h\left\{\frac{1}{1-A} + \frac{hQ}{\lfloor 1}\frac{\partial}{\partial A}\left(\frac{1}{1-A}\right) + \cdots + \frac{(hQ)^{n-1}}{\lfloor n-1}\frac{\partial^{n-1}}{\partial A^{n-1}}\left(\frac{1}{1-A}\right)\right\}(f)$$

$$= h\left\{\frac{1}{1-A} + \left(\frac{1}{1-A}\right)^2 hQ + \cdots + \left(\frac{1}{1-A}\right)^n (hQ)^{n-1}\right\}(f)$$

$$= \left(\frac{1}{a} + \frac{Q}{a^2} + \cdots + \frac{Q^{n-1}}{a^n}\right)(f)$$

as far as $|A|=|1-ah|<1$.

This solution takes the eliminated form in regard to the agent function $h(t)$, but it is not always the case. For instance, if we take the case $\boldsymbol{P}=d/dx$ and $f=x$ we have

$$w(x,t)=h\sum_{k=0}^{\infty}(x-kh)=xh\infty-h^2\infty^2.$$

So we can have no $h(t)$ to be effective for this case.

Ex. 2) A formal solution of "reiterated type" of the equation

$$c^2\Delta u-u_{tt}=0\,;\ \Delta\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2},\ c=\text{const.}$$

is given by

$$u(x,y,z,t)=\sum_{k=0}^{\infty}\left(\frac{a}{\lfloor 2k}+\frac{bt}{\lfloor 2k+1}\right)t^{2k}c^{2k}\Delta^k g$$

where a, b are arbitrary constant numbers and $g(x,y,z)$ is an arbitrary operatable function for Δ.

Ex. 3) If $v(x,y,z)$ is another solution of (A) from $u(x,y,z)$ defined by (2, 2),

$$\boldsymbol{P}(v)=f \qquad u=\sum_{k=0}^{\infty}(1-\boldsymbol{P})^k\boldsymbol{P}(v)$$

and moreover it can be shown that the relation

$$u=\sum_{k=0}^{\infty}(1-\boldsymbol{P})^k\boldsymbol{P}^2\sum_{j=0}^{\infty}(1-\boldsymbol{P})^j(v)$$

is effective. Therefore if we denote

$$\sum_{k=0}^{\infty}(1-\boldsymbol{P})^k\boldsymbol{P}\equiv\mathfrak{E}_1,\qquad \sum_{k=0}^{\infty}(1-\boldsymbol{P})^k\boldsymbol{P}^2\sum_{j=0}^{\infty}(1-\boldsymbol{P})^j\equiv\mathfrak{E}_2,$$

for an arbitrary solution of (A) we see

$$u=\mathfrak{E}_1(v)=\mathfrak{E}_2(v).$$

Ex. 4) If (2, 2) converges, both of

$$\hat{u}(\xi)=f+\xi\boldsymbol{R}(f)+\xi^2\boldsymbol{R}^2(f)+\cdots$$
$$\psi(\xi)=u+\xi\boldsymbol{R}(u)+\xi^2\boldsymbol{R}^2(u)+\cdots$$
$$;\ f=f(x,y,z),\quad \boldsymbol{R}\equiv(1-\boldsymbol{P}),\quad |\xi|<1$$

converge and $\psi(\xi)=\sum_{k=0}^{\infty}\boldsymbol{R}^k(\hat{u})$. (I cannot afford to demonstrate this theorem throughly in this limited paper, and I wish to leave it off here, expecting another opportunity of publishing it in detail.)

Ex. 5) For the equation

$$\boldsymbol{P}(u)=u_{tt}\,;\quad \boldsymbol{P},\ \text{independent of } t$$

there can be shown two solutions of our type, viz.

$$u = \sum_{k=0}^{\infty} \frac{t^{2k+1}}{\underline{|2k+1}} \boldsymbol{P}^k(g) = \frac{\sinh(t\sqrt{\boldsymbol{P}})}{\sqrt{\boldsymbol{P}}}(g)$$

$$u = \sum_{k=0}^{\infty} \frac{t^{2k}}{\underline{|2k}} \boldsymbol{P}^k(g) = \cosh(t\sqrt{\boldsymbol{P}})(g)$$

where g is an arbitrary operatable function for $\boldsymbol{P}$.

4. On $(\boldsymbol{P}-\boldsymbol{D}_t^n)(u) = 0$ ...... Initial-value-Problem. Setting $\boldsymbol{D}_t^n \equiv \frac{\partial^n}{\partial t^n}$ and 1 in place of $\boldsymbol{S}$ ane $h(t)$ in (2, 5) and (2, 6) respectively, the simultaneous relation:

$$\left.\begin{aligned} \hat{u}_n &= \sum_{k=0}^{\infty} \boldsymbol{P}^k(g)t^{nk}/\underline{|nk} \\ \boldsymbol{P}(\hat{u}_n) &= \boldsymbol{D}_t^n(\hat{u}_n);\ n,\ a \text{ positive integer} \end{aligned}\right\} \tag{4, 1}$$

is brought about.

According to Stirling's formula

$$\underline{|nk} = \sqrt{2\pi nk}\,(nk)^{nk}e^{-nk}$$

hence $\sqrt[k]{\underline{|uk}} = \sqrt[2k]{2\pi gk}\,(nk)^n e^{-n} \sim (nk/e)^n$ as $k \to \infty$

therefosre if $\lim_{k=\infty} \sqrt[k]{|\boldsymbol{P}^k(g)|/\underline{|nk}} \leqq 1$ [1] (4, 2)

$\sqrt[k]{|\boldsymbol{P}^k(g)|} \lesssim (n/e)^n$ for big k (4, 3)

when (4, 1) converges for $|t| < 1$. Then, as it is, $\boldsymbol{P}$ may be called "equivalent to $\boldsymbol{D}_t^n$" as far as related to $\hat{u}_n$ in the above-mentioned sense, and henceforward we may say $g(x,y,z)$ is "of *n-th* order for $\boldsymbol{P}$" in the equivalency when (4, 2) or (4, 3) is satisfied.

$g(x,y,z)$ be of *n-th* order for $\boldsymbol{P}$, then $(\hat{u}_{n+i})(i = 0,1,2,\cdots)$ all converge and

$$\boldsymbol{P}(\hat{u}_{n+i}) = \boldsymbol{D}_t^{n+i}(\hat{u}_{n+i})$$

that is easily verificated. And by some analytic-function-theoreical considerations the following fact is demonstrated: ...... *If $u(x,y,z;t)$ is regular for $|t| < 1$ when x, y, z belong to certain domains respectively and*

$$\boldsymbol{P}(u) = \boldsymbol{D}^t(u);\quad \boldsymbol{P},\ \textit{independent of } t,$$

$u(x,y,z;0)$ is of 1-st order for $\boldsymbol{P}$ and

$$u = \sum_{k=0}^{\infty} t^k [\boldsymbol{P}^k(u)/\underline{|k}]_{t=0}$$

The inverse case of this theorem is true, too, i.e.: ......*If $g(x,y,z)$ is operatable and of 1-st order for $\boldsymbol{P}$, the equation $\boldsymbol{P}(u) = \boldsymbol{D}^t(u)$ is solved uniquely on condition that $u(x,y,z;t)$ has his initial value at $t = 0$ as*

$$u(x,y,z;0) = g(x,y,z)$$

1) Cf. CAUCHY-HADAMARD's theorem, e.g. KNOPP: Funktionentheorie, I (1937), S. 68, or T. TAKENOUCHI: Kansu Ron, I (1937), p. 235.

and is regular for $|t| < 1$.

But as far $n>1$, the solutions of

$$P(u) = D_t^n(u)\,;\quad P, \text{ independent of } t \tag{C}$$

are no longer unique even if $u(x,y,z;0)$ is given. In this case, it is important to take up an analytic regular solution of (C) $u_n(x,y,z;t)$ in the expression:

$$u_n(x,y,z;t) = \sum_{k=0}^{\infty} g_n^{(k)} t/\underline{|k}$$

where $$g_n^{(k)} = [(D_t^k u_n(x,y,z;t)]_{t=0} \tag{4, 4}$$

Let us posit; $$u_{n,i}(x,y,z;t) = \sum_{k=0}^{\infty} g_n^{(nk+i)} t^{nk+i}/\underline{|nk+i}\,, \tag{4, 5}$$

then it can be verficated that

$$P(u_{n,i}) = D_t^n(u_{n,i}) \qquad (i = 0,1,2,\cdots, u-1)\,.$$

Therefore

$$u = \sum_{i=0}^{n-1} \lambda_i u_{n,i} \tag{4, 6}$$

solves (C) on every parameter's combination $(\lambda_0, \lambda_1,\cdots, \lambda_{n-1})$ on condition that

$$u(x,y,z;0) = g_n^{(0)}(x,y,z).$$

Of the solutions (4, 6) the primary one $u = u_n(x,y,z;t)$ makes only a special case for $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = 1$.

Ultimately, in the case $n > 1$, the unicity is really brought about as follows.

THEOREM: *If* $g_n^{(i)}(x,y,z)$ $(i = 0,1,2,\cdots, n-1)$ *are arbitrarily given as of n-th order for* P, (C) *is solved uniquely by*

$$u = \sum_{k=0}^{\infty} g_n^{(k)} t^k/\underline{|k}\;;\quad g_n^{(n+j)} = P(g_n^{(j)})\ (j = 0,1,2,\cdots)$$

on condition that $g_n^{(i)}(x,y,z) = [D_t^i(u)]_{t=0}$ $(i = 0,1,2,\cdots,n-1)$.

Chapter II Function-Theoretical Calculus.

1. Definitions. Denoting by E the set of combinations (x,y,z) of the three variables: $x \varepsilon D_1$, $y \varepsilon D_2$, $z \varepsilon D_3$
where D_1, D_2, D_3 are certain domains in x-, y-, z-planes of complex number respectively, let us call a function $f(x,y,z)$ "regular in E", when $f(x,y,z)$ is regular for $x\varepsilon D_1$, for $y\varepsilon D_2$, and $z\varepsilon D_3$.

In this chapter the operator P is supposed as expressed by a certain rational integral form of finite degree of

$$D_x \equiv \partial/\partial x, \quad D_y \equiv \partial/\partial y, \quad D_z \equiv \partial/\partial z$$

the coefficients of which are all regular functions in E. And the function

$\phi(x,y,z;u)$ is posited as regular for (x,y,z) and u, a complex value belongs to a certain domain D^* of the Gauss-plane. If we denote by G the set of $(x,y,z;u)$ of the above-stated conditions, ϕ may be called "regular in G".

2. Composition of the Operator $\boldsymbol{P}$. To investigate the solutions of the equation $\qquad \boldsymbol{P}(u) = \phi(x,y,z;u) \qquad$ (B)
which are regular in E, it is convenient firstly to take simply closed analytic curves c_1, c_2, c_3 in domains D_1, D_2, D_3 respectively and denote by ϑ_1, ϑ_2, ϑ_3 domains enclosed by c_1, c_2, c_3, because by the Cauchy's theorem[1] the relation:

$$u(x,y,z) = \frac{-1}{8\pi^3 i}\int_{c_1}\frac{d\xi}{\xi-x}\int_{c_2}\frac{d\eta}{\eta-y}\int_{c_3}\frac{d\zeta}{\zeta-z}\cdot u(\xi,\eta,\zeta) \qquad (2,1)$$

is effected when $x\varepsilon\vartheta_1$, $y\varepsilon\vartheta_2$, $z\varepsilon\vartheta_3$.

If (2, 1) is effected

$$\boldsymbol{P}(u) = \frac{-1}{8\pi^3 i}\int_{c_1} d\xi \int_{c_2} d\eta \int_{c_3} d\zeta\cdot u(\xi,\eta,\zeta)\, \boldsymbol{P}\left\{\frac{1}{(\xi-x)(\eta-y)(\zeta-z)}\right\}$$

Then if we set:

$$\boldsymbol{P}\left\{\frac{1}{(\xi-x)(\eta-y)(\zeta-z)}\right\} = \frac{p(\xi,\eta,\zeta;x,y,z)}{(\xi-x)^{\lambda+1}(\eta-y)^{\mu+1}(\zeta-z)^{\nu+1}} \qquad (2,2)$$

we can possibly associate λ, μ, ν as non-negative integers and $p(\xi,\eta,\zeta;x,y,z)$ has non of $(\xi-x)$, $(\eta-y)$, $(\zeta-z)$ as factor……that is to say

$$(p)_{\xi=x} \neq 0, \quad (p)_{\eta=y} \neq 0, \quad (p)_{\zeta=z} \neq 0, \quad \text{if } p \not\equiv 0\,. \qquad (2,3)$$

By the way, if $u(x,y,z)$ is regular in E

$$\Phi(x,y,z) = \phi(x,y,z;u(x,y,z)) \qquad (2,4)$$

must be regular in E too, hence by the Cauchy's theorem

$$\Phi(x,y,z) = \frac{-1}{8\pi^3 i}\int_{c_1} d\xi \int_{c_2} d\eta \int_{c_3} d\zeta \cdot \frac{\Phi(\xi,\eta,\zeta)}{(\xi-x)(\eta-y)(\zeta-z)}\,. \qquad (2,5)$$

Then if we set

$$w(\xi,\eta,\zeta;z,y,z) \equiv \frac{\Phi(\xi,\eta,\zeta)}{(\xi-x)(\eta-y)(\zeta-z)} - \frac{p(\xi,\eta,\zeta;x,y,z)\,u(\xi,\eta,\zeta)}{(\xi-x)^{\lambda+1}(\eta-y)^{\mu+1}(\zeta-z)^{\nu+1}} \qquad (2,6)$$

$$\int_{c_1} d\xi \int_{c_2} d\eta \int_{c_3} d\zeta \cdot w(\xi,\eta,\zeta;x,y,z) = 0 \qquad (3,7)$$

because

$$\boldsymbol{P}(u) = \frac{-1}{8\pi^3 i}\int_{c_1} d\xi \int_{c_2} d\eta \int_{c_3} d\zeta \cdot \frac{p(\xi,\eta,\zeta;x,y,z)\,u(\xi,\eta,\zeta)}{(\xi-x)^{\lambda+1}(\eta-y)^{\mu+1}(\zeta-z)^{\nu+1}}$$

and according to (B) and (2, 4)

$$\Phi(x,y,z) - \boldsymbol{P}(u) = 0. \qquad (2,8)$$

Moreover, if we set

1) Knopp: Funktionentheorie, I, S. 61; also T. Takenouchi: Kansu Ron, I, p. 199.

$$\psi(\xi,\eta,\zeta;x,y,z) \equiv (\xi-x)^{\lambda+1}(\eta-y)^{\mu+1}(\zeta-z)^{\nu+1}w(\xi,\eta,\zeta;x,y,z)$$
$$= (\xi-x)^{\lambda}(\eta-y)^{\mu}(\zeta-z)^{\nu}\cdot\Phi(\xi,\eta,\zeta)-p(\xi,\eta,\zeta;x,y,z)u(\xi,\eta,\zeta) \quad (2,9)$$

$\psi(\xi,\eta,\zeta;x,y,z)$ is regular for (ξ,η,ζ), $(x,y,z)\,\epsilon E$, because $\Phi(\xi,\eta,\zeta)$, $(\xi-x)$, $(\eta-y)$, $(\zeta-z)$, $p(\xi,\eta,\zeta;x,y,z)$ and $u(\xi,\eta,\zeta)$ are all regular in E. Hence by the Cauchy's theorem

$$\left[\frac{\partial^{\lambda+\mu+\nu}\psi}{\partial\xi^{\lambda}\,\partial\eta^{\mu}\,\partial\zeta^{\nu}}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} = \frac{-\underline{|\lambda}\,\underline{|\mu}\,\underline{|\nu}}{8\pi^3 i}\int_{c_1}d\xi\int_{c_2}d\eta\int_{c_3}d\zeta\cdot\frac{\psi(\xi,\eta,\zeta;x,y,z)}{(\xi-x)^{\lambda+1}(\eta-y)^{\mu+1}(\zeta-z)^{\nu+1}}$$
$$= \frac{-\underline{|\lambda}\,\underline{|\mu}\,\underline{|\nu}}{8\pi^3 i}\int_{c_1}d\xi\int_{c_2}d\eta\int_{c_3}d\zeta\cdot w(\xi,\eta,\zeta;x,y,z)$$

Then according to (2, 7)

$$\left[\frac{\partial^{\lambda+\mu+\nu}\psi}{\partial\xi^{\lambda}\,\partial\eta^{\mu}\,\partial\zeta^{\nu}}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} = 0\ , \quad (2,10)$$

while by (2, 9)

$$\left[\frac{\partial^{\lambda+\mu+\nu}\psi}{\partial\xi^{\lambda}\,\partial\eta^{\mu}\,\partial\zeta^{\nu}}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} = \underline{|\lambda}\,\underline{|\mu}\,\underline{|\nu}\cdot\Phi(x,y,z) - \left[\frac{\partial^{\lambda+\mu+\nu}\partial p(\xi,\eta,\eta;x,y,z)u(\xi,\eta,\zeta)}{\partial\xi^{\lambda}\partial\eta^{\mu}\partial\zeta^{\nu}}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}}$$

Therefore

$$\Phi(x,y,z) = \frac{1}{\underline{|\lambda}\,\underline{|\mu}\,\underline{|\nu}}\left[\frac{\partial^{\lambda+\mu+\nu}pu}{\partial\xi^{\lambda}\partial\eta^{\mu}\partial\zeta^{\nu}}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}}\ ,$$

then comparing with (2, 8) we see directly

$$\boldsymbol{P} = \frac{1}{\underline{|\lambda}\,\underline{|\mu}\,\underline{|\nu}}\sum_{i=0}^{\lambda}\sum_{j=0}^{\mu}\sum_{k=0}^{\nu}c_i^{\lambda}c_j^{\mu}c_k^{\nu}\left[\frac{\partial^{\lambda+\mu+\nu-i-j-k}p(\xi,\eta,\zeta;x,y,z)}{\partial\xi^{\lambda-i}\partial\eta^{\mu-j}\partial\zeta^{\nu-k}}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}}\cdot\boldsymbol{D}_x^i\boldsymbol{D}_y^j\boldsymbol{D}_z^k \quad (2,11)$$
$$;\ C_k^{\nu} \equiv \underline{|\nu}\big/\underline{|k}\,\underline{|\nu-k}$$

If the case $\boldsymbol{P} = p(x,y,z;x,y,z)$, where $\boldsymbol{P}$ contains non of $\boldsymbol{D}_x$, $\boldsymbol{D}_y$, $\boldsymbol{D}_z$ really, is omitted as a trivial case, we can go with the supposition that

λ, μ, ν *are three non-negative integers which do not all vanish.* (2, 12)

3. Loosened Relation. Let us denote as

$$\omega(\xi,\eta,\zeta;x,y,z) \equiv (\xi-x)^{\lambda}(\eta-y)^{\mu}(\zeta-z)^{\nu}$$
$$H(\xi,\eta,\zeta;x,y,z) \equiv \omega\phi(u)-p(\xi,\eta,\zeta;x,y,z)\cdot u \quad (3,1)$$
$$\phi(u) \equiv \phi(\xi,\eta,\zeta;u);\quad u = u(\xi,\eta,\zeta)$$

then (2, 9) effects the relation

$$H(\xi,\eta,\zeta;x,y,z;u) = \psi(\xi,\eta,\zeta;x,y,z)$$
$$;\ u = u(\xi,\eta,\zeta)\ , \quad (3,2)$$

when H is to be considered naturally accompanied only to $\boldsymbol{P}$ and ϕ and therefore in our case (3, 2) is equivalent to (B) itself.

$\psi(\xi,\eta,\zeta;x,y,z)$ is considered as a regular function of the restriction (2, 10), therefore if for a function $h(\xi,\eta,\zeta;x,y,z)$ the relation

$$h(\xi,\eta,\zeta;x,y,z)=H^{-1}(\xi,\eta,\zeta;x,y,z;\psi) \tag{3, 3}$$

is found effective, by (2, 10) and (3, 1)

$$\left[\phi(x,y,z;h\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}}-\frac{1}{\lfloor\lambda\ \lfloor\mu\ \lfloor\nu}\left[\frac{\partial^{\lambda+\mu+\beta}ph}{\partial\xi^{\lambda}\,\partial\eta^{\mu}\,\partial\zeta^{\nu}}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}}=0$$

Then according to (2, 11)

$$\left[\boldsymbol{P}\left\{h(x,y,z;\bar{x},\bar{y},\bar{z},)\right\}\right]_{\substack{\bar{x}=x\\ \bar{y}=y\\ \bar{z}=z}}=\phi(x,y,z;h(x,y,z)). \tag{3, 4}$$

This is the "loosened relation".

4. ψ and the Solutions. The function's set of $h(\xi,\eta,\zeta;x,y,z)$ in the loosened relation (3, 4) is given by means of all the regnlar functions as far as their values belong to the domain D^{*}[1], and generally bigger than the set of regular solutions of (B). This is realized when $\boldsymbol{P}$ is given as $\boldsymbol{P}=\partial/\partial x+\partial/\partial y+\partial/\partial z$ and $\phi=u^2$ for example, where $p=(\xi-x)(\eta-y)+(\eta-y)(\zeta-z)+(\zeta-z)(\xi-x)$, $\lambda=\mu=\nu=1$ and $u=1/(A-ax-by-cz)$ makes a solution and for

$$\frac{(\xi-x)(\eta-y)(\zeta-z)}{(A-a\xi-b\eta-c\zeta)^2}-\frac{(\xi-x)(\eta-y)+(\eta-y)(\zeta-z)+(\zeta-z)(\xi-x)}{A-a\xi-b\eta-c\zeta}$$

$\left[\dfrac{\partial^3\psi}{\partial\xi\partial\eta\partial\zeta}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}}$ vanishes if $A-ax-by-cz\neq0$ and $a+b+c=1$, but does not generally vanish if $a+b+c\neq1$, whereas h is given for any value of $a+b+c$.

Moreover the relation (3, 2) which can be written as

$$\omega(\xi,\eta,\zeta;x,y,z)\phi(\xi,\eta,\zeta;u)-p(\xi,\eta,\zeta;x,y,z)u(\xi,\eta,\zeta)=\psi(\xi,\eta,\zeta;x,y,z) \tag{4, 1}$$

allows not two different regular solutions of (B) for the same ψ. This is demonstrated as follows: ……If there exist two different regular solution u_1 and u_2

$$\omega(\xi,\eta,\zeta;x,y,z)\phi(\xi,\eta,\zeta;u_i)-p(\xi,\eta,\zeta;x,y,z)u_i=\psi(\xi,\eta,\zeta;x,y,z)$$
$$;\quad u_i=u_i(\xi,\eta,\zeta)\quad(i=1,2)$$

i.e. $\qquad \omega\phi(u_2)-pu_2=\omega\phi(u_1)-pu_1\,;\quad \phi(u)\equiv\phi(\xi,\eta,\zeta;u)\,,$

then $\qquad \{\phi(u_2)-\phi(u_1)\}/(u_2-u_1)=p/\omega.$

The left hand is dependent only on ξ,η,ζ and independent of x,y,z, therefore the right hand must be naturally independent of x,y,z, too, that can happen only when $\lambda=\mu=\nu=0$, hence

$$\boldsymbol{P}\left\{\frac{1}{(\xi-x)(\eta-y)(\zeta-z)}\right\}=\frac{p}{(\xi-x)(\eta-p)(\zeta-z)}$$

1) See N° 1 of this chapter.

but this has been omitted from our cases as related in (2, 12). q.e.d.

Denoting here by G_ψ the set of possible $\psi(\xi,\eta,\zeta;x,y,z)$ related to the solutions of (B), the above-stated fact can be enounced as follows.

THEOREM: *ψ and u correspond one-to-one, as far as u is a regular solution of* (B) *and* $\psi\varepsilon G_\psi$.

Next, let us take up the case ψ = const. $= c$ viz.

$$\omega\Phi - pu = c \tag{4, 2}$$

$$;\quad u = u(\xi,\eta,\zeta),\ \Phi = \Phi(\xi,\eta,\zeta).$$

Then according to (3, 1)

$$\left.\begin{aligned}\left[\frac{\partial^{i+j+k}\omega\Phi}{\partial x^i\partial y^j\partial z^k}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} &= \Phi(x,y,z)\left[\frac{\partial^{i+j+k}\omega}{\partial x^i\partial y^j\partial z^k}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}}\\ &= 0 \quad \text{if } i \neq \lambda \text{ or } j \neq \mu \text{ or } k \neq \nu\\ &= \Phi(x,y,z,) \quad \text{if } i=\lambda,\ j=\mu,\ k=\nu\ ;\end{aligned}\right\} \tag{4, 3}$$

On the other hand

$$\left[\frac{\partial^{i+j+k}pu}{\partial x^i\partial y^j\partial z^k}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} = u(x,y,z)\left[\frac{\partial^{i+j+k}p}{\partial x^i\partial y^j\partial z^k}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} \tag{4, 4}$$

And by (4, 2) (4, 3) (4,4) togather it is seen that the regular function pu can be written in the form

$$pu = u(\xi,\eta,\zeta)\{B(\xi,\eta,\zeta)+\omega(\xi,\eta,\zeta;x,y,z)A(\xi,\eta,\zeta)\} \tag{4, 5}$$

where A, B are certain regular functions of $(\xi,\eta,\zeta)\varepsilon E$.

Substituting (4, 5) in (4, 2), we get

$$\omega\Phi - uB - u\omega A = c$$

from this it is clear that: $\Phi = uA$ and $-uB = c$. And in this case by (4, 5) and (2, 11)

$$\boldsymbol{P} = A(x,y,z) + \frac{B(x,y,z)\boldsymbol{D}_x^\lambda\boldsymbol{D}_y^\mu\boldsymbol{D}_z^\nu}{\lfloor\lambda\ \lfloor\mu\ \lfloor\nu}. \tag{4, 6}$$

Inversely, in the case $\boldsymbol{P}$ is given by (4, 6) and

$$\phi = A(x,y,z)u \tag{4, 7}$$

for $H(\xi,\eta,\zeta;x,y,z;u)$ defined by (3, 1) it runs

$$\left.\begin{aligned}H &= \{\omega(\xi,\eta,\zeta;x,y,z)A(\xi,\eta,\zeta) - B(x,y,z) - \omega A(x,y,z)\}\,u(\xi,\eta,\zeta)\\ &= \psi(\xi,\eta,\zeta;x,y,z).\end{aligned}\right\} \tag{4, 8}$$

Then apparently the restriction (2, 10) is altered by

$$\left[\frac{\partial^{\lambda+\mu+\nu}u(\xi,\eta,\zeta)B(x,y,z)}{\partial\xi^\lambda\partial\eta^\mu\partial\zeta^\nu}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} = 0$$

i. e.

$$\boldsymbol{D}_x^\lambda\boldsymbol{D}_y^\mu\boldsymbol{D}_z^\nu u = 0 \tag{4, 9}$$

Here we see that (4, 9) is equivalent to the case where $\boldsymbol{P}$ is given by (4, 6) and ϕ by (4, 7). And comparing (4,8)with (4,5) it is seen that

$$A, B = \text{const.}$$

for the exact possibility of the case (4, 2), when $u = \dfrac{-c}{B} = \text{const.}$

5. Sequence of Uniform Increments. Differentiating (3, 2) by u successively with regard to (3,1), we obtain the relations

$$\frac{\partial \phi}{\partial u} - p = \frac{\partial \psi}{\partial u}, \quad \omega \frac{\partial^k \phi}{\partial u^k} = \frac{\partial^k \psi}{\partial u^k} \qquad (5,1)$$

$$; \quad k = 1, 2, 3, \cdots$$

Then, if there is another regular solution $\hat{u}$ of (B) which satiafies the relation:

$$\hat{\psi}(\xi,\eta,\zeta\,;x,y,z) = H(\xi,\eta,\zeta\,;x,y,z\,;\hat{u}) \qquad (5,2)$$

$$; \quad \hat{u} = \hat{u}(\xi,\eta,\zeta)$$

and if the expression:

$$\hat{\psi} = \sum_{k=0}^{\infty} \frac{d^k \psi}{du^k} \frac{(\hat{u}-u)^k}{\lfloor k} \qquad (5,3)$$

is effected for every fixed $(\xi,\eta,\zeta,)$, $(x,y,z)\ \varepsilon E$, it must coincide with the series

$$\hat{\psi} = \psi - (\hat{u}-u)p + \omega \sum_{k=1}^{\infty} \frac{\partial^k \phi}{\partial u^k} \frac{(\hat{u}-u)^k}{\lfloor k} \qquad (5,4)$$

In this case, if we suppose

$$\hat{u} = u + \varepsilon \qquad (\varepsilon = \text{const.}) \qquad (5,5)$$

$$\left[\frac{\partial^{\lambda+\eta+\nu}\hat{\psi}}{\partial\xi^\lambda \partial\eta^\mu \partial\zeta^\nu}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} = \left[\frac{\partial^{\lambda+\eta+\nu}\psi}{\partial\xi^\lambda \partial\eta^\mu \partial\zeta^\nu}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} - \varepsilon\left[\frac{\partial^{\lambda+\mu+\nu}p}{\partial\xi^\lambda \partial\eta^\mu \partial\zeta^\nu}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}},$$

$$-\frac{\lfloor\lambda \lfloor\mu \lfloor\nu}{8\pi^3 i} \sum_{k=1}^{\infty} \frac{\varepsilon^k}{\lfloor k} \int_{c_1}\int_{c_2}\int_{c_3} \frac{\omega \dfrac{\partial^k \phi}{\partial u^k} \cdot d\xi d\eta d\zeta}{(\xi-x)^{\lambda+1}(\eta-y)^{\mu+1}(\zeta-z)^{\nu+1}}$$

The left hand and the first term of the right hand vanish according to (2, 10) and as seen in (2, 11)

$$\left[\frac{\partial^{\lambda+\mu+\nu}p}{\partial\xi^\lambda \partial\eta^\mu \partial\zeta^\nu}\right]_{\substack{\xi=x\\ \eta=y\\ \zeta=z}} = \lfloor\lambda \lfloor\mu \lfloor\nu \cdot A_0 ,$$

where A_0, the coefficient of the zero-th power of $\boldsymbol{D}_x$, $\boldsymbol{D}_y$, $\boldsymbol{D}_z$ in the composition of $\boldsymbol{P}$. Hence

$$\varepsilon A_0 = \frac{-1}{8\pi^3 i} \sum_{k=1}^{\infty} \frac{\varepsilon^k}{\lfloor k} \int_{c_1}\int_{c_2}\int_{c_3} \frac{\omega \dfrac{\partial^k \phi}{\partial d^k} \cdot \xi d\eta d\zeta}{(\xi-x)^{\lambda+1}(\eta-y)^{\mu+1}(\zeta-z)^{\nu+1}}$$

by the definition for ω in (3, 1)

$$= \frac{-1}{8\pi^3 i} \sum_{k=1}^{\infty} \frac{\varepsilon^k}{\lfloor k} \int_{c_1}\int_{c_2}\int_{c_3} \frac{\frac{\partial^k \phi}{\partial u^k} \cdot d\xi d\eta d\zeta}{(\xi-x)(\eta-y)(\zeta-z)}$$

$$= \sum_{k=1}^{\infty} \frac{\varepsilon^k}{\lfloor k} \frac{\partial^k \phi}{\partial u^k} = \phi(u+\varepsilon)-\phi(u)$$

i. e. $$\varepsilon A_0 = \phi(u+\varepsilon)-\phi(u) = \phi(\hat{u})-\phi(u).$$

This relation is exactly effected if $A_0 = 0$ and $\phi(u)$ is a function which has ε as a period, but the periodic case does not essentially refer to the condition (2, 10). Leaving out the periodic case, if (5, 3) and (5, 5) are applicable and $\hat{u}$ changes continuously with the uniform increment

$$A_0 = \left[\frac{\partial \phi}{\partial \hat{u}} \frac{d\hat{u}}{d\varepsilon}\right]_{\varepsilon=0} = \left[\frac{\partial \phi}{\partial u} \frac{d\hat{u}}{d\varepsilon}\right]_{\substack{\hat{u}=u \\ \varepsilon=0}} .$$

In this last case, it is written

$$\hat{u}(x,y,z) = u(x,y,z) - \frac{A_0}{\frac{\partial \phi}{\partial u}} \cdot \varepsilon + <\varepsilon^2> . \tag{5, 6}$$

Then, substituting (5, 5) we see

$$A_0 \Big/ \frac{\partial \phi}{\partial u} = 1 \qquad \text{i. e.} \qquad A_0 = \frac{\partial \phi}{\partial u} .$$

It is notable that the case of (5, 6) is the case where u is a solution of the implicite relation

$$\frac{\partial \phi(x,y,z\,;u)}{\partial u} - A_0(x,y,z) = 0 . \tag{5, 7}$$

Such a result is important when ε is put as an infinitesimal quantity majorating the increment of u and (5, 6) is approximately effected.

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