On Operational Calculus

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Abstract

Let $F(\boldsymbol{P})$ be the abridgement of $F(\boldsymbol{P})(\varphi)$ and $\boldsymbol{P}$, $\boldsymbol{Q}$ be commutative operators, then if $F(y)$ is analytic the mean-value theorem and the Taylor-expansion have their analogies for $F(\boldsymbol{P}+t\boldsymbol{Q})$. If $F(\boldsymbol{P})G(\boldsymbol{Q})$ converges $F(x\boldsymbol{P})G(x\boldsymbol{Q})$ is regular in $|x|<1$. And some other important facts are shown basing on the fact that when some of the operators are being principally investigated the rest can be regarded as parametric elements. In the last paragraph, two generalizations of the Laplace-transformation are shown.

Introduction

In algebra an operation which is defined by the equality

$$\boldsymbol{d}(uv) = \boldsymbol{d}(u)\,v + u\boldsymbol{d}(v)$$

is called derivation[1]. This is a generalization for differentiation of the first order, so it will be very natural if we define the operator by the following three conditions: (i) if $\boldsymbol{P}(uv) = \boldsymbol{P}(u)\,v + u\boldsymbol{P}(v)$, $\boldsymbol{P}$ is called *of the first order*; (ii) if an operator $\boldsymbol{Q}$ is expressed by a rational integral form of the n-th degree of some operators of the first order, $\boldsymbol{Q}$ is called of the n-th order; (iii) for any operator of the first order $\boldsymbol{Q}$, $\boldsymbol{Q}^{-1}(u)$ cannot be discontinuous in the range where its values are bounded.

The present author has been interested to name these operators as *derivers* (of finite order, say) and make a study of them, but he has come across a difficulty, that the process

$$\varphi = \boldsymbol{P}^2(u) \longrightarrow \boldsymbol{P}^{-1}(\varphi) = \boldsymbol{P}(u)$$

cannot be posited as a unique correspondence. So the operators mean in this paper derivers and not inverse derivers.

If a function $\varphi(x_1, \cdots, x_n)$ and the result $\boldsymbol{P}(\varphi)$ are analytic both in a domain (of $(x_1, \cdots, x_n)$), φ is called operatable[2] for $\boldsymbol{P}$ there.

To build up an algebraic system of operators and to study its topological proprieties is of course important, but when we are going to research about unknown practical principles, it seems rather circuitous to mannage

1) The word "*differentiation*" can be used, too, instead of "*derivation*".

2) Cf. Y. Kinokuniya: On Operational Equations, Mem. Muroran Coll. Tch. Vol. I, No. 1 (1950) p. 13.

our eyes only upon the proprieties of the topological structure. So the author attempts in this paper, to show a simple composition of neighbouring, starting from the fact, that when some of the operators are being principally investigated the rest of them can be regarded as parametric elements.

§1. Analogies to Differential Calculi

Suppose the operator $\boldsymbol{P}$ relates to $(\xi) \equiv (\xi_1, \cdots, \xi_n)$ and is independent of x and the function $\varphi(\xi) \equiv \varphi(\xi_1, \cdots, \xi_n)$ independent of x is operatable for $\boldsymbol{P}$, then positing

$$f(x, y) = x^k y^l\ , \qquad u = f(x, \boldsymbol{P})(\varphi) \tag{1,1}$$

and denoting

$$\boldsymbol{D}_x \equiv \partial/\partial x$$

we have

$$\begin{aligned}\boldsymbol{D}_x^i \boldsymbol{P}^j(u) &= \boldsymbol{D}_x^i \boldsymbol{P}^j \{x^k \boldsymbol{P}^l(\varphi)\} = k(k-1)\cdots(k-i+1)\, x^{k-i} \boldsymbol{P}^{l+j}(\varphi) \\ &= \boldsymbol{P}^j \boldsymbol{D}_x^i (x^k)\, \boldsymbol{P}^l(\varphi) = \boldsymbol{P}^j \boldsymbol{D}_x^i (u)\end{aligned}$$

i.e.

$$\boldsymbol{D}_x^i \boldsymbol{P}^j(u) = \boldsymbol{P}^j \boldsymbol{D}_x^i(u) = \boldsymbol{P}^j \frac{\partial^i f}{\partial x^i}(\varphi) = \left\{\frac{\partial^i f}{\partial x^i} \boldsymbol{P}^j\right\}(\varphi)\,. \tag{1,2}$$

This relation gives an important operation by which we may reach a definition of neighbourhood of any operator.

If $F(y)$ is an integral function of finite degree, we can write

$$F(\boldsymbol{P} + t\boldsymbol{R}) = \sum a_{ij}\, t^j \boldsymbol{P}^i \boldsymbol{R}^j\ ,$$

then if the function f is operatable for $\boldsymbol{P}$ which is commutative with $\boldsymbol{R}$ (say, $\boldsymbol{PR} = \boldsymbol{RP}$) and the variable t is independent of f, $\boldsymbol{P}$ and $\boldsymbol{R}$, the function $g(t)$ defined by

$$g(t) = F(\boldsymbol{P} + t\boldsymbol{R})(f) \tag{1,3}$$

is an integral function of the same degree with $F(y)$. Hence, by the mean-value theorem

$$g(t) - g(0) = g'(\theta_1 t)\, t \qquad (0 < \theta_1 < 1)$$

where it holds that

$$\begin{aligned}g'(t) &= \boldsymbol{D}_t F(\boldsymbol{P} + t\boldsymbol{R})(f) = \frac{\partial F(\boldsymbol{P}+t\boldsymbol{R})(f)}{\partial t} \\ &= F'(\boldsymbol{P}+t\boldsymbol{R})\, \boldsymbol{R}(f)\ ,\end{aligned}$$

so we have

$$\{F(\boldsymbol{P}+t\boldsymbol{R}) - F(\boldsymbol{P})\}(f) = F'(\boldsymbol{P}+\theta_1 t\boldsymbol{R})\, t\boldsymbol{R}(f)\ . \tag{1,4}$$

This theorem may be called the *mean-value theorem* too, and if some suitable conditions are given on the convergency, we may put $F(y)$ as a general regular function of y, for which the theorem (1, 4) effects. It will need no further explanation that the TAYLOR-expansion can be applied too, viz.

$$\Big\{F(\boldsymbol{P}+t\boldsymbol{R})-F(\boldsymbol{P})\Big\}(f)=\Big\{\sum_{k=1}^{n-1} F^{(k)}(\boldsymbol{P})\frac{t^k\boldsymbol{R}^k}{\lfloor k}+\frac{F^{(n)}(\boldsymbol{P}+\theta_n t\boldsymbol{R})}{\lfloor n}t^n\boldsymbol{R}^n\Big\}(f) \quad (1,5)$$

$$(0<\theta_n<1)$$

in the range where $F(\boldsymbol{Y})(f)$ is continuous $(D,\ n)$[1] $(\boldsymbol{Y}\equiv\boldsymbol{P}+t\boldsymbol{R})$.

It may be very convenient, when we ommit the explanative symbol (f) and simply use the abridged notation

$$F(\boldsymbol{P}+t\boldsymbol{R})-F(\boldsymbol{P})=F'(\boldsymbol{P}+\theta_1 t\boldsymbol{R})\,t\boldsymbol{R} \quad (1,6)$$

instead of (1, 4). And it will be rather general if we say (1, 6) is effective for a certain range of operation (which means (1, 6) holds for any operatable function f the value of which belongs to a certain range), because we can thus associate with a manifold of functions f.

On putting $n\to\infty$ in (1, 5), we gain an operational series and then, if the range of convergence is found to be $|t|<\rho$, the operator $\rho\boldsymbol{R}$ may be called the radius of convergence though it is of course an abstract one. Thus we have a system of neighbourhood.

§ 2. On the Convergence Problem

In this paragraph the summation theorem of WEIERSTRASS gives important helps for our aspects. Two lemmas can be induced by this theorem and we gain some methods of investigation on the convergence of our operational series.

WEIERSTRASS' THEOREM[2]: *If $f_n(z)$ $(n=1,\ 2,\ \cdots)$ are the functions which are all regular throughout a contour C and its interior and the series*

$$f(z)=\sum_{n=1}^{\infty}f_n(z)$$

converges uniformly on C, $f(z)$ converges uniformly in the generalized sense and regular inside C. Moreover, on termwise differentiation,

$$f^{(k)}(z)=\sum_{n=1}^{\infty}f_n^{(k)}(z)$$

holds with the uniform convergence in the generalized sense inside C.

Lemma 1: *If the series $\sum_{n=0}^{\infty}a_n$ is convergent*

1) This means: the given function has its n-th derivative as continuous in the given range. Then, consequently, $F, F', \cdots, F^{(n-1)}$ are all continuous in the same range, too.

2) Cf. L. BIEBERBACH: Lehrbuch der Funktionentheorie I (1930), S. 155.

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{2, 1}$$

converges and regular in $|x| < 1$.

Lemma 2: *When* $f_k(z) = \sum_{n=0}^{\infty} a_n^{(k)} z^n$ $(k = 0, 1, 2, \cdots)$ *are all convergent and the series* $f(z) = \sum f_k(z)$ *converges uniformly in the generalised sense in* $|z| < r$, $f(z)$ *is regular in the same range and for its expansion*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2, 2}$$

we have

$$a_n = \sum_{k=0}^{\infty} a_n^{(k)} \qquad (n = 0,\ 1,\ 2,\ \cdots).$$

Proof of Lemma 1[1]: Since $\sum a_n$ is convergent, we can find an integer N for any $\varepsilon > 0$ such as

$$|a_n| < \varepsilon \qquad \text{for every } n > N,$$

then if we write

$$f_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

we have

$$|f_{n+i}(x) - f_n(x)| < \varepsilon\rho^{n+1}(1 + \rho + \rho^2 + \cdots) = \varepsilon\rho^{n+1}/(1-\rho)$$

for any $i (= 0, 1, 2, \cdots)$ and any $n > N$, in $|x| < \rho < 1$.

Besides, we can choose an integer k for which

$$\frac{\rho^k}{1-\rho} < 1$$

if we fix $\rho < 1$. So it follows that

$$|f_{n+i}(x) - f_n(x)| < \varepsilon \frac{\rho^n}{\rho^{k-1}} \to 0 \qquad \text{as } n \mapsto \infty$$

independently of i, hence $f_n(x)$ converges uniformly in $|x| < \rho$, i.e. on denoting by $f(x)$ the limiting function we see, $f_n(x)$ converges to $f(x)$ uniformly in the generalized sense in $|x| < 1$. Then, by WEIERSTRASS' theorem, $f(x)$ is regular in $|x| < 1$. Q.E.D.

Proof of Lemma 2: Directly by WEIERSTRASS' theorem $f(z)$ is regular and

$$a_n = \frac{1}{\underline{|n}} f^{(n)}(0) = \frac{1}{\underline{|n}} \sum_{k=0}^{\infty} f_k^{(n)}(0) = \sum_{k=0}^{\infty} a_n^{(k)}. \qquad \text{Q.E.D.}$$

Let us posit two operational series

1) Lemma 1 is well-known as ABEL's first theorem, but the author gives its demonstration to show it can be thought as involved in WEIERSTRASS'.

$$\left.\begin{aligned} F(x\boldsymbol{P}) &= \sum_{k=0}^{\infty} a_k (x\boldsymbol{P})^k , \\ &\text{and} \\ G(y\boldsymbol{Q}) &= \sum_{k=0}^{\infty} b_k (y\boldsymbol{Q})^k \end{aligned}\right\} \tag{2, 3}$$

where a_k, b_k $(k = 0, 1, 2, \cdots)$ are constant coefficients and suppose that: (i) $\boldsymbol{P}$ and $\boldsymbol{Q}$ are commutative (i.e. $\boldsymbol{PQ} = \boldsymbol{QP}$); (ii) $G(\boldsymbol{Q})$ gives an operatable function for $F(\boldsymbol{P})$. Then it may give a critical problem how we shall understand the range of $G(\boldsymbol{Q})$, because, as has been remarked in the previous paragraph, we associate with the manifold of functions φ when we regard $G(\boldsymbol{Q})$ as the value of $G(\boldsymbol{Q})(\varphi)$. But in this paper let us adopt as $G(\boldsymbol{Q})$ the value of a given function at a given (therefore fixed) point $(\xi_1, \cdots, \xi_n)$, whose cordinates are all independent of x and y.

By the supposition (ii) the series

$$F(\boldsymbol{P})\, G(\boldsymbol{Q}) = \sum_{k=0}^{\infty} a_k\, \boldsymbol{P}^k\, (G(\boldsymbol{Q}))$$

is convergent, which implies, by the supposition (i), that

$$F(\boldsymbol{P})\, G(\boldsymbol{Q}) = F(\boldsymbol{P}) \sum_{k=0}^{\infty} b_k\, \boldsymbol{Q}^k = \sum_{k=0}^{\infty} b_k\, \boldsymbol{Q}^k\; F(\boldsymbol{P})$$

is convergent. So, in regard to Lemma 1, we have $F(x\boldsymbol{P})\,G(\boldsymbol{Q})$ to be convergent and regular for x in $|x| < 1$ and $F(\boldsymbol{P})\,G(y\boldsymbol{Q})$ to be convergent for y in $|y| < 1$; and then by the similar considerations on $F(x\boldsymbol{P})\; G(\boldsymbol{Q})$ and $F(\boldsymbol{P})\,G(y\boldsymbol{Q})$ instead of $F(\boldsymbol{P})\,G(\boldsymbol{Q})$ we conclude that $F(x\boldsymbol{P})\; G(y\boldsymbol{Q})$ is regular in $|x| < 1$ and $|y| < 1$, and for fixed y (or x) the convergence of the series (with respect to x (or y)) is uniform in the generalized sense. Hence, $F(x\boldsymbol{P})\,G(x\boldsymbol{Q})$ converges throughout $|x| < 1$ and regular there, because by WEIERSTRASS' theorem

$$\frac{\partial F}{\partial x}\, G + F\, \frac{\partial G}{\partial x}$$

is convergent, so that $\dfrac{dFG}{dx}$ exists and is equal to this value. Especially on setting $F(z) = G(z) = 1/(1-z)$, we shall obtain the result which was shown in p. 16 of the preceeding number of this memoirs as Ex. 4.

Now on denoting

$$f_k(x) = a_k\, \boldsymbol{P}^k \sum_{i=0}^{\infty} b_i\, \boldsymbol{Q}^i\, x^{k+i} \qquad (k = 0, 1, 2, \cdots) ,$$

by VITALI's theorem[1] we find the series

$$f(x) = f_1(x) + f_2(x) + \cdots$$

1) Cf. L. BIEBERBACH: Lehrbuch der Funktionentheorie I, S. 170.

converges uniformly in the generalized sense in $|x| < 1$, because by the foregoing result $f(x) = F(x\boldsymbol{P})G(x\boldsymbol{Q})$ and each of $f_n(x)$ $(n = 0, 1, \cdots)$ is regular in $|x| < 1$. So, according to Lemma 2, on expanding $f(x)$ in the form

$$f(x) = A_0 + A_1 x + A_2 x^2 + \cdots \tag{2, 4}$$

we find that this series converges uniformly in the generalized sense in $|x| < 1$ and

$$\begin{aligned} A_0 &= a_0 b_0 \\ A_1 &= a_1 b_0 \boldsymbol{P} + a_0 b_1 \boldsymbol{Q} \\ A_2 &= a_2 b_0 \boldsymbol{P}^2 + a_1 b_1' \boldsymbol{P}\boldsymbol{Q} + a_0 b_2 \boldsymbol{Q}^2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ A_n &= \sum_{k=0}^{n} a_k b_{n-k} \boldsymbol{P}^k \boldsymbol{Q}^{n-k} \qquad (n = 0, 1, \cdots) \end{aligned} \tag{2, 5}$$

THEOREM: *If $F(\boldsymbol{P})G(\boldsymbol{Q})$ converges, $F(x\boldsymbol{P})G(x\boldsymbol{Q})$ is regular in $|x| < 1$, and its expansion* (2, 4) *converges uniformly in the generalized sense in the same range, given its coefficients by* (2, 5). *It is notable that the radius of convergence of* (2, 4) *is not less than* 1.

§ 3. Parametric Effects of Operators

To solve an operational (especially differential) equations regarding some operators as parametric elements, is not a new idea in our calculus. In the theory of HEAVISIDE's operator, the operator $p \equiv \partial/\partial t$ is supposed as a positive number in the formal process of calculation and is regarded as a parameter when it is combined with another differential operator $\partial/\partial x$, where the solution is first put in the form $u = u(p, x)$. If we associate a special abstract field of operators and put them in some process as parametric elements, there will happen many extensions of calculi of this kind and the investigations thus will find an important field. So, in this paragraph let us have some examples of this idea.

Ex. 1) For the equation

$$\left(\boldsymbol{R}\frac{\partial}{\partial t} - \boldsymbol{P}\frac{\partial}{\partial \xi}\right)\left(u\right) = 0 , \tag{3, 1$_1$}$$

positing $\boldsymbol{R}$, $\boldsymbol{P}$ as two commutative operators in respect to only (x), which are the variables independent of t and ξ, simply we may have a solution by

$$u(x\,;\,\xi, t) = f(t\boldsymbol{P} + \xi\boldsymbol{R})\ (\varphi(x)) \tag{3, 1$_2$}$$

when f is an analytic function for its argument and $\varphi(x)$ is operatable for $\boldsymbol{P}$ and $\boldsymbol{R}$.

Besides, if we set $[u]_{\xi=0} = u(x;\,0,t)$ in the form

$$u(x;\,0,t) = f(t\boldsymbol{P})\,(\varphi(x)) \tag{3, 1$_3$}$$

we have

$$\left[\boldsymbol{R}\frac{\partial u}{\partial t}\right]_{\xi=0} = f'(t\boldsymbol{P})\,\boldsymbol{P}\boldsymbol{R}(\varphi) = \boldsymbol{P}f'(t\boldsymbol{P})\,\boldsymbol{R}(\varphi) = \left[\boldsymbol{P}\frac{\partial u}{\partial \xi}\right]_{\xi=0}.$$

On repeating this process, we have

$$\boldsymbol{R}^{\nu}\left[\frac{\partial^{\nu} u}{\partial t^{\nu}}\right]_{\xi=0} = f^{(\nu)}(t\boldsymbol{P})\,(\boldsymbol{P}\boldsymbol{R})^{\nu}(\varphi) = \boldsymbol{P}^{\nu}\left[\frac{\partial^{\nu} u}{\partial \xi^{\nu}}\right]_{\xi=0}$$

i.e.

$$\boldsymbol{R}^{\nu} f^{(\nu)}(t\boldsymbol{P})\,(\varphi) = \left[\frac{\partial^{\nu} u}{\partial \xi^{\nu}}\right]_{\xi=0} - s_{\nu} \qquad \text{with} \quad \boldsymbol{P}^{\nu}(s_{\nu}) = 0\,.$$

Then,

$$u = \left\{f(t\boldsymbol{P}) + \sum_{i=1}^{\infty}\frac{f^{(i)}(t\boldsymbol{P})}{\underline{|i}}\boldsymbol{R}^{i}\xi^{i}\right\}(\varphi) + S$$

$$;\quad S = \sum_{i=1}^{\infty}\frac{s_i}{\underline{|i}}\xi^{i}\,,$$

which means:

$$u = f(t\boldsymbol{P} + \xi\boldsymbol{R})\,(\varphi) + S\,. \tag{a}$$

And if we take the similar consideration on

$$u = f(x;\,\xi,0)\,(\varphi)$$

we shall gain another solution

$$u = f(t\boldsymbol{P} + \xi\boldsymbol{R})\,(\varphi) + \bar{S} \tag{b}$$

where $\bar{S} = \sum_{i=0}^{\infty}\frac{\bar{s}_i}{\underline{|i}}t^{i}$ with $\boldsymbol{R}^{i}(\bar{s}_i) = 0$ $(i = 1, 2, \cdots)$.

To regard (a) and (b) as the same solution we should have $S = \bar{S}$ to be described in the form

$$S = \bar{S} = \sum \sigma_{ij}\,\xi^{i}\,t^{j} \qquad \text{with} \quad \boldsymbol{P}\left(\frac{\sigma_{ij}}{j+1}\right) = \boldsymbol{R}\left(\frac{\sigma_{i-1,\,j+1}}{i}\right),$$

while the solution $(3, 1)_1$ makes a special case for

$$\sigma_{ij} = 0 \qquad \text{for every} \quad i, j = 0, 1, 2, \cdots\cdots$$

Ex. 2) For the equation

$$\left\{\left(\boldsymbol{P}\frac{\partial}{\partial\xi}\right)^{2} + \left(\boldsymbol{R}\frac{\partial}{\partial t}\right)^{2}\right\}(u) = 0 \tag{3, 2$_1$}$$

we have a solution by

$$u = f(t\boldsymbol{P} + i\xi\boldsymbol{R})\,(\varphi) + g(t\boldsymbol{P} - i\xi\boldsymbol{R})\,(\psi) \qquad (i = \sqrt{-1}) \tag{3, 2$_2$}$$

under suitable conditions on f, g, φ and ψ.

Ex. 3) To solve the equation

$$\left(\frac{\partial^2}{\partial\xi\,\partial x}+\frac{\partial^2}{\partial\eta\,\partial y}+\frac{\partial^2}{\partial\zeta\,\partial z}\right)(u)=0 \tag{3, 3$_1$}$$

ξ, η, ζ, x, y and z being independent, we have a convenience to write $\frac{\partial}{\partial\xi}=\theta_1, \cdots, \frac{\partial}{\partial\zeta}=\theta_2, \frac{\partial}{\partial x}=D_1, \cdots, \frac{\partial}{\partial z}=D_3$ and suppose u in the form

$$u=f(\lambda_1x+\lambda_2y+\lambda_3z)\ ,$$

because for this case, we may set

$$\lambda_1=\kappa_1\varphi(\xi,\eta,\zeta)\,,\quad \lambda_2=\kappa_2\varphi\,,\quad \lambda_3=\kappa_3\varphi \qquad (\kappa_i=\text{const.},\ i=1,2,3)$$

which leads to a solution

$$u=f(\kappa_1x+\kappa_2y+\kappa_3z)\,g\left(\frac{\xi}{\kappa_1}-\frac{\eta}{\kappa_2},\ \frac{\eta}{\kappa_2}-\frac{\zeta}{\kappa_3},\ \frac{\zeta}{\kappa_3}-\frac{\xi}{\kappa_1}\right) \tag{3, 3$_2$}$$

when f, g are arbitrary functions being differentiable for their arguements respectively.

Ex. 4) If the operators $\boldsymbol{P}_k\ (k=1, 2, 3, \cdots, s)$ are expected with the probabilities $w_k(x_1, \cdots, x_\nu)$ respectively, we may have the expected operator

$$\boldsymbol{P}=w_1\boldsymbol{P}_1+\ \cdots\ +w_s\boldsymbol{P}_s\ . \tag{3, 4$_1$}$$

Putting here

$$\boldsymbol{P}(\varphi)=\Phi \tag{3, 4$_2$}$$

and

$$\boldsymbol{D}_k=\frac{\partial}{\partial x_k}$$

we have:

$$\boldsymbol{D}_k\Phi(x_1,\ \cdots,\ x_\nu)=\sum_{i=1}^{s}(\boldsymbol{D}_kw_i)\boldsymbol{P}_i(\varphi)+\sum_{i=1}^{s}w_i\boldsymbol{P}_i(\boldsymbol{D}_k(\varphi))\ ,$$

on condition that $\boldsymbol{D}_i$ are all commutative with $\boldsymbol{P}_k\ (i=1, 2, \cdots, \nu\,;\ k=1, 2, \cdots, s)$; so we may write symbolically

$$d\Phi=(d\boldsymbol{P})(\varphi)+\boldsymbol{P}(d\varphi)\ . \tag{3, 4$_3$}$$

Then, under the restriction that $d\Phi=0$, we may have

$$(d\boldsymbol{P})(\varphi)+\boldsymbol{P}(d\varphi)=0$$

i.e.

$$-\left\{\int d\boldsymbol{P}\right\}(\varphi)=\int\boldsymbol{P}(d\varphi)=\text{constant.} \tag{3, 4$_4$}$$

Ex. 5) We may have a new type of operational equation in

$$u_{\theta_1}\boldsymbol{P}_1(w_1)+u_{\theta_2}\boldsymbol{P}_2(w_2)+\cdots+u_{\theta_n}\boldsymbol{P}_n(w_n)=f(u) \tag{3, 5$_1$}$$

where w_k are operatable functions for $\boldsymbol{P}_k$, $k=1, 2, \cdots, n$.

For this case, it will be very convenient if we can find the difierentiable functions $a_k(\theta_1, \cdots, \theta_n, u)$ such as

$$\boldsymbol{P}_k(w_k) = a_k \ , \qquad (k = 1, 2, \cdots, n) \tag{3,5$_2$}$$

because we have then the well-known equation of the 1-st order

$$u_{\theta_n} a_1 + u_{\theta_n} a_2 + \cdots + u_{\theta_n} a_n = f(u) \, .$$

So, the system $(3, 5)_2$ will be called a subsidiary system of the equation $(3, 5)_1$.

This idea will be applicable in the equations of higher order, for instance

$$\boldsymbol{P}(u) \equiv \sum_{k=0}^{n} u^{(k)}(\theta) \, \boldsymbol{P}_k(w_k) = f(u) \tag{3,6$_1$}$$

and in this case we may have a subsidiary system by

$$\boldsymbol{P}_k(w_k) = a_k(\theta) \ , \qquad (k = 1, 2, \cdots, n) \tag{3,6$_2$}$$

and consequently reach at the equation of one-dimension

$$\sum_{k=0}^{n} \frac{d^k u}{d\vartheta^k} a_k(\theta) = f(u) \ .$$

But, we are not dealing with a new idea in the above course of calculation. In fact, we learn in the theory of primitive solutions[1] we are given with the type of solutions

$$u = u(\theta) \ ; \quad \theta = \theta(x_1, \cdots, x_m)$$

and this case gives more complex types of operations, when we posit as

$$\boldsymbol{P}(u) = \sum_{k=1}^{n} \frac{d^k u}{d\theta^k} \boldsymbol{P}_k(\theta)$$

because $\boldsymbol{P}_k(\theta)$ $(k = 1, 2, \cdots, n)$ cannot be regarded then as linear operations but give ones which have ont yet been introduced by the author in this series of papers. So we see: it should give a natural source of calculations if we put up the metamorphisms of the theory of primitive solutions to find the way of calulation for the complex operations of higher order.

§ 4. On Limiting Metamorphoses

If there is a sequence of functions $f_n (n = 1, 2, \cdots)$, which are operatable for the same operator $\boldsymbol{P}$ and converges to a function $f(x)$, it is very convenient if we can have the relation

$$\lim_{n=\infty} \boldsymbol{P}(f_n) = \boldsymbol{P}(f) \tag{4,1}$$

But it is sometimes not simple, especially for the case when the equation is satisfied but bit by bit in the given domain; we have some well-known types of examples which do not conform to the expectation (4, 1) but are

1) Cf. H. BATEMAN: Partial Differential Equations (1931), p. 95.

very interesting. For instance, by the function defined as

$$g(\xi, x) = x(1-\xi) \qquad (x \leqslant \xi)$$
$$= \xi(1-x) \qquad (x \geqslant \xi)$$

which is called GREEN's function, the equation $y''=0$ is solved except the point $x=\xi$ in the interval $-\infty < x < \infty$ and by the function

$$y = \sum_{S=1}^{n} f_S\, g(x, \xi_S)$$

is solved the same equation bit by bit (say: except $x = \xi_1, \cdots, \xi_n$). But

$$y = \int_0^1 g(x, \xi) f(\xi)\, d\xi$$

does no more solve the equation but gives a solution of the equation $y''(x) + f(x) = 0$. In the following the author will show an analogy of this fact in the operational calculus.

Suppose that: (i) the function $u(\xi, x)$ is operatable for $\boldsymbol{R}$ which is independent of ξ; (ii)

$$\boldsymbol{R}(u) = a_\xi \qquad (0 \leqslant x < \xi),$$

and

$$\boldsymbol{R}(u) = b_\xi \qquad (\xi \leqslant x \leqslant 1), \tag{4, 2}$$

where both a_ξ and b_ξ are constants as far as x does not move beyond ξ, and $f(\xi) = a_\xi - b_\xi$ is a continuous function of ξ in $0 \leqslant \xi \leqslant 1$; (iii) the integral

$$\int_0^1 u(\xi, x)\, d\xi = U(x) \tag{4, 3}$$

exists. Then by (iii) we can choose $\xi_{n,k}$ $(k = 0, 1, \cdots, n)$ such as

$$0 = \xi_{n,0} < \xi_{n,1} < \cdots < \xi_{n,n} = 1$$

and

$$\lim_{n=\infty} (\xi_{n,k} - \xi_{n,k-1}) = 0 \qquad (k = 1, 2, \cdots)$$

so that on positing

$$U_n(x) = \sum_{k=1}^{n} (\xi_{n,k} - \xi_{n,k-1})\, u(\xi_{n,k}, x)$$

we may have

$$\lim_{n=\infty} U_n(x) = U(x)\,. \tag{4, 4}$$

In this case, if $\boldsymbol{D} \equiv \partial/\partial x$ is commutative with $\boldsymbol{R}$, for

$$\boldsymbol{P} \equiv \boldsymbol{D}\boldsymbol{R} \tag{4, 5}$$

it may hold that

$$\boldsymbol{P}(U(x)) = \lim_{\varepsilon=0} \frac{1}{\varepsilon}\Big\{\boldsymbol{R}(U(x+\varepsilon)) - \boldsymbol{R}(U(x))\Big\} = \lim_{\varepsilon=0} \frac{1}{\varepsilon}\boldsymbol{R}\Big\{U(x+\varepsilon) - U(x)\Big\}\,.$$

So, if $\lim \boldsymbol{P}(U_n) = \boldsymbol{P}(U)$, we may have

$$\boldsymbol{P}(U(x)) = \lim_{\varepsilon=0} \frac{1}{\varepsilon} \boldsymbol{R}\left[\lim_{n=\infty} \sum_{k=1}^{n} \left\{u(\xi_{n,k}, x+\varepsilon) - u(\xi_{n,k}, x)\right\}(\xi_k - \xi_{k-1})\right]$$
$$= \lim_{\varepsilon=0} \frac{1}{\varepsilon}\left[\lim_{n=\infty} \sum_{k=1}^{n} \left\{\boldsymbol{R}(u(\xi_{n,k}, x+\varepsilon)) - \boldsymbol{R}(u(\xi_{n,k}, x))\right\}(\xi_k - \xi_{k-1})\right]$$

where we may possibly suppose $\xi_{n,k} \neq x,\ x+\varepsilon$ for $k, n = 1, 2, \cdots$, hence in accordance with the suppositions (4, 2) and (4, 3)

$$\boldsymbol{P}(U(x)) = \lim_{\varepsilon=0} \frac{1}{\varepsilon}\left[\int_0^{x+\varepsilon} a_\xi\, d\xi + \int_{x+\varepsilon}^{1} b_\xi\, d\xi - \int_0^{x} a_\xi\, d\xi - \int_x^1 b_\xi\, d\xi\right]$$
$$= \lim_{\varepsilon=0} \frac{1}{\varepsilon}\left[\int_x^{x+\varepsilon} a_\xi\, d\xi - \int_x^{x+\varepsilon} b_\xi\, d\xi\right]$$
$$= \lim_{\varepsilon=0} \frac{1}{\varepsilon} \int_x^{x+\varepsilon} f(\xi)\, d\xi\ .$$

Since $f(x)$ is continuous we gain

$$\boldsymbol{P}(U(x)) = f(x) \tag{4, 6}$$

while directly from (4, 2) $\boldsymbol{P}\{U_n(x)\} = 0$ for every $x \neq \xi_{n,k}$ $(k = 0, 1, \cdots, n)$.

On the loosened relation

$$[\boldsymbol{P}(h(x, \xi))]_{x=\xi} = \varphi(h(x, x)) \tag{4, 7}$$

where $h(x, \xi)$ is a solution of the system

$$\left.\begin{aligned} &\omega(x, \xi)\, \varphi(h) - p(x, \xi)\, h = \psi(x, \xi)^{1)} \\ &\left(\frac{\partial^{\nu_1+\nu_2+\cdots+\nu_n} \psi(x, \xi)}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_n^{\nu_n}}\right)_{(x)=(\xi)} = 0 \end{aligned}\right\} \tag{4, 8}$$

we may find another similarity of GREEN's case, positing

$$U_j(x) = \sum_{\nu=1}^{N_j} \eta_{j,k}(x)\, h(x, \xi_{j,k})$$

where

$$\begin{aligned} \eta_{j,k} &= 1 \qquad \text{when} \quad (x) \in D_{j,k} \\ &= 0 \qquad \text{when} \quad (x) \,\bar{\in}\, D_{j,k} \qquad (k = 1, 2, \cdots, N_j)\ ; \end{aligned}$$

$(x) \equiv (x_1, \cdots, x_n)$; $\xi_{j,k} \in D_{j,k}$, $D_{j,k} \cap D_{j,k'} = 0$ if $k \neq k'$ and $D = D_{j,1} + \cdots + D_{j,N_j}$; diameter of $D_{j,k} \leqslant 1/j$ $(k = 1, 2, \cdots, N_j)$. In this case, if $h(x, \xi)$ is bounded and regular in D U_j converges to a limiting function $u(x, x)$, but we cannot say generally $\lim_{j=\infty} \boldsymbol{P}(U_j(x)) = \boldsymbol{P}(u(x, x))$.

§ 5. Generalizations of Laplace-transformation

In operational analysis, especially in practical mathematics, the theory

1) For the details cf. Y. KINOKUNIYA: On Operational Equations. Mem. Muroran Coll. Tch. Vol. I, No. 1.

of LAPLACE-transformation[1] has given us many contributions. So, it will be a natural try to research for any analogous course or generalization of this standpoint.

For such a purpose as merely to raise the dimension of the variable's set we may have an answer simply by the following means, when we define the symbol $\mathfrak{L}$ (or $\mathfrak{L}_\nu$) by

$$\mathfrak{L}A(t_1, \cdots, t_\nu) = p_1 \cdots p_\nu \int_0^\infty \cdots \int_0^\infty e^{-p_2 t_1 - \cdots - p_\nu t_\nu} A(t_1, \cdots, t_\nu) dt_1 \cdots dt_\nu \quad (5,1)$$

and call the transformation

$$f(p_1, \cdots, p_\nu) = \mathfrak{L}A(t_1, \cdots, t_\nu) \quad (5,2)$$

the *Laplace-transformation of ν-dimension.* In this case we have the inverse transformation

$$A(t_1, \cdots, t_\nu) = \left(\frac{1}{2\pi i}\right)^\nu \int_{c_1 - i\infty}^{c_1 + i\infty} dp_1 \cdots \int_{c_\nu - i\infty}^{c_\nu + i\infty} \frac{f(p_1, \cdots, p_\nu)}{p_1 \cdots p_\nu} e^{p_1 t_1 + \cdots + p_\nu t_\nu} dp_\nu \quad (5,3)$$

which is to be named *Mellin's formula of ν-dimension.* The unit-function $\boldsymbol{1}$ will be defined as

$$\begin{aligned} \boldsymbol{1} &= \left(\frac{1}{2\pi i}\right)^\nu \int_{-i\infty}^{i\infty} dp_1 \cdots \int_{-i\infty}^{i\infty} \frac{e^{p_1 x_1 + \cdots + p_\nu x_\nu}}{p_1 \cdots p_\nu} dp_\nu \, , \\ &= 1 \quad \text{when } x_1 > 0, \, x_2 > 0, \, \cdots, \, x_\nu > 0 \, , \\ &= 0 \quad \text{when } x_j < 0 \text{ for at least one } j \, (= 1, 2, \cdots, \nu) \, . \end{aligned}$$

And we may have many results which are induced analogously from the theory of L-transformation of 1-dimension.

But, if we persist on our standpoint which has been shown in Paragraph 2, the conditions would differ accordingly. When two commutative operators $\boldsymbol{P}$ and $\boldsymbol{R}$ are independent of the variable t, we have the difinition in the form

$$\mathfrak{L}A(\boldsymbol{P}, t\boldsymbol{R}) = p \int_0^\infty A(\boldsymbol{P}, t\boldsymbol{R}) \, e^{-pt} \, dt \quad (5,3)$$

the inversion of which is

$$\mathfrak{L}^{-1} f(p; \boldsymbol{P}, \boldsymbol{R}) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(p; \boldsymbol{P}, \boldsymbol{R})}{p} e^{pt} \, dp \, . \quad (5,5)$$

To deal with the last generalization, we have the equation

$$(p - \boldsymbol{P}) \, (u) = f(t; \cdots)$$

to be solved by

$$u = \frac{1}{p - \boldsymbol{P}} (f(t)) = e^{t\boldsymbol{P}} \int_0^t e^{-x\boldsymbol{P}} f(x; \cdots) \, dx \quad (5,6)$$

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1) Cf. G. DOETSCH: Theorie und Anwendung der LAPLACE-Transformation)1937).

and as for the unicity problem of the solutions KINO-theorem on the equation $\boldsymbol{P}(u) = p^{\nu}(u)$[1] will come in close relation.

Instead of the form $A(\boldsymbol{P}, t\boldsymbol{R})$ we can adopt different types $A(\boldsymbol{P}_1, \cdots, \boldsymbol{P}_l; t\boldsymbol{R})$, $G(\boldsymbol{P})\,\varphi(t; \cdots)$ and so on, but these will make no difference to our principle.

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1) Y. KINOKUNIYA: On Operational Equations. Mem. Muroran Coll. Tch. Vol. I, No. 1.