On Continuum

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Abstract

This paper will serve to report my results of study, if we may find another peep into the mystic land of aggregates introducing the point-measure theory (or the theory of point-dimensions).

I. Introduction.

In the case of an infinite series:

$$a_0+a_1+a_2+\ldots,$$

writing $$s_n=a_0+a_1+\ldots+a_n\ ,$$

there had been established a criterion for convergency that

$$s_n-s_m \to 0 \text{ as } n,\ m \to \infty\ ;$$

but, when we had built the conception of the set

$$\{P_k\}_{k=1,2,3,\cdots}$$

as the collected whole of these elements, it was not the analogon of the above; it was the conception of enumerability. In the similar way, we promised the set

$$M=M_1+M_2+M_3+\ldots$$

to exist when each of M_k ($k=1, 2, 3, \ldots$) is considered to exist.

As Zenon asserted, we cannot have the conception of the set (0, 1) as the collected whole of the points $0<x<1$, within enumerability, because

$$0+0+0+\ldots=0.$$

But as far as we may not deny the conception of the set (1, 0), there must be promissed a way of collection $\mathfrak{S}$ which asserts that:

$$\mathop{\mathfrak{S}}_{0}^{1} 0=1. \tag{I, 1}$$

He was G. Cantor who had shown the collection of the continuum very exactly for the first time.

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Though this is so, G. Cantor showed another character too, i. e.: One-to-one correspondence is not enough to make the measure of set fixed, whereas (0, 1) can be put in one-to-one correspondence with (a, b), by arbitrarily given finite points a, b. It is a kind of rectifiable propriety of correspondence. To look on this propriety as a clear structive one, there is a convenient representation of measures.

Let us write

$$\mathfrak{S}_{A} \mu(P) = m(A) \tag{I, 2}$$

instead of (I, 1), $m(A)$ being the measure of the set A.[1] $\mu(P)$ be called the *dimension* of the point P. Then, the rectifiable propriety will be sufficiently described by the formulation

$$\mu_1(P_1) = \lambda(P)\mu(P) \tag{I, 3}$$

P_1 being the image of P and $\lambda(P)$ being a non negative number, when we take $\mu_1(P_1)$ as the transformed dimension and write

$$m(A_1) = \mathfrak{S}_{P_1^{-1} \varepsilon A} \mu_1(P_1) = \mathfrak{S}_{A} \lambda(P)\mu(P).$$

In the special case $\lambda(P) = k = \text{const.}$, it will be

$$m(A_1) = k\, m(A).$$

II. Null Measure Assertion.

In this paper, we mean by mapping, a one-to-one correspondence by which

$$P_1 \prec Q_1 \text{ in } I_1, \quad \text{whenever} \quad P \prec Q \text{ in } I,$$

where P_1, Q_1 and I_1 denote the images of P, Q and I respectively. Then, for a mapping described by (I, 3), if we give as

$$\lambda(P) = 1 \quad \text{for each} \quad P \in A\,;$$
$$\lambda(Q) = \varepsilon > 0 \quad \text{for each} \quad Q \in B \equiv I - A,$$

we gain:

$$m(A_1) = m(A) \quad \text{and} \quad m(B_1) = \varepsilon\, m(B),$$

where $I_1 = A_1 + B_1$ and $I = A + B$.

By the way, there will be no difficulty if we give the axiom: *If the power (or, the cardinal) of the set B is not smaller than that of the set A, in every neighbourhood, then*

$$m(A) \leqq m(B).$$

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1) Of course, there arise many questions on our measure, but their discussions shall be left for the future. The measure will be then called *a priori measure.*

Now, let us suppose the cardinal number of the set A is really smaller than that of continuum, so that the power of B is equal to that of continuum in every neighbourhood in I. Then the power of A_1 must be really smaller than that of B_1, which must be equal to the power of continuum in every neighbourhood in I_1. Therefore

$$m(A)=m(A_1)\leqq m(B_1)=\varepsilon\cdot m(B)\leqq\varepsilon\cdot m(I).$$

ε being arbitrary, it must be

$$m(A)=0.$$

In the case A is not bounded, we may take a sequence $A^{(k)}=A\frown(-k,\ k)$ and gain the same result. So we conclude:

THEOREM: *If the power of the set A is really smaller than that of continuum,* $m(A)=0$.

III. On Point Dimension.

The conception of a point may not consist without the formulation

$$P=\lim_{\varepsilon\to 0}(P-\varepsilon,\ P+\varepsilon)$$

when we look on any interval $(a,\ b)$ as the collected whole of its inner points.

On $\mu(P)$, we will associate with the formulation

$$\mu(P)=\mu(Q) \tag{III, 1}$$

for each pair of points P, Q of $(-\infty,\ \infty)$, but by any mapping

$$\mu_1(P_1)=\lambda(P)\,\mu(P),$$

we no more take $\mu_1(P_1)$ to be *uniform*. This coincides with the character that: Though I be put in a one-to-one correspondence to I_1, it cannot always imply $m(I)=m(I_1)$.

Especially it is very important that: To assert sumability of (I, 2) with the character (III, 1), we must understand the dimension $\mu(P)$ as

$$\mu(P)\sim 1/\mathfrak{C} \tag{III, 2}$$

$\mathfrak{C}$ denoting the cardinal number of continuum, though it is not so exact as 1/5. If we do not permit the formulation (III, 2), we will lose our essential idea to take (0, 1) as the collected whole of the continuum. At a fixed point P_1, let $\mu_1(P_1)$ be fixed, then if we extend this dimension by the formulation

$$\mu_1(P_1)=\mu_1(Q_1)\ ^{2)}$$

2) This means, we take $\lambda(P)=$const. everywhere.

it will mean that we give another unit of measure on our linear space. This being so, the formulation

$$\mu_1(P_1) \sim 1/\mathfrak{C}$$

should hold always, too.

Now, on dividing as

$$I \equiv (0,\ 1) = \overset{(k)}{I_1} + \overset{(k)}{I_2} + \ldots + \overset{(k)}{I_{2^k}} + \overset{(k)}{R},$$
$$\overset{(k)}{I_i} \equiv \{(i-1)/2^k,\ i/2^k\},$$
$$\overset{(k)}{R} = \{i/2^k\}_{i=1,2,\ldots,2^k-1},$$

we may have $\mu(P)$ to be not less than $\lim m(\overset{(k)}{I_i})$; then according to (III, 2) $1/\mathfrak{C} \gtrsim \lim m(\overset{(k)}{I_i})$, while $\lim\limits_{k=\infty} m(\overset{(k)}{I_i}) = \lim\limits_{k=\infty} \{1/2^k\} \gtrsim 1/2^{\mathfrak{A}} \sim 1/\mathfrak{C}$.[3] Hence we gain

$$\mu(P) \sim \lim_{k=\infty} \{1/2^k\} \sim 1/2^{\mathfrak{A}} \sim 1/\mathfrak{C}\ ^{4)} \tag{III, 3}$$

so that this formulation will naturally give the definite structure of our dimension.

Through this consideration, it will be remarkable that we restrict us within the uniform system of dimension by means of the binary scale. When a system of dimension is given as every-where uniform, we will say the system is *normal*.

IV. Several Remarks.

We can take $\{1/n^k\}_{k=1,2,\ldots}$ instead of $\{1/2^k\}$, but we cannot take $\{1/n\}_{n=1,2,\ldots}$[5]. Moreover, when applying (III, 3) we conclude as

$$2^k < \mathfrak{A}^k < \mathfrak{C} \quad \rightarrow \quad \mathfrak{A}^{\mathfrak{A}} \sim \mathfrak{C}$$

there can be no impediment, but we can never determine as

$$\mathfrak{C} \lesssim \mathfrak{A}$$

though $2^k < \mathfrak{A}$ $(k=1, 2, \ldots)$ and $2^{\mathfrak{A}} \sim \mathfrak{C}$. These are the circumstances concerned with the representation structure of our points.

In abstract considerations, it will be difficult to look over the destinations of collecting elements, because there can be interpreted no point-dimension such as is of our sense. For such an example, we may take the problem of well-ordered set, on our continuum.

3) $\mathfrak{A}$ denotes the cardinal number of enumerable infinity.

4) When the cardinal number is considered as the inversion of the point dimension, it will be called the *inversion number*.

5) On $\int_0^1 dx = 1$, $dx \sim 1/\mathfrak{A}$.

If we give an order to the set (0, 1), it must be operated on condition that the formulation (III, 1) might be hold. But, it is well-known that, we cannot regard the set (0, 1) as well-ordered in its present structure (say: $x \prec y$, if $x < y$), so that many changes of elements on their ordering will be necessarily needed, whereas our point-dimensions may lose their senses described in (I, 2), (III, 1) and (III, 2) by these changes, because by such an abstract treatment no exact structure shall be maintained on these formulations. Besides, if we persist to believe in (I, 3), $\lambda(P)$ must emerge to be too random to be caught by any means. Such must be a terribly chaotic state to our reason.

On the number-theoretical points of view, we shall find important distinctions from the classical ones introduced by R. Dedekind and G. Cantor, who showed the positions of the real numbers but completely neglected the propriety of the measure of point which cannot be posited as empty. For instance, Cantor posited as:

$$1 \equiv 0.999\ldots,$$

but this is not evident. As 0.9, 0.99,... are all different from 1, if we posit the limiting position of the sequence (0.999...), it will give us the point $1-o$ very naturally, but it must not overlap with the point 1 itself. This is not a new idea, but is to be considered as of Zenon, who asserted that Achilles might not outrun Hector. With Zenon, we may assert that $1-o \not\equiv 1$, whereas in the classical theory of numbers it has been guessed that $1-o \equiv 1 \equiv 1+o$.

The idea "*zero*" as the measure of point will not be a naturally evident one, but it will be interpreted as the inversion of "*infinity*" as the number of points to be summed up to make the measure of the set of them, i. e. the inversion of the cardinal number. To complete the conception $(O, 1)$ as a continuum, we must define the scale of the point P as:

$$((P)) = (P-o) + (P) + (P+o).$$

This is the ground on which we posit the point-dimensions to be flexible in the sense of the transformation (I, 3).

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