Some Mathematical Investigations on Doppler-Effect

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Abstract

Supposing a moving point emitting spherical waves continuously, which reach another moving point, we have a relative relation about which several interesting things can be calculated.

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1. Supposing a moving point E emitting spherical waves continuously which propagate with a constant velocity c, and reach another moving point A at the time τ, while t be the time of emission of E, we have the formula due to Doppler

$$\mu = \frac{d\tau}{dt} = \frac{c-e_\rho}{c-a_\rho} \tag{1}$$

where e_ρ and a_ρ are the projections on $\rho=\overrightarrow{EA}$ of the velocities $\boldsymbol{e}$ of E and $\boldsymbol{a}$ of A respectively.

To prove (1), let us denote the coordinates of A and E as (ξ, η, ζ) and (x, y, z) respectively, and we have

$$c(\tau-t) = \rho; \tag{2}$$

$$\rho^2 = X^2 + Y^2 + Z^2, \quad X = \xi - x, \quad Y = \eta - y, \quad Z = \zeta - z.$$

By differentiation

$$\frac{d\rho}{dt} = c(\mu - 1),$$

and

$$\frac{d\rho}{dt} = \frac{d}{dt}\sqrt{X^2+Y^2+Z^2} = \frac{1}{\rho}\Sigma X\frac{dX}{dt} \tag{2'}$$

$$= \mu\Sigma\frac{X}{\rho}\frac{d\xi}{d\tau} - \Sigma\frac{X}{\rho}\frac{dx}{dt} = \mu a_\rho - e_\rho,$$

hence

$$c(\mu-1) - \mu a_\rho + e_\rho = 0$$

i.e.

$$\mu(c-a_\rho) = c - e_\rho \quad \text{q.e.d..}$$

The relative motion of A to E is completely determined when the (differentiable) curve of its motion and the ratio of the times μ are given. Let the curve be given by

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$$F(X, Y, Z) = 0, \quad G(X, Y, Z) = 0 \tag{3}$$

then we have

$$\frac{\partial F}{\partial X} dX + \frac{\partial F}{\partial Y} dY + \frac{\partial F}{\partial Z} dZ = 0$$
$$\frac{\partial G}{\partial X} dX + \frac{\partial G}{\partial Y} dY + \frac{\partial G}{\partial Z} dZ = 0$$

and on the other hand, according to (2′),

$$\frac{X}{\rho} dX + \frac{Y}{\rho} dY + \frac{Z}{\rho} dZ = c(\mu - 1)\, dt$$

so that we may solve dX, dY, dZ as linear forms of dt. As for the case

$$\begin{vmatrix} \frac{\partial F}{\partial X} & \frac{\partial F}{\partial Y} & \frac{\partial F}{\partial Z} \\ \frac{\partial G}{\partial X} & \frac{\partial G}{\partial Y} & \frac{\partial G}{\partial Z} \\ \frac{X}{\rho} & \frac{Y}{\rho} & \frac{Z}{\rho} \end{vmatrix} = 0,$$

if (3) gives the curve as the section of the two surfaces at least one of

$$\frac{\partial(F, G)}{\partial(X, Y)}, \quad \frac{\partial(F, G)}{\partial(Y, Z)}, \quad \frac{\partial(F, G)}{\partial(Z, X)}$$

does not vanish. So it must be

$$\mu = 1$$

because, then $X/\rho = \lambda \cdot (\partial F/\partial X) + \nu \cdot (\partial G/\partial X)$, $Y/\rho = \lambda \cdot (\partial F/\partial Y) + \nu \cdot (\partial G/\partial Y)$, $Z/\rho = \lambda \cdot (\partial F/\partial Z) + \nu \cdot (\partial G/\partial Z)$ are necessarily demanded.

2. The projection on $\overrightarrow{\rho}$ of the relative velocity $\mathfrak{v}$ of **A** to **E** $\mathfrak{v}_\rho(\tau)$ satisfies the relation

$$v_\rho\, d\tau = a_\rho\, d\tau - e_\rho\, dt$$

while a_ρ can be given in the form

$$a_\rho = \frac{X}{\rho} \cdot \frac{d\xi}{d\tau} + \frac{Y}{\rho} \cdot \frac{d\eta}{d\tau} + \frac{Z}{\rho} \cdot \frac{d\zeta}{d\tau}$$
$$= \Sigma \frac{X}{\rho} \cdot \frac{d}{d\tau}(X + x) = \Sigma \frac{X}{\rho}\left(\frac{dX}{d\tau} + \frac{dx}{dt} \cdot \frac{dt}{d\tau}\right)$$
$$= \frac{d\rho}{d\tau} + e_\rho \cdot \frac{dt}{d\tau}$$

i.e.
$$\frac{d\rho}{d\tau} = a_\rho - e_\rho \cdot \frac{dt}{d\tau}.$$

Hence we have
$$v_\rho\, d\tau = d\rho$$

i.e.
$$\int^\tau v_\rho\, d\tau = \rho + \text{const.}.$$

This relation depends on v_ρ alone so that any orthogonal component of $\boldsymbol{v}$ to v_ρ may be left as free from restriction.

But, if we take the system (ρ, θ) (θ: the integrating angle of $\vec{\rho}$) in place of (X, Y, Z) to represent the relative position of **A** to **E** and define the quantity

$$L(\tau) = \int^{\tau} v_\theta \, d\tau$$

v_θ being the θ-component of $\boldsymbol{v}$, then we shall find an important significance in this quantity. The element of the real length of the relative motion of **A** to **E** ds is given then, by

$$ds = \sqrt{(d\rho)^2 + \rho^2 (d\theta)^2},$$

$$\frac{dL}{ds} = \frac{\rho d\theta}{\sqrt{(d\rho)^2 + \rho^2 (d\theta)^2}}.$$

Since $d\rho = (d\tau - dt)\, c = (\mu - 1)\, cdt$, if we write

$$\frac{dL}{ds} = \frac{\theta_t}{\sqrt{\left(\frac{\mu - 1}{\tau - t}\right)^2 + \theta_t^2}}$$

$$= \frac{\theta_t}{\sqrt{\theta_t^2 + \left(\frac{dlg(\tau - t)}{dt}\right)^2}} \equiv cos\, \omega$$

we reach
$$lg(\tau - t) = \int^t \theta_t \cdot tg\omega \cdot dt. \tag{4}$$

3. In the above, we have calculated for the case $c =$ const., but if we posit the expression

$$(\tau - t)\, C(t) = \rho(t)$$

$C(t)$ being the mean value of c in the interval (t, τ), in place of (2), we have the formula

$$\mu = \frac{d\tau}{dt} = \frac{(t - \tau) C' + C - e_\rho}{C - a_\rho};$$

$$C' = dC/dt$$

in place of (1). In this case, we may apply

$$lg(\tau - t) + lgC = \int^t \theta_t \cdot tg\omega \cdot dt$$

in place of (4).

If we posit the function

$$\varphi(t) = lg\, \rho$$

we can take an interesting evaluation. On differentiating, we gain

$$\varphi'(t) = \frac{1-\mu}{t-\tau} + \frac{C'}{C} = Ce^{-\varphi(t)}(\mu-1) + \frac{C'}{C}.$$

So, if $Ce^{-\varphi(t)}$ is bounded (for instance: $\varphi(t) \geqslant 0$) and $1-\mu \sim 0$

$$\varphi'(t) \sim C'/C$$

can be used.

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