Linear Topologies on Semi-ordered Linear Spaces and their Regularity

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Abstract

It is desirable that linear topologies on semi-orderd linear spaces have continuity-properties of join and meet.

In this paper, we have introduced a linear topology on semi-orderd linear spaces, so that we have investigated some properties, especially those of regularity of the space.

1. Introduction.

By a *semi-ordered linear space* we mean a vector lattice in the sense of Birkhoff[1].

Let **R** be a linear space.

A topology on **R** for which the additive operation + and the scalar multiplication (i. e. ξx for $x \in \mathbf{R}$, ξ is real number) are continuous is called a *linear topology*.

Prof. H. Nakano has introduced a linear topology on universally semi-ordered linear space[2] as follows.

A set of positive elements V is said to be a *positive vicinity*, if

1) *for any* $a \geqq 0$ *we can find* $\varepsilon > 0$ *such that* $\varepsilon a \in V$.
2) $0 \leqq b \leqq a \in V$ *implies* $b \in V$.
3) $V \in a_\lambda \uparrow_\lambda a$ *implies* $a \in V$.

Above linear topology was defined by a collection $\mathfrak{V}$ of positive vicinities satisfying the following conditions :

1. $V \in \mathfrak{V}$, $V \subset U$ *implies* $U \in \mathfrak{V}$
2. U, V *implies* $UV \in \mathfrak{V}$.
3. $V \in \mathfrak{V}$ *implies* $\xi V \in \mathfrak{V}$ *for every* $\xi > 0$.
4. *for any* $V \in \mathfrak{V}$ we *can find* $U \in \mathfrak{V}$ *such that* $U \times U \subset V$.

Then he has proved that $\{x; a \leqq x \leqq b\}$ is complete by thus linear topology for every two element $a \leqq b$.

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1 Cf. G. Birkhoff : *Lattice theory*, Amer. Math, Soc. Colloqium Publ. vol. 1, 25(1949)

2 Cf. H. Nakano : *Linear topologies on semi-ordered linear spaces*, J. Fac. Sci. Hokkaido Univ. vol XII, no. 3, (1953), pp. 87-104. This paper will be denoted by H. Nakano [4] in this paper.

In this paper, considering only the *continuous semi-ordered linear space*[3] **R** we have introduced a linear topology on **R** as described in the section 2. Furthermore, in the section 2 we have proved that the operation $\smile$, $\frown$ are continuous by such linear topology (Theorem 1.1).

In the section 3, we have shown the relations between the *order-topology*[4] and the linear topology that satisfies some conditions and that its linear topology is sequential complete[5] (Theorem 3.3).

In the section 4, we shall introduce a topology on the set of continuous linear functionals on **R** and refer to *regularity* of **R**.

The notations used in this paper follow those in .H Nakano [1], [2], [3][6].

It is a pleasure to record here a debt of gratitude to Professor H. Nakano for his kindness in reading the original manuscript, and to Mr. Amamiya to his helpful advices.

2. The definitions and remarks.

In this section we shall introduce a linear topology by the notion of *vicinitor.*

Let **R** be a semi-ordered linear speace.

A manifold V in **R** is called a *vicinitor*, if it satisfies the following conditions :

(i) *for any* $x \in \mathbf{R}$, *there exists a positive number* α *such that* $\xi x \in V$ *for* $0 \leqq \xi \leqq \alpha$.

(ii) *if* $x \in V$, $|y| \leqq |x|$ *implies* $y \in V$.

By definition we easily see that vicinitor has the following properties :

(1) for every vicinftor V, αV is a vicinitor for every real nunber $\alpha \neq 0$.

(2) for two vicinitors U, V, their intersection is also a vicinitor.

(3) every vicinitor V is symmetric ; $(-1)V = V$.

(4) every vicinitor V is a star ; $\xi V \subset V$ for $0 \leqq \xi \leqq 1$.

We can define a linear topology by a collection $\mathfrak{V}$ of viciniters V in **R** satisfing the next conditions :

(1′) $V \in \mathfrak{V}$, $V \subset U$ *implies* $U \in \mathfrak{V}$.

(2′) U, $V \in \mathfrak{V}$ *implies* $UV \in \mathfrak{V}$.

(3′) $V \in \mathfrak{V}$ *implies* $\xi V \in \mathfrak{V}$ *for every* $\xi \neq 0$.

(4′) *for any* $V \in \mathfrak{V}$, *we can find* $U \in \mathfrak{V}$ *such that* $V \subset U \times U = \{x+y; x, y \in U\}$.

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3 **R** is called a continuous semi-ordered linear space, if $0 \leqq a_\nu$ $(\nu = 1, 2, \ldots\ldots)$ implies $\bigcap_{\nu=1}^{\infty} a_\nu \in \mathbf{R}$.

4 G. Birkhoff . loc. cit. p. 60 and H. Nakano : *Modullared semi-ordered linear spaces.* Tokyo. Math. Book Ser. I, Tokyo (1950). This book will be denoted by H. Nakano [1].

5 As to this terminology, cf. J. v. Neumann : *On complete topological spaces,* Trans. Amer. Math. Soc. 37 1-20. (1935)

6 H. Nakano [2] and [3] represent the following books : *Modern spectral theory,* Tokyo. Math. Book Ser. II, Tokyo (1950) and *Topology and linear topological spaces,* Tokyo Math Book Ser, III, Tokyo(1951).

As a basis of $\mathfrak{B}$, we can take a collection $\mathfrak{B}$ of vicinitors in **R** satisfyng

(1″) *for every U, $V\varepsilon\mathfrak{B}$ we can find $W\varepsilon\mathfrak{B}$ and $\lambda>0$ such that $\lambda W\subset UV$.*

(2″) *for any $V\varepsilon\mathfrak{B}$ we can find $U\varepsilon\mathfrak{B}$ and $\lambda>0$ such that $\lambda U\times\lambda U\subset V$.*

we can uniquely introduce uniformity $\mathfrak{U}^{\mathfrak{B}}$ in **R** of which $\mathfrak{B}$ is basis. The induced topology $\mathfrak{T}^{\mathfrak{U}}$ by this uniformity $\mathfrak{U}^{\mathfrak{B}}$ is called the *induced toporogy*[7] $\mathfrak{T}^{\mathfrak{B}}$ *by a linear toporogy* $\mathfrak{B}$. In this paper, saying we merely a linear topology we mean the above linear topology.

$\mathfrak{B}$ is called *separative linear topology*, if $\prod_{V\varepsilon\mathfrak{B}} V=\{0\}$.

A mapping a defined on **R** is said to be continuous by $\mathfrak{B}$, if a is continuous by $\mathfrak{T}^{\mathfrak{B}}$.

THEOREM 1.1 : *The operations ; addition, scalar multiplication, $\smile$, $\frown$, are continuous by $\mathfrak{B}$ respectively.*

Proof. The continuities of the addition and of the scalar multiplication are obvious by the definition of a linear topology.

(i) **The continuity of the opration $\frown$.**

For any a, $b\varepsilon\mathbf{R}$, $V\varepsilon\mathfrak{B}$, there exists $U\varepsilon\mathfrak{B}$ such that $U\times U\times U\subset V$.

Therefore, for any u_1, $u_2\varepsilon U$, since

$$\begin{aligned}&(u_1+a)\frown(u_2+b)-a\frown b\\&=(u_1+a)\frown(u_2+b)+(-a)\smile(-b)\\&\{u_1+a+(-a)\smile(-b)\}\frown\{u_2+b+(-a)\smile(-b)\}\\&=\{u_1+(a-b)\smile 0\}\frown\{u_2+(b-a)\smile 0\}\\&=\{u_1+(a-b)^+\}\frown\{u_2+(a-b)^-\}\end{aligned}$$

and

$$|u_1+(a-b)^+|\leqq|u_1|+(a-b)^+$$
$$|u_2+(a-b)^-|\leqq|u_2|+(a-b)^-,$$

we have

$$\begin{aligned}&|\{u_1+(a-b)^+\}\frown\{u_2+(a-b)^-\}|\\&\leqq\{|u_1|+(a-b)^+\}\frown\{|u_2|+(a-b)^-\}\\&\leqq|u_1|\frown|u_2|+|u_1|\frown(a-b)^-+|u_2|\frown(a-b)^++(a-b)^+\frown(a-b)^-\\&\leqq|u_1|\frown|u_2|+|u_1|+|u_2|\,\varepsilon\, V\end{aligned}$$

and consequently $(U+a)\frown(U+b)\subset V+a\frown b$.

(ii) **The contiouity of the operation $\smile$.**

On account of the formulation $a\smile b=(a+b)-(a\frown b)$ we can easily demonstration it. Q. E. D.

Let V be a vicinitor.

Putting $$\|x\|_V=\inf_{x\,\varepsilon\,\xi V}|\xi| \qquad \text{for } x\varepsilon\mathbf{R},$$

we obtain a *pseudo-noum*[8] on **R** satisfying next property :

7 Cf. H. Nakano [3], §54.

8 A functional $\lambda(x)$ on R is called a pseudo-norm on R, if $\lambda(x)\geqq 0$ for every $x\varepsilon$ R and $\lambda(\xi x)=|\xi|\lambda(x)$ for all real number ξ.

$$|x| \leqq |y| \quad \text{implies} \quad \|x\|_V \leqq \|y\|_V.$$

Conversely, let $\lambda(x)$ be a pseudo-norm on **R** satisfying aboave property. We see that

$$V = \{x; \lambda(x) \leqq 1\}$$

is a vicinitor.

3. The reletions between the linear topology and the order-topology.

On account of Theorem 1.1, we can prove easily the analogy to the properties valid in the normed semi-ordered linear space[9].

When a_ν is convergent to a by $\mathfrak{V}$, we may write $\text{T-}\lim_{\nu\to\infty} a_\nu = a$.

THEOEM 2.1 ; *Let $\mathfrak{V}$ be a separative linear topology on* R.

$a_\nu \downarrow_{\nu=1}^{\infty}$, $\text{T-}\lim_{\nu\to\infty} a_\nu = a$ *implies* $a_\nu \downarrow_{\nu=1}^{\infty} a$ *and* $a_\nu \uparrow_{\nu=1}^{\infty}$, $\text{T-}\lim a_\nu = a$ *implies* $a_\nu \uparrow_{\nu=1}^{\infty} a$.

THEOREM 2.2 ; *If* **R** *is a continuous semi-ordered linear space, then in order that* $0 \leqq a_\nu \uparrow_{\nu=1}^{\infty} a$ *implies* $\text{T-}\lim_{\nu\to\infty} a_\nu = a$, *it is necessary and sufficient that any* $b \geqq 0$, $[p_\nu] \uparrow_{\nu=1}^{\infty} [p]$ *implies* $\text{T-}\lim_{\nu\to\infty} [p_\nu]b = [p]b$.

Let **R** be continuous semi-ordered linear space. We shall consider a linear topology $\mathfrak{V}$ on **R** of which basis consists only of vicinitors satisfying the following condition ;

$$(\#) \qquad a_\nu \in V (\nu = 1, 2, \cdots\cdots) \textit{ implies } \bigcup_{\nu=1}^{\infty} a_\nu \in V.$$

In the sequel, $\mathfrak{V}\sigma$ and $\mathfrak{V}\sigma$ denote the above linear topology and its basis respectively.

THEOREM 2.3 ; *When $\mathfrak{V}\sigma$ is separative, if* $a_\nu (\nu = 1, 2, \cdots\cdots)$ is convergent to a by $\mathfrak{V}\sigma$, then $a_\nu \in \mathbf{R}$ $(\nu = 1, 2 \cdots\cdots)$ is order-convergent to a.

Proof. $\text{T-}\lim_{\nu\to\infty} a_u = a$, for any $V \in \mathfrak{V}\sigma$ we can find n_0 such that $a_\nu - a \in V$ for all $\nu \geqq n_0$. Now, setting $\bigcup_{u=k}^{\infty} (|a_\mu - a|) = \varepsilon_k$ $(k = 1, 2, \cdots\cdots)$ we see that $V \ni \varepsilon_k \downarrow_{k=1}^{\infty}$ and $|a_\nu - a| \leqq \varepsilon_\nu$ $(\nu = 1, 2 \cdots\cdots)$. Furrthermore, by theorem 2.1, since $\text{T-}\lim_{\nu\to\infty} \varepsilon_\nu = 0$ we have $\varepsilon_k \downarrow_{k=1}^{\infty} 0$. Q. E. D.

THEOREM 2.4 ; *if* $a_\nu \in \mathrm{R}$ $(\nu = 1, 2, \cdots\cdots)$ *is uniformly order-convergent*[10] *to* a, *then* $a_\nu \in \mathbf{R}$ $(\nu = 1, 2, \cdots\cdots)$ *is convergent to a by $\mathfrak{V}$.*

9 Cf. H. Nakano [1], Theorem 30.1 and 30.5, 126—127.

10 Cf. H. Nakano [1], §2.

Proof. If $a_\nu \varepsilon \mathbf{R}$ $(\nu=1,2,\cdots\cdots)$ is uniformly order-convergent to a, there exists $l \varepsilon \mathbf{R}, \varepsilon_\nu \downarrow_{\nu=1}^{\infty} 0$ such that $|a_\nu - a| \leqq \varepsilon_\nu l$ $(\nu=1,2,\cdots\cdots)$.

For any $V \varepsilon \mathfrak{B}$, by the definition of vicinitor we can find μ such that $\frac{1}{\mu} l \varepsilon V$ and further we can find n_0 such that $|a_\nu - a| \leqq \varepsilon_{n_0} l \leqq \frac{1}{\mu} l$ for all $\nu \geqq n_0$.

Therefore $\text{T-}\lim_{\nu\to\infty} a_\nu = a$. Q.E.D.

THEOREM 2.5 ; *A continuous semi-ordered linear space is sequential complete by $\mathfrak{B}\sigma$, if $\mathfrak{B}\sigma$ is sparative.*

Proof. Let sequence $a_\nu \varepsilon \mathbf{R}$ $(\nu=1,2,\cdots\cdots)$ be a Cauchy sequence by $\mathfrak{B}\sigma$. For any $U \varepsilon \mathfrak{B}\sigma$, there exists $V \varepsilon \mathfrak{B}\sigma$, $\lambda>0$ such that $\lambda V \times \lambda V \subset U$. By assumption we can find n_0 such that $a_\nu - a_\mu \varepsilon \lambda V$ for all μ, $\nu \geqq n_0$. Setting $n_k = n_0 + k$, we see first that a subsequence a_{n_k} $(k=1,2,\cdots\cdots)$ is a Cauchy sequence by the order-topology, because, setting $\bigcup_{\nu,\mu \geqq n_k} (|a_\nu - a_\mu|) = \varepsilon_k \varepsilon \lambda V$, we have $\text{T-}\lim_{k\to\infty} \varepsilon_k = 0$ and hence by theorem 2,1, we have $\varepsilon_k \downarrow_{k=1}^{\infty} 0$ and $|a_{n_k} - a_{n_j}| \leqq \varepsilon_k$ for all $j \geqq k$. Therefore there exists a such that $\lim_{k\to\infty}$[11] $a_{n_k} = a$, then we have $a = \bigcup_{\nu=1}^{\infty} (\bigcap_{i\geqq\nu} a_{n_j})$ and hence

$$|a_{n_k} - a| \leqq |\bigcup_{\nu=1}^{\infty} (\bigcap_{i\geqq\nu} (a_{n_k} - a_{n_j}))| \leqq \bigcup_{\nu=1}^{\infty} (\bigcap_{j=\nu} |a_{n_k} - a_{n_j}|) \leqq \varepsilon_1.$$

Accordingly, we have $a_{n_k} - a \varepsilon \lambda V$ for all $k=1,2,\cdots\cdots$, namely, $\text{T-}\lim_{\nu\to\infty} a_{n_\nu} = a$

Therefor, we may conclude that $a_\mu - a = (a_\mu - a_{n_k}) + (a_{n_k} - a) \varepsilon \lambda V \times \lambda V \subset U$ for all $\mu \geqq n_1$. The proof is completed.

4. Adjoint space.

A manifold A of $\mathbf{R}$ is said to be *topologically bounded* by $\mathfrak{B}$, if for any $V \varepsilon \mathfrak{B}$, there exists $\lambda>0$ such that $A \subset \lambda V$.

A linear functional φ on $\mathbf{R}$ is said to be *topologically bounded* if $\sup_{x \varepsilon A} |\varphi(x)| < +\infty$ for every topologically bounded manifold A.

The totality of the topologecally bounded linear functionals on $\mathbf{R}$ is called the *associated space* of $\mathbf{R}$ by $\mathfrak{B}$ and we may write $\tilde{\mathbf{R}}^{\mathfrak{B}}$.

We shall be able to consider as semi-ordered linear space by the next semi-order $\geqq$;

for $\tilde{\mathbf{R}}^{\mathfrak{B}} \varepsilon L, F$, $L \geqq F$ *means* $L(x) \geqq F(x)$ *for all* $x \geqq 0$.

Because, for any $\tilde{a} \varepsilon \tilde{\mathbf{R}}$, setting

11 $\lim a_\nu = a$ means that a_ν is order-convergent to a. *"The order-topology is sequental complete"*, As to this theorem, cf, H, Nakano [2], Theorem 6. 4.

$$P(a)=\sup_{0\leqq x\leqq a}\tilde{a}(x) \quad \textit{for any } a\geqq 0,$$

obviously $P(\alpha a)=\alpha P(a)$ for any $\alpha\geqq 0$, $a\geqq 0$.
Furthermore, for any a, $b\geqq 0$ we have

$$P(a+b)=\sup_{x\leqq a+b}\tilde{a}(x)\geqq\sup_{0\leqq y\leqq b}\tilde{a}(x+y)$$

$$=\sup_{0\leqq x\leqq a}\tilde{a}(x)+\sup_{0\leqq y\leqq b}\tilde{a}(y)=P(a)+P(b).$$

On the other hand , if $0\leqslant z\leqslant a+b$, then putting $x=a\frown z$, $y=z-x$ we obtain $0\leqq x\leqslant a$, $z=x+y$ and $0\leqq y\leqq b$ and consequently $P(a+b)=P(a)+P(b)$. Therfore, for any $x\varepsilon\mathbf{R}$, setting $L(x)=P(x^+)-P(x^-)$, we obtain a topologically bounded linear functional L on $\mathbf{R}$. (*for any* $a\geqq 0$, $V\varepsilon\mathfrak{B}$, *there exists a* $\lambda>0$ *such that* $\lambda V\supset\{x\,;\,0\leqslant x\leqslant a\}$) Furthermore, it is obvious that other postulates[12] on the semi-order are satisfied.

A linear functional φ on $\mathbf{R}$ is said to be *topologically continuous* by $\mathfrak{B}$, if we can find $V\varepsilon\mathfrak{B}$, $\alpha>0$ such that

$$|\varphi(x)|\leqq\alpha\|x\|_V \qquad \text{for every } x\varepsilon\mathbf{R}.$$

The totality of the topologically continuous linear fnnctionals on $\mathbf{R}$ is called the adjoint space of $\mathbf{R}$ by $\mathfrak{B}$.
By the definition the adjoint space is oviously a semi-ordered linear space and we shall write $\overline{\mathbf{R}}^{\mathfrak{B}}$.

Setting $\bar{V}=\{\bar{x}\,;\,|\bar{x}(x)|\leqq 1 \text{ for } x\varepsilon\mathbf{R}\}$, $\bar{V}$ satisgies the condition (ii) in § 2.

Let $\mathfrak{D}$ be a collection satisfying the condition that there exists a vicinitor W in $\mathfrak{D}$ such that $W\supset V+U$ for any V, $U\varepsilon\mathfrak{D}$ and $\xi V\varepsilon\mathfrak{D}$ for $V\varepsilon\mathfrak{D}$, $\xi\neq 0$.

Furthermore, $(V/r)=\{\bar{x}\,;\,\sup_{x\varepsilon V}|\bar{x}(x)|<r\}$ $(V\varepsilon\mathfrak{D})$ defienes a convex topology on $\overline{\mathbf{R}}^{\mathfrak{B}}$ and we may write $\mathfrak{T}^{\overline{\mathfrak{B}}}$ this topology.

LEMMA ; *The linear topology* $\mathfrak{T}^{\overline{\mathfrak{B}}}$ *on* $\overline{\mathbf{R}}^{\mathfrak{B}}$ *is convex and separative.*

proof. It is obvious that $\mathfrak{T}^{\overline{\mathfrak{B}}}$ is convex by the definition. For $0\neq\bar{a}\varepsilon\overline{\mathbf{R}}^{\mathfrak{B}}$, we can find $a\varepsilon\mathbf{R}$ such that $\bar{a}(a)>1$. Therefore, for any V such that $a\varepsilon V\varepsilon\mathfrak{B}$, $\bar{a}\bar{\varepsilon}\bar{V}=\{\bar{x}\,;\,\sup_{x\varepsilon V}|x(x)|<1\}$ namely, $\mathfrak{T}^{\mathfrak{B}}$ is separative. Q. E. D.

Let $\overline{\overline{\mathbf{R}}}^{3}$ be the totality of the continnous linear functionals on $\overline{\mathbf{R}}^{\mathfrak{B}}$ dy $\mathfrak{T}^{\overline{\mathfrak{B}}}$,

12 This postulates are furnished in H, Nakano [1], § 1

Any of the elements $x \in \mathbf{R}$ is considered as a topologically continuous linear functional on $\overline{\mathbf{R}}^{\mathfrak{V}}$ by the relation ;

$$(※) \quad x(\bar{x})=\bar{x}(x) \quad \textit{for evesy } \bar{x} \in \overline{\mathbf{R}}^{\mathfrak{V}},$$

R is said to *be regular*, if it satisfies the following conditions ;

(i) *the correspondence* (※) *from* R *to* $\overline{\overline{\mathbf{R}}}^{\mathfrak{V}}$ *is a isomorphism, that is,*

(1) *for any* $\bar{\bar{a}} \in \overline{\overline{\mathbf{R}}}^{\mathfrak{V}}$, *there exists* $a \in \mathbf{R}$ *satisfying* (※) and $\overline{\overline{\mathbf{R}}}^{\mathfrak{V}}$ is the semi -ordered linear space.

(2) $\bar{\bar{a}} \geqq 0$ *if and only if* $a \geqq 0$.

(ii) $\mathfrak{V}$ *is a reflexive linear topology*[13], *that is, for any* $V \in \mathfrak{V}$, *if for* $\overline{V}=\{\bar{x} ; \sup_{\|x\|_V \leqq 1} |\bar{x}(x)| \leqq 1\}$, *we have* $\|x\|_V = \sup_{|x| \in V} |\bar{x}(x)|$ *for every* $x \in \mathbf{R}$.

Remark. If $\bar{a} \in \overline{V}$, since $|\bar{a}|(a)=\sup_{|x| \leqq a} \bar{a}(x)$ for $a \geqq 0$, then for any $y \in V$ we have $|\bar{a}|(|y|) \leqq 1$ further

$$|\bar{a}|(|y|) \geqq |\bar{a}|(y) \geqq |\bar{a}|(-|y|) = -|\bar{a}|(|y|)$$

namely, $||\bar{a}|(y)| \leqq |\bar{a}|(|y|) \leqq 1$ and hence $|\bar{a}| \in \overline{V}$ and consequently $\overline{V}=\{x : |x(|\bar{x}|)| \leqq 1$ for $x \in V\}$.

A linear topology $\mathfrak{W}$ on **R** such that the system $\overline{V}_{\bar{a}} =\{\bar{x} ; |\bar{a}|(|x|) \leqq 1\} (\bar{a} \in \overline{\mathbf{R}}^{\mathfrak{V}})$ is a basis of $\mathfrak{W}$ is called the *absolute weak topology* of **R** by $\overline{\mathbf{R}}^{\mathfrak{V}}$.

THEOREM 3.1[14] : *A manifold A of* **R** *is topologically bounded by the absolute weak topology* **R** *by* $\overline{\mathbf{R}}^{\mathfrak{V}}$ *if and only if* $\sup_{x \in A} |\bar{x}(x)| < +\infty$ *for every* $\bar{x} \in \overline{\mathbf{R}}^{\mathfrak{V}}$.

For every manifoid A, A^- denotes the *closure* of A by the induced topology $\mathfrak{T}^{\mathfrak{V}}$.

Let $\mathbf{F}$ be a linear subspace of the set of all linear functionals on a linear space **R**. For any finite subset $f_1, f_2, \cdots\cdots, f_n$ of $\mathbf{F}$, $x_a \in \mathbf{R}$ and real positive number $r>0$, $\{x ; |f_i(x)-f_i(x_0)|<r, \ i=1,2,\cdots\cdots,n\}$ is regarded as neighorhoods of x_0 in **R**. This topology is obviously a linear toporogy on **R** and we may say the weak topology of **R** by $\mathbf{F}$.

A manifold K of **R** will be said to be *weakly bounded, weakly closed, weakly compact*, if K is so respectively by the weak topology of **R**.

In theorem 4 of § 65 in H. Nakano [3], modifying we its proof, we have

THEOREM 3.2 : *For every vicinitor* $V \in \mathfrak{V}$,

$\overline{V}=\{\bar{x} ; |\bar{x}|(|x|) \leqq 1$ *for* $x \in V\}$ *is absolutely weakly compact by* $\overline{\mathbf{R}}^{\mathfrak{V}}$.

Let $\mathbf{D}_X$ be a family of subsets of **R** such that there exists a subsets K_3 in $\mathbf{D}_X$ satisfying the condition $K_3 \supset K_1 \dotplus K_2$ for any $K_1, K_2 \in \mathbf{D}_X$. (where notation $\dotplus$

13 Cf. H, Nakano [4], 103—104.

14 Cf. H. Nakano [4], Theorem 8.2

means union of sets)

Denoting we $\{\bar{x}; \sup_{x \varepsilon K} |\bar{x}(x)| < r\}$ by (K/r) for $r>0, \mathbf{D}_X \ni K, \{(K/r) : K \varepsilon \mathbf{D}_X, r>0\}$ defines a convex linear topology on $\overline{\mathbf{R}}^{\mathfrak{B}}$ and this topology will be called the $\mathbf{D}_X$ -topology.

We may say *p-topology* and *κ-topology* if $\mathbf{D}_X$ is the class of all finite sets of $\mathbf{R}$ and the chass of convex weakly compact subsets of $\mathbf{R}$ by $\overline{\mathbf{R}}^{\mathfrak{B}}$ respectively.

THEOREM 3.3 : *In order that* $\mathbf{R}$ *is regular, it is necessary and sufficient that* $\mathfrak{B}$ *is convex, separative and every* V^- $(V \varepsilon \mathfrak{D})$ *is weakly compact by* $\overline{\mathbf{R}}^{\mathfrak{B}}$

Proof.

Necessity. If $\mathbf{R}$ is regular, then for any $V \varepsilon \mathfrak{D}$, since $V^- = \{x ; \|x\|_{V^-} \leqq 1\}$[15] and $\|x\|_{V^-} = \sup_{\bar{x} \varepsilon \bar{V}} |\bar{x}(x)|$ for $\bar{V} = \{\bar{x} ; \sup_{\|x\|_{V^-} \leqq 1} |\bar{x}(x)| \leqq 1\} = \{\bar{x} ; |\bar{x}(x)| \leqq 1$ for every $x \varepsilon V^-\}$, we have

$$
\begin{aligned}
(\bar{\bar{V}})^- &= \{\bar{\bar{x}} ; |\bar{\bar{x}}(\bar{x})| \leqq 1 \text{ for every } \bar{x} \varepsilon \bar{V}\} \\
&= \{x ; |\bar{x}(x)| \leqq 1 \text{ for every } \bar{x} \varepsilon \bar{V}\} \\
&= \{x ; \sup_{\|x\|_{\bar{V}} \leqq 1} |\bar{x}(x)| \leqq 1\} = V^-
\end{aligned}
$$

and hence V^- is weakly compact[16] by $\overline{\mathbf{R}}^{\mathfrak{B}}$. Further $\mathfrak{B}$ is convex, separative by Lemma.

Sufficiency. Let $\mathbf{R}$ be to satisfy the conditions of the theorem.

At first, we shall show that for any $\bar{\bar{x}} \varepsilon \overline{\overline{\mathbf{R}}}^{\mathfrak{B}}$, there exists $x_0 \varepsilon \mathbf{R}$ such that $\bar{\bar{x}}_0(\bar{x}) = \bar{x}(x_0)$ for every $\bar{x} \varepsilon \overline{\mathbf{R}}^{\mathfrak{B}}$. For any $\bar{V} \varepsilon \bar{\mathfrak{X}}^{\mathfrak{B}}$, there exists $V \varepsilon \mathfrak{D}$ such that $\bar{V} \supset \{\bar{x} ; \sup_{x \varepsilon V^-} |\bar{x}(x)| < 1\}$, namely, $\mathfrak{X}^{\overline{\overline{\mathfrak{B}}}}$ is weaker than κ-topology. It is obvious that $\mathfrak{X}^{\overline{\overline{\mathfrak{B}}}}$ is stronger than p-topology. Therefore, by theorem 2 in R. Arens's paper[17], the elements of $\mathbf{R}$ represent precisely the continuous linear functionals of $\overline{\mathbf{R}}^{\mathfrak{B}}$. Futhermore, for any $V \varepsilon \mathfrak{B}$, $a \varepsilon \mathbf{R}$, by Banach's extension theorem[18], there exists a linear functional φ on R such that

$$\varphi(a) = \|a\|_V, \quad |\varphi(x)| \leqq \|x\|_V \qquad \text{for every } x \varepsilon \mathbf{R}.$$

This φ is obviously topologically continuous and hence $\varphi \varepsilon \overline{\mathbf{R}}^{\mathfrak{B}}$ and that $\varphi \varepsilon \bar{A}$ for $\bar{A} = \{\bar{x} ; \sup_{\|x\|_V \leqq 1} |\bar{x}(x)| \leqq 1\}$. Accordingly, $\sup_{\bar{x} \varepsilon \bar{V}} |\bar{x}(a)| \geqq \varphi(a) = \|a\|_V$.

Opposite inequality is evident. Thus we conclude

$$\|a\|_V = \sup_{\bar{x} \varepsilon \bar{A}} |\bar{x}(a)| \qquad \text{for every } a \varepsilon \mathbf{R}.$$

It is obvious that $\mathbf{R} \varepsilon a \geqq 0$ implies $0 \leqq \bar{\bar{a}} \varepsilon \overline{\overline{\mathbf{R}}}$ and $a \neq 0$ implies $\bar{\bar{a}} \neq 0$ ($\bar{\bar{a}}$ is element

15 This equality is obtained easily by theorem 3 of §49 and theorem 2 of §54 in H. Nakano (3)

16 This conclusion depends on theorem 4 of §65 in H. Nakano (3).

17 Cf. R. Arens : *Duality in linear spaces*, Duke Math. J. 14, 787—794 (1947)

18 Cf. S. Banach : *Théorie des opérations linéaires*, Warsaw, Theorem 1, 27—28 (1932)

of $\overline{R}^{\mathfrak{B}}$ corresponding to a by (※). Further, if $a \neq 0$, there exists $\bar{a} \in \overline{R}^{\mathfrak{B}}$ such that $\bar{a}(a^-) > 0$; hence $|\bar{a}|(a^-) > 0$. We have then $|\bar{a}|[a^-](a^-) > 0$ and $|\bar{a}|[a](a^+) = 0$. Since $|\bar{a}|[a]$[19] is topologically continuous positive functional, we have that $\bar{\bar{R}} \ni \bar{\bar{a}} > 0$ implies $R \ni a > 0$.

Remark. By theorem 3.2 we see that if **R** is regular, V^- $(V \in \mathfrak{D})$ is absolutely weakly compact $\overline{R}^{\mathfrak{B}}$.

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19 If **R** is continuous, then all elements are normalable and consequently we can consider the projection $[a^-]$.
Cf. H. Nakano [1], Th. 6.14 and Th. 5.5., 19—28.