On Term by Term Integration

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Masao Yoshida

Abstract

In this note the writer proves some theorems on integration of a sequence of the functions integrable in the sense of Perron, analogus to those in the theory of the Lebesgue integral.

In the following arguments we treat, for simplicity, exclusively finite functions of a real variable, integrable in the sense of Perron and $\int_a^b f(x)dx$ denotes the Perron integral of the function $f(x)$ ovre a finite interval $[a, b]$. The results remain true for $n(\geqq 2)$ dimensional integrals.

Theorem I. Let $\{f_n(x)\}$ be a sequence of the integrable functions defined on $[a, b]$ and converges to a function $f(x)$ as $n \to \infty$ almost everywhere in $[a, b]$. If there exist two functions $g(x)$ and $h(x)$ intergrable over $[a, b]$ such that $g(x) \leqq f_n(x) \leqq h(x)$ for all n and x in $[a, b]$, then we have

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$$

Proof. We define functions $\varphi_n(x)$, $\varphi(x)$, and $\psi(x)$ by putting

$$\varphi_n(x) = f_n(x) - g(x), \quad \varphi(x) = f(x) - g(x), \quad (n=1, 2, \cdots),$$

and

$$\psi(x) = h(x) - g(x).$$

Since the functions $\varphi_n(x)$ and $\psi(x)$ non-negative and integrable are summable over $[a, b]$ and satisfy the inequality $0 \leqq \varphi_n(x) \leqq \psi(x)$ for all n and x in $[a, b]$, the sequence $\{\varphi_n(x)\}$ converges as $n \to \infty$ to a summable function $\varphi(x)$ almost evrywhere in $[a, b]$. Then, by Lebesgue's theorem, we have

$$\lim_{n\to\infty} (L)\int_a^b \varphi_n(x)dx = (L)\int_a^b \varphi(x)dx$$

where $(L)\int_a^b \varphi_n(x)dx$ and $(L)\int_a^b \varphi(x)dx$ denote the Lebesgue integrals of $\varphi_n(x)$

and $\varphi(x)$ respectively.

Now, if we substitute $f_n(x)-g(x)$ and $f(x)-g(x)$ respectively for $\varphi_n(x)$ and for $\varphi(x)$ into the above equation, recalling that the summable function $\varphi_n(x)$ is integrable, and that the function $f(x)$ is also integrable as sum of the two integrable functions $\varphi(x)$ and $g(x)$, we can immediately obtain the desired result.

Corollary. If $\{f_n(x)\}$ is a uniformly bounded sequence of the integrable functions defined on $[a, b]$ such that $\lim_{n\to\infty} f_n(x)=f(x)$ exists almost everywhere in $[a, b]$, then we have

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx.$$

Teorem II. Let $\{f_n(x)\}$ be a monotone sequence of the integrable functions defined on $[a, b]$ and converges to a integrable function $f(x)$ almost everywhere in $[a, b]$. Then we have

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx.$$

Proof. We first prove the theorem for a monotone increasing sequence $\{f_n(x)\}$. Putting

$$g_n(x)=f_n(x)-f_1(x) \qquad (n=2,\ 3,\ \cdots),$$

the sequence $\{g_n(x)\}$ of the integrable functions converges, by assumption, to a integrable function $g(x)=f(x)-f_1(x)$ almost everywhere in $[a, b]$ and the inequality $0\leqq g_n(x)\leqq g(x)$ holds for $n=2,\ 3,\ \cdots$ almost everywhere in $[a, b]$. Then by Theorem I. we have

$$\lim_{n\to\infty}\int_a^b g_n(x)dx=\int_a^b g(x)dx.$$

Substituting $f_n(x)-f_1(x)$ and $f(x)-f_1(x)$ respectively for $g_n(x)$ and for $g(x)$ into the above equation, we have at once

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b f(x)dx$$

For a monotone decreasing sequence $\{f_n(x)\}$ we may proceed by precisely the same argument with $g_n(x)=f_1(x)-f_n(x)$, $n=2,\ 3,\ \cdots$ and the result is true in this case.

Theorem III. Let $\{f_n(x)\}$ be a monotone sequence of the integrable functions defined on $[a, b]$. If $\lim_{n\to\infty}\int_a^b f_n(x)dx$ exists and is finite, then we have

$$\lim_{n\to\infty}\int_a^b f_n(x)dx=\int_a^b \lim_{n\to\infty} f_n(x)dx$$

Proof. It is sufficient to prove the result in the case of a monotone increasing sequence. Define the functions $g_n(x)$ as follows :

$$g_n(x)=f_n(x)-f_1(x) \qquad (n=2,\ 3,\ \cdots).$$

The sequence $\{g_n(x)\}$ of the summable functions converges to a function $g(x)=f(x)-f_1(x)$, where $f(x)=\lim_{n\to\infty} f_n(x)$. Since $\lim_{n\to\infty}\int_a^b f_n(x)dx$ exists and is finite by assumption, $\lim_{n\to\infty}\int_a^b g_n(x)dx$ also exists and is finite. Then by Fatou's lemma the function $g(x)$ is summable. Accordingly the function $f(x)$, as sum of the two integrable functions $g(x)$ and $f_1(x)$, must be integrable. It is clear that immediate application of Theorem II. proves the validity of the result.

Theorem IV. Let $\{f_n(x)\}$ be a monotone sequence of the measurable functions of the constant sign defined on $[a, b]$ such that the limit function $f(x)=\lim_{n\to\infty} f_n(x)$ and $f_1(x)$ are integrable over $[a, b]$. Then we have

$$\lim_{n\to\infty}\int_a^b f_n(x)\,dx=\int_a^b f(x)dx.$$

Proof. For a monotone non-negative sequence inequality

$$0\leqq f_n(x)\leqq 2f(x)+f_1(x)$$

holds for all n and x in $[a, b]$.

Since the functions $f_n(x)$ $(n=1,\ 2,\ \cdots)$ are measurable and the function $2f(x)+f_1(x)$, as the funtion non-negative and integrable, is summable. Then $f_n(x)$ are integrable. Accordingly by Theorem I. we have desired result.

In the case of a non-positive monotone sequence we may verify the result, using $\{-f_n(x)\}$ instead of $\{f_n(x)\}$.

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