A Constructive Study of the Vector Space of Real Functions

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Abstract

In this paper the author aims to show his some important results about the vector space of real functions from the standpoint of infinitesimal analysis. The coefficients of a linear combination are especially studied in some details.

1. Preliminaries

When a function $f(x)$ is defined in the form

$$f(x)=\sum_{1}^{\infty} c_k \varphi_k(x), \tag{1,1}$$

it is very inconvenient if the convergence of the right hand is not given as absolute. So, the summations in this papar are supposed to be absolutely convergent when they are not divergent.

To generalize the formula (1,1), the family of the orthonormal functions $\{\varphi_\alpha\}$ $(0 \leqq \alpha \leqq 1)$ is given to define a function $f(x)$ in the form as unique representation

$$f(x)=\underset{\alpha}{\mathfrak{S}} c_\alpha \varphi_\alpha(x) \tag{1,2}$$

on condition that

$$\mathfrak{S}|c_\alpha \varphi_\alpha(x)|=f_{||}(x)$$

is convergent; orthonormality of $\{\varphi_\alpha\}$ is given such as

$$(\varphi_\alpha, \varphi_\beta)=0 \quad \text{when} \quad \alpha \neq \beta$$

and

$$\|\varphi_\alpha\|^2 \equiv (\varphi_\alpha, \varphi_\alpha)=1$$

in respect to a certain scalar product form (,), which satisfies the conditions: (i) $(f_1 \pm f_2, g)=(f_1, g) \pm (f_2, g)$; (ii) $(f, g)=\overline{(g, f)}$; (iii) $(cf, g)=c(f, g)$.

In this case, we may have an infinitesimal quantity as scalar value, because $(f, \varphi_\beta)=\mathfrak{S} c_\alpha(\varphi_\alpha, \varphi_\beta)=c_\beta \|\varphi_\beta\|^2=c_\beta$; the investigations are much simpli-

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fied if we confine the functions to the type of decomposition

$$f(x)=\mu\underset{\alpha}{\mathfrak{S}}\lambda(\alpha)\,\varphi_\alpha(x)+\underset{\beta}{\mathfrak{S}}\gamma_\beta\,\varphi_\beta(x) \tag{1,3}$$

instead of (1,2), where $\lambda(\alpha)$ and γ_β take finite real values for each α,β, and μ indicates an infinitesimal quantity independent of $f(x)$ and α,β.

In this paper, all the functions are given as real functions of a real variable x to make the space V, all the values are expected to be observed as real numbers, and any linear transformation T of V is supposed to be obliged to the following hypothesis: if the relation

$$(Tf, f)=0$$

is observed for each $f\in$V, it must be

$$T=0.$$

Then, the condition (ii) is naturally altered by

(ii′) $(f, g)=(g, f)$,

and moreover, it may be proved that if a (linear) transformation T has its adjoint T^* [which is defined by the relation $(Tf, g)=\overline{(f, T^*g)}$], T must be a *symmetric* one, i. e. $T=T^*$.

Besides (1,2), the decomposition formula

$$f(x)=\underset{\xi}{\mathfrak{S}}f(\xi)\,\partial_\xi\boldsymbol{1} \tag{1,4}$$

is adopted, where the symbol $\partial_\xi\boldsymbol{1}$ designates the characteristic function of the point-set (ξ); i. e.

$$\partial_\xi\boldsymbol{1}(x)=0 \quad \text{when} \quad x\neq\xi$$

and $\partial_\xi\boldsymbol{1}(\xi)=1$. On account of the formula (1,4) the scalar product (f,g) may be symbolically written in the form

$$(f, g)=\underset{\xi}{\mathfrak{S}}\underset{\eta}{\mathfrak{S}}f(\xi)\,g(\eta)(\partial_\xi\boldsymbol{1}, \partial_\eta\boldsymbol{1})$$

which leads to the general representation

$$(f, g)=\underset{x}{\mathfrak{S}}\underset{y}{\mathfrak{S}}f(x)g(y)\delta(x, y).$$

If $\delta(x, y)=0$ when $x\neq y$, the space V is called *canonical*, and then the scalar product (f, g) is given in the form

$$(f, g)=\underset{x}{\mathfrak{S}}f(x)g(x)\delta(x) \tag{1,5}$$

where $\delta(x)$ is an non-negative quantity which may be infinitesimal.

2. Spectral Definition

Let us suppose that all of the functions $\varphi_\alpha (0 \leqq \alpha \leqq 1)$ can be regarded as proper functions with respect to a certain transformation T, and denote the proper value of φ_α by ω_α respectively; i.e.

$$T\varphi_\alpha = \omega_\alpha \varphi_\alpha. \tag{2,1}$$

Then, if we define the transformations E_ω by the formula

$$E_\omega f = \underset{\omega_\alpha \leqq \omega}{\mathfrak{S}} c_\alpha \varphi_\alpha, \qquad (-\infty < \omega < \infty)$$

for the functions $f(x)$ given by (1,2), it can be proved that $\{E_\omega\}$ make a spectral family of T and T may be written in the symbolical form

$$\begin{aligned} T &= \underset{\omega}{\mathfrak{S}}\, \omega (E_\omega - E_{\omega-0}) \\ &\equiv \underset{\omega}{\mathfrak{S}}\, \omega \delta E_\omega. \end{aligned} \tag{2,2}$$

It is evidently verified that

$$(E_\omega - E_{\omega-0})(E_{\omega}' - E_{\omega}'_{-0}) f = 0$$

for any $f(x)$ given by (1,2) when $\omega \neq \omega'$, in regard to the fact: the set $\boldsymbol{M}_\omega$ of the values α for which $\omega_\alpha = \omega$, and the set $\boldsymbol{M}_\omega'$ of the values α for which $\omega_\alpha = \omega'$ are distinct (i.e. $\boldsymbol{M}_\omega \cap \boldsymbol{M}_\omega' = 0$) when $\omega \neq \omega'$. Then, in case $\omega_\alpha \neq \omega_\beta$ when $\alpha \neq \beta$, we have

$$\begin{aligned} (\varphi_\alpha, \varphi_\beta) &= ((E_{\omega_\alpha} - E_{\omega_{\alpha-0}}) f, (E_{\omega_\beta} - E_{\omega_{\beta-0}}) f) \\ &= (f, (E_{\omega_\alpha} - E_{\omega_{\alpha-0}})^* (E_{\omega_\beta} - E_{\omega_{\beta-0}}) f) \\ &= (f, (E_{\omega_\alpha} - E_{\omega_{\alpha-0}})(E_{\omega_\beta} - E_{\omega_{\beta-0}}) f) \\ &= (f, 0) = 0 \end{aligned}$$

hence

$$(\varphi_\alpha, \varphi_\beta) = 0, \quad \text{when} \quad \alpha \neq \beta.$$

Therefore, the orthogonality of the family $\{\varphi_\alpha\}$ does not harm the generality of our space **V**, as long as it accompanies the condition

$$\omega_\alpha \neq \omega_\beta \quad \text{when} \quad \alpha \neq \beta. [1] \tag{2,3}$$

On the above-stated structure, we can inversely define a (linear) trans-

1 This condition may be important to distinguish the class of such transformations from others.

formation T of our space $\mathbf{V}$ by the formula (2,1) to be accompanied with the spectral decomposition (2,2), on condition (2,3). Then, if

$$Tf = \mathfrak{S}_{\alpha} \omega_{\alpha} c_{\alpha} \varphi_{\alpha} = \omega f = \omega \mathfrak{S}_{\alpha} c_{\alpha} \varphi_{\alpha},$$

we have

$$\mathfrak{S}_{\alpha} (\omega_{\alpha} - \omega) c_{\alpha} \varphi_{\alpha} = 0.$$

Hence, in accordance to the unicity of the representation of a function by (1,2), it must be

$$(\omega_{\alpha} - \omega) c_{\alpha} = 0 \quad \text{for each} \quad \alpha,$$

so that there may be no other α for which $c_{\alpha} \neq 0$,[2] than the values of $\mathbf{M}_{\omega}$.

THEOREM. *If T is defined by (2,1), there can be no other proper function than $\{\varphi_{\alpha}\}$ with respect to T, on condition (2,3).*

3. Canonical Space

In this section the canonical space $\mathbf{L}$ will be investigated specially in regard to the consistency with the hypothesis: the relation

$$(f_1, g) = (f_2, g) \quad \text{for each} \quad g \in \mathbf{L} \tag{3,1}$$

implies $f_1 = f_2$; this will be called the hypothesis of *weak coincidence*. If there are two functions f_1, $f_2 \in \mathbf{L}$ for which $f_1(x) = f_2(x)$ when $x \neq \xi$ and $f_1(\xi) \neq f_2(\xi)$, and the relation (3,1) is satisfied, apparently it must be $\delta(\xi) = 0$. On the other hand, if the above-mentioned hypothesis is demanded to be consistent even in case such two functions belong to $\mathbf{L}$, it is necessary that $|\delta(\xi)| > 0$.

In regard to the unicity of the representation of a function (1,2), it is sufficient for the coincidence $f_1(x) = f_2(x)$ that

$$(f_1, \varphi_{\alpha}) = (f_2, \varphi_{\alpha})$$

for each α. But, in this connexion, difficulty lies in that the quantity $(f, \varphi_{\alpha}) = = c_{\alpha}$ is possibly infinitesimal, so that, to avoid the uncertainty, the restricted representaion given in (1,3)

2 Here, it must be promised that in case $c_{\alpha} = \mu\lambda(\alpha)$ the notations $c_{\alpha} \neq 0$ or $= 0$ mean $\lambda(\alpha) \neq 0$ or $= 0$ respectively.

$$f(x)=\mu\mathfrak{S}\lambda(\alpha)\varphi_\alpha(x)+\mathfrak{S}\gamma_\beta\varphi_\beta(x)$$

may be found very convenient, though it gives not a sufficiently general space of functions.

In regard to the representation (1,5), we have

$$\|f\|^2=\mathfrak{S}f^2(x)\delta(x).$$

Hereupon, for a set $\boldsymbol{E}$ of x let us define the function

$$\delta(\boldsymbol{E})=\underset{x\in\boldsymbol{E}}{\mathfrak{S}}\delta(x)$$

which may be called the *δ-measure* of $\boldsymbol{E}$. Then, in regard to the step decomposition of $\boldsymbol{I}=(-\infty,\infty)$ according to the values of a function $f(x)$

$$\boldsymbol{I}=\boldsymbol{I}_0+\boldsymbol{I}_1+\boldsymbol{I}_2+\cdots$$

where $\boldsymbol{I}_0=\{x:\ f(x)=0\}$, $\boldsymbol{I}_n=\{x:\ n-1<|f(x)|\leqq n\}$ $(n=1,2,\cdots\cdots)$, we have

$$\sum_{n=1}^{\infty}(n-1)^2\,\delta(\boldsymbol{I}_n)\leqq\|f\|^2\leqq\sum_{n=1}^{\infty}n^2\,\delta(\boldsymbol{I}_n) \tag{3,2}$$

so that we may have

$$n^2\,\delta(\boldsymbol{I}_n)\to 0\quad\text{as}\quad n\to\infty.$$

From the relation (3,2) itself, we directly induce:

THEOREM. *In the canonical space* L, *if* $\|f\|^2>0$ *it must be that*

$$\delta(\boldsymbol{S}_f)>0$$

where S_f *denotes the least support*[3] *of the function* $f(x)$.

The inverse expression of this theorem is: *if* $\delta(\boldsymbol{S}_f)$ *vanishes,* $\|f\|$ *vanishes too.*

4. Transformation of $\partial_\xi \boldsymbol{I}$

If we aim to find any discriminating discussion on the possibility of a space to be transformed to a canonical space, we may not avoid merely hypothetical conditions which will be imposed on the primitive space with no steady reason based on the construction of the space itself. Accordingly, it may be preferable to start with a canonical space from which the given space is to be investigated as transformed.

3 This is the set in which $f(x)$ does not vanish but out of which $f(x)$ vanishes.

First, let us posit the canonical space **L** as imposed by the following group of formulas:

$$\text{(i)}\quad (f, g)=\mathfrak{S} f(x)g(x)\delta(x);$$

$$\text{(ii)}\quad f(x)=\mathfrak{S} f(\xi)\partial_\xi \mathbf{1};$$

$$\text{(iii)}\quad 1=\mathfrak{S}\delta(x).$$

To elucidate that a quantity q may be infinitesimal but non-negative, we will introduce a symbolical expression

$$q \geqq \circledcirc.$$

When q is a non-negative quantity which cannot make any positive quantity by multiplication of any infinity, we will write

$$q=\circledcirc,$$

and when $q \geqq \circledcirc$ but $q \neq \circledcirc$ we will write

$$q>\circledcirc.$$

In this paper, we associate the space **L** with

$$\delta(x)>\circledcirc.$$

Next, let us take a linear transformation T of **L** and write

$$T\partial_\xi \mathbf{1}=\rho_\xi(x),$$

then, applying this on the formula (ii) we have

$$Tf(x)=\underset{\xi}{\mathfrak{S}} f(\xi)\rho_\xi(x).$$

It is evident that $\partial_\xi \mathbf{1} \perp \partial_\eta \mathbf{1}$ for $\xi \neq \eta$ in a canonical space, so that if the transformation T holds this orthogonality as invariant we may go with the formulas:

$$(\partial_\xi \mathbf{1}, \partial_\eta \mathbf{1})=\circledcirc, \quad (\rho_\xi, \rho_\eta)=\circledcirc \quad \text{for} \quad \xi \neq \eta \tag{4,1}$$

$$\|\partial_\xi \mathbf{1}\|=\sqrt{\delta(\xi)}>\circledcirc, \quad \|\rho_\xi\|>\circledcirc.$$

Let us call such a transformation T a *point-wise orhtogonal* transformation, and suppose here T to be so.

On inversion of T, $\partial_\xi \mathbf{1}$ may be expressed in the form

$$\partial_\xi \mathbf{1}=\underset{\eta \in \mathbf{Y}(\xi)}{\mathfrak{S}} c_{\xi\eta}\rho_\eta(x) \tag{4,2}$$

where $Y(\xi)$ denotes the set of η for which $c_{\xi\eta} \neq \circledcirc$. If $\eta' \bar{\in} Y(\xi)$, accrding to (4,1) we have $(\rho_\eta, \rho_\eta{}')=\circledcirc$ for each $\eta \in Y(\xi)$ so that, on account of (4,2), $(\partial_\xi \mathbf{1}, \rho_\eta{}')=\circledcirc$ i.e. we have:

$$Y(\xi) \bar{\ni} \eta' \rightarrow (\partial_\xi \mathbf{1}, \rho_\eta{}')=\circledcirc. \tag{4,3}$$

If $\eta \in \boldsymbol{Y}(\xi)$, according to (4,2) we have

$$(\partial_\xi \boldsymbol{1}, \rho_\eta) = c_{\xi\eta} \|\rho_\eta\|^2. \tag{4,4}$$

In this place, it is very efficient if the symbol ◎ conforms to the rule : if $p \neq$ ◎ and $q \neq$ ◎, then $pq \neq$ ◎ ; let this assumption be called the hypothesis of *algebraic nullity*. If this hypothesis is consistent, (4,4) directly implies $(\partial_\xi \boldsymbol{1}, \rho_\eta) \neq$ ◎, so that we may have:

$$\boldsymbol{Y}(\xi) \ni \eta \rightarrow (\partial_\xi \boldsymbol{1}, \rho_\eta) \neq \text{◎}. \tag{4.5}$$

If $\rho_{\eta_1}(\xi) \neq$ ◎, by the definition of $\partial_\xi \boldsymbol{1}(x)$ we see $(\partial_\xi \boldsymbol{1}, \rho_{\eta_1}) = \rho_{\eta_1}(\xi)\delta(\xi) \neq$ ◎. Then, in regard to (4,3) and (4,5), we see $\eta_1 \in \boldsymbol{Y}(\xi)$. Consequently we have:

THEOREM. *Let the transformation T be point-wise orthogonal in the canonical space* L, *with respect to the symbol ◎ conforms to the hypothesis of algebraic nullity, and $\boldsymbol{Y}(\xi)$ be the set of the values of η for which $c_{\xi\eta} \neq$ ◎ in the expression* (4,2); *then $\boldsymbol{Y}(\xi)$ is the total aggregation of the values of η for which $(\partial_\xi \boldsymbol{1}, \rho_\eta) \neq$ ◎ (i. e. $\partial_\xi \boldsymbol{1} \not\perp \rho_\eta$), and moreover $\boldsymbol{Y}(\xi)$ is the total aggregation of the values of η for which $\rho_\eta(\xi) \neq$ ◎.*

We have the relation

$$\boldsymbol{Y}(\xi) = \sum_0^\infty \boldsymbol{E}_k$$

on the assignations : $\boldsymbol{E}_0 = \{\eta : \delta(\xi) < c^2_{\xi\eta} \|\rho_\eta\|^2 \leqq \infty\}$, and $\boldsymbol{E}_k = \left\{\eta : \frac{\delta(\xi)}{k+1} < c^2_{\xi\eta} \|\rho_\eta\|^2 \leqq \frac{\delta(\xi)}{k}\right\}$ $(k=1,2,\cdots)$. Besides, as

$$\delta(\xi) = \underset{\eta}{\mathfrak{S}}\, c^2_{\xi\eta} \|\rho_\eta\|^2,$$

we directly see that each of $\boldsymbol{E}_k$ consists of a finite number of elements $(k=0, 1,2,\cdots)$. Hence $\boldsymbol{Y}(\xi)$ is found to be a set at most enumerable when

$$\boldsymbol{Y}(\xi) = \sum_0^\infty \boldsymbol{E}_k.$$

This relation may be a trivial one, when $\delta(x)$ is given as a finite positive quantity as well as $c^2_{\xi\eta}$ and $\|\rho_\eta\|^2$, but it may not be generally brought about when $\delta(x)$ is allowed to be infinitesimal. Hence, this relation may be regarded as a sort of equi-measure condition between the systems $\{\partial_\xi \boldsymbol{1}\}$ and $\{\rho_\xi\}$.

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