On Elastic Behavior along Simply Supported Edges of Rectangular Plates

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Abstract

In this paper, the three-dimensional stress problems concerning the rectangular plate with two of the opposite edges simply supported and the other two edges hinged, are dealt with. In order to criticise Kirchhoff's assumption[1~4] concerning the boundary of the plate, the variations of the shearing forces in the vicinity of the simply supported edge are presented here. Along the simply supported edge, both the bending moments and the shearing forces acting on the cross section paralell to the same edge, always vanish by means of the thin plate theory. The results obtained here, however, indicate that they still remain at the edge.

Introduction

In an elastic body subjected to given forces the tractions specified by X_ν, Y_ν, Z_ν where ν denotes the normal to the boundary surfaces, have prescribed values at every point of those surfaces. The equilibrium of the forces at the boundary takes the following forms

$$\left.\begin{aligned} l\sigma_x+m\tau_{xy}+n\tau_{zx} &= X_\nu\ , \\ l\tau_{xy}+m\sigma_y+n\tau_{yz} &= Y_\nu\ , \\ l\tau_{zx}+m\tau_{yz}+n\sigma_z &= Z_\nu\ , \end{aligned}\right\} \tag{1}$$

where σ_x, σ_y, and σ_z denote normal stresses by x, y, and z axes respectively; and τ_{yz}, τ_{zx}, and τ_{xy} denote shearing stresses. And l, m, and n are the direction cosine of the normal to the boundary. It shows us that three boundary conditions should be necessary at each point of surfaces bounding the elastic body. When the plates is thin, the actual distribution of the tractions applied to the edge regarded as a parabolic surface is no practical importance. We represent therefore the tractions applied to the edges by their force and couple-resultants, estimated per unit of length of the edge-line. Let T, S, N denote the components of force, H, G those of the couple: T is a tension, S and N are shearing force tangential and normal to the middle plane, G is a flexial couple, and H a torsional couple.

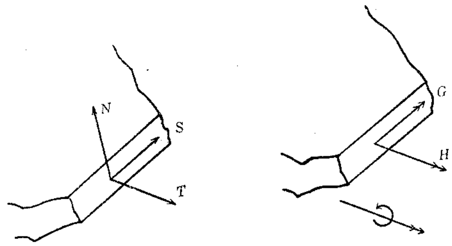

Fig. 1

We also let the stress-resultants and stress-couples belonging to a curve parallel to the edge-line, be denoted $\overline{T}$, $\overline{S}$, $\overline{N}$ and $\overline{G}$, $\overline{H}$. The statical equivalence at the edge-line is expressed as

$$T=\overline{T}\,,\quad S=\overline{S}\,,\quad N=\overline{N}\,,\quad H=\overline{H}\,,\quad G=\overline{G}\,. \tag{2}$$

When the distribution of the stresses by T, S are antisymmetrical* with respect to the middle plane by its thickness, the five conditions mentioned above, are diminished to the three conditions:

$$\left.\begin{aligned} N&=\overline{N}\,,\\ H&=\overline{H}\,,\\ G&=\overline{G}\,.\end{aligned}\right\} \tag{3}$$

This represents the case where the plate is subjected by the equal tractions normal to its upper and lower surface respectively.

In this case a system of two boundary conditions was obtained by Kirchhoff, who set out from a special assumption as to the nature of the strain within the plate, and proceeded by the method of variation of the energy-function. The couple on any finite length might be applied by means of tractions directed at right angles to force- and couple-resultant, estimated per unit of length of the edge-line, would be equivalent to a distribution of shearing force of type N instead of torsional couple of the type H. The required shearing force is easily found to be $-\dfrac{\partial H}{\partial s}$, in which s denotes the length along the edge. The boundary conditions are thus found to be

$$\left.\begin{aligned} N-\frac{\partial H}{\partial s}&=\overline{N}-\frac{\partial \overline{H}}{\partial s}\,,\\ G&=\overline{G}\,.\end{aligned}\right\} \tag{4}$$

These conditions are generally adopted in the theory of plate and shell.

* If the stresses by T, S are symmentrical with respect to the middle plane the plate would be in the state of plane stress.

Components of Stress and Displacement from Three-Dimensional Stress Problems

From the equilibrium condition applied to a cubical element of the elastic medium, equations for the statical equivalence of forces take the well-known forms

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}&=X,\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}&=Y,\\ \frac{\partial\tau_{zx}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}&=Z,\end{aligned}\right\} \tag{5}$$

where X, Y, Z denote the components of the body force per unit volume in x, y, z directions respectively.

The stresses are related to the displacements u, v, and w

$$\left.\begin{aligned}\sigma_x&=(2\mu+\lambda)\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}+\lambda\frac{\partial w}{\partial z},\\ \sigma_y&=\lambda\frac{\partial u}{\partial x}+(2\mu+\lambda)\frac{\partial v}{\partial y}+\lambda\frac{\partial w}{\partial z},\\ \sigma_z&=\lambda\frac{\partial u}{\partial x}+\lambda\frac{\partial v}{\partial y}+(2\mu+\lambda)\frac{\partial w}{\partial z},\\ \tau_{xy}&=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right),\\ \tau_{yz}&=\mu\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right),\\ \tau_{zx}&=\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right),\end{aligned}\right\} \tag{6}$$

where u, v, w are the components of displacement in x, y, z directions respectively; and μ, λ are Lame's constants.

As the author have already described in the previous paper[5], Eq. (5) and Eq. (6) by means of finite Fourier transforms, yield the results without the body forces X, Y, and Z, as follows:

$$\begin{aligned}u=\sum_m\sum_n\frac{c^2m}{2a\gamma_{mn}}\Big[&K_{mn}\left\{-\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}\zeta)\right\}\\ &+K'_{mn}\left\{-\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\gamma_{mn}\zeta)\right\}\Big]\cos\frac{m\pi}{a}x\end{aligned}$$

$$
\begin{aligned}
&\times \sin\frac{n\pi}{b}y+\sum_m\sum_n\frac{b^3mr}{2ca\beta_{mr}^2}\frac{\mu+\lambda}{2\mu+\lambda}\Big\{G_{mr}P^{(1)}(\beta_{mr}\eta)+G'_{mr}P^{(2)}(\beta_{mr}\eta)\Big\}\\
&\times \cos\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_n\sum_r\frac{ra^2}{2c\alpha_{nr}}\Big[E_{nr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\alpha_{nr}\xi)\\
&+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\alpha_{nr}\xi)\Big\}+E'_{nr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\\
&\times \sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z+\sum_m\sum_r\frac{b}{2}\Big[H_{mr}\Big\{Q^{(1)}(\beta_{mr}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{1}{\beta_{mr}^2}\Big(\frac{mb}{a}\Big)^2\\
&\times P^{(1)}(\beta_{mr}\eta)\Big\}+H'_{mr}\Big\{Q^{(2)}(\beta_{mr}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{1}{\beta_{mr}^2}\Big(\frac{mb}{a}\Big)^2P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\\
&\times \cos\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_n\sum_r\frac{a^2n}{2b\alpha_{nr}}\Big[J_{nr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times \Psi^{(1)}(\alpha_{nr}\xi)\Big\}+J'_{nr}\Big\{\frac{\mu}{2\mu+\lambda}P^{(2)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\\
&\times \cos\frac{r\pi}{c}z-\sum_m\sum_r\frac{b^3m}{4a\beta_{mr}^2}\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Big\{B_{mr}P^{(1)}(\beta_{mr}\eta)+B'_{mr}P^{(2)}(\beta_{mr}\eta)\Big\}\\
&\times \cos\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_n\frac{c^4mn}{4ab(\gamma_{mn})^3}\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Big[F_{mn}\Big\{\phi^{(1)}(\gamma_{mn}\zeta)\\
&+\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+F'_{mn}\Big\{\phi^{(2)}(\gamma_{mn}\zeta)+(\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y\\
&+\sum_n\sum_r\frac{a^2}{4\alpha_{nr}}\Big[A_{nr}\Big\{\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\phi^{(1)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Psi^{(1)}(\alpha_{nr}\xi)\Big\}\\
&+A'_{nr}\Big\{\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\phi^{(2)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Psi^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z\\
&+\sum_m\sum_n\frac{c^2}{4\gamma_{mn}\mu}\Big[D_{mn}\Big\{\Big(2-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma_{mn}^2}\Big)\phi^{(1)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times \frac{c^2m^2}{a^2\gamma_{mn}^2}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+D'_{mn}\Big\{\Big(2-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma_{mn}^2}\Big)\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times \frac{c^2m^2}{a^2\gamma_{mn}^2}\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y+\frac{a^2}{8\mu}\sum_n\frac{b}{an}\Big[A_{n0}\Big\{\frac{3\mu+\lambda}{2\mu+\lambda}\\
&\times \phi^{(2)}(\alpha_{n0}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\alpha_{n0}\xi)\Big\}+A'_{n0}\Big\{\frac{3\mu+\lambda}{2\mu+\lambda}\phi^{(2)}(\alpha_{n0}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times \Psi^{(2)}(\alpha_{n0}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y+\frac{b}{4}\sum_m\Big[H_{m0}\Big\{Q^{(1)}(\beta_{m0}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\beta_{m0}\eta)\Big\}\\
&+H'_{m0}\Big\{Q^{(2)}(\beta_{m0}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\beta_{m0}\eta)\Big\}\Big]\cos\frac{m\pi}{a}x+\frac{a}{4}\sum_n\Big[J_{n0}\Big\{\frac{\mu}{2\mu+\lambda}\\
&\times \phi^{(1)}(\alpha_{n0}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\alpha_{n0}\xi)\Big\}+J'_{n0}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\alpha_{n0}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times \Psi^{(2)}(\alpha_{n0}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y-\sum_m\frac{ab}{8m}\frac{1}{2\mu+\lambda}\Big\{B_{m0}P^{(1)}(\beta_{m0}\eta)+B'_{m0}
\end{aligned}
\qquad (7)
$$

$$\times P^{(2)}(\beta_{m0}\eta)\Big\}\cos\frac{m\pi}{a}x+\sum_n\frac{bc}{4n}\Big\{D_{0n}\phi^{(1)}(\gamma_{0n}\zeta)+D'_{0n}\phi^{(2)}(\gamma_{0n}\zeta)\Big\}\sin\frac{n\pi}{b}y$$
$$-\frac{b}{4}\sum_r\Big\{J_{0r}Q^{(1)}(\beta_{0r}\eta)+J'_{0r}Q^{(2)}\beta_{0r}\eta)\Big\}\cos\frac{r\pi}{c}z-\frac{b}{8}\Big\{J_{00}(1-2\eta)+J'_{00}\Big\}$$
$$m,\ n,\ r=1,\ 2,\ 3,\ 4,\cdots\cdots,$$

$$v=\sum_m\sum_n\frac{c^2n}{2b\gamma_{mn}}\Big[K_{mn}\Big\{-\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}$$
$$+K'_{mn}\Big\{-\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y$$
$$+\sum_n\sum_r\frac{a^3nr}{2cba_{nr}^2}\frac{\mu+\lambda}{2\mu+\lambda}\Big\{E_{nr}P^{(1)}(\alpha_{nr}\xi)+E'_{nr}P^{(2)}(\alpha_{nr}\xi)\Big\}\cos\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z$$
$$-\sum_n\sum_r\frac{a^3n}{4b\alpha_{nr}^2}\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Big\{A_{nr}P^{(1)}(\alpha_{nr}\xi)+A'_{nr}P^{(2)}(\alpha_{nr}\xi)\Big\}\cos\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z$$
$$-\sum_m\sum_r\frac{rb^2}{2c\beta_{mr}}\Big[G_{mr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{mr}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\beta_{mr}\eta)\Big\}$$
$$+G'_{mr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\beta_{mr}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z$$
$$+\sum_n\sum_r\frac{a}{2}\Big[J_{nr}\Big\{Q^{(1)}(\alpha_{nr}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{n^2a^2}{b^2\alpha_{nr}^2}P^{(1)}(\alpha_{nr}\xi)\Big\}+J'_{nr}\Big\{Q^{(2)}(\alpha_{nr}\xi)$$
$$-\frac{\mu+\lambda}{2\mu+\lambda}\frac{n^2a^2}{b^2\alpha_{nr}^2}P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\cos\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_n\frac{b^2m}{2a\beta_{mr}}$$
$$\times\Big[H_{mr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{mr}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\beta_{mr}\eta)\Big\}+H'_{mr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\beta_{mr}\eta)$$
$$+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_n\frac{c^4mn}{4ab(\gamma_{mn}^3)}$$
$$\times\frac{\mu+\lambda}{2\mu+\lambda}\Big[D_{mn}\Big\{\phi^{(1)}(\gamma_{mn}\zeta)+\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+D'_{mn}\Big\{\phi^{(1)}(\gamma_{mn}\zeta)+\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}\Big]$$
$$\times\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y+\sum_m\sum_r\frac{b^2}{4\beta_{mr}}\Big[B_{mr}\Big\{\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\phi^{(1)}(\beta_{mr}\eta)\qquad(8)$$
$$+\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Psi^{(1)}(\beta_{mr}\eta)\Big\}+B'_{mr}\Big\{\frac{3\mu+\lambda}{\mu(2\mu+\lambda)}\phi^{(2)}(\beta_{mr}\eta)+\frac{\mu+\lambda}{\mu(2\mu+\lambda)}$$
$$\times\Psi^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{\gamma\pi}{c}z+\sum_m\sum_n\frac{c^2}{4\gamma_{mn}\mu}\Big[F_{mn}\Big\{\Big(2-\frac{\mu+\lambda}{2\mu+\lambda}$$
$$\times\frac{c^2n^2}{b^2\gamma_{mn}^2}\Big)\phi^{(1)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma_{mn}^2}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+F'_{mn}\Big\{\Big(2-\frac{\mu+\lambda}{2\mu+\lambda}$$
$$\times\frac{c^2n^2}{b^2\gamma_{mn}^2}\Big)\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma_{mn}^2}\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y$$
$$-\frac{b^2}{8\mu}\sum_m\frac{a}{bm}\Big[B_{m0}\Big\{\frac{3\mu+\lambda}{2\mu+\lambda}\phi^{(1)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\beta_{m0}\eta)\Big\}+B'_{m0}$$

$$\times \Big\{\frac{3\mu+\lambda}{2\mu+\lambda}\phi^{(2)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\psi^{(2)}(\beta_{m0}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x+\frac{a}{4}\sum_n\Big[J_{n0}$$

$$\times\Big\{Q^{(1)}(\alpha_{n0}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\alpha_{n0}\xi)\Big\}+J'_{n0}\Big\{Q^{(2)}(\alpha_{n0}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\alpha_{n0}\xi)\Big\}\Big]$$

$$\times\cos\frac{n\pi}{b}y+\frac{b}{4}\sum_m\Big[H_{m0}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\psi^{(1)}(\beta_{m0}\eta)\Big\}$$

$$+H'_{m0}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\psi^{(2)}(\beta_{m0}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x-\sum_n\frac{ba}{8n}$$

$$\times\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Big\{A_{n0}P^{(1)}(\alpha_{n0}\xi)+A'_{n0}P^{(2)}(\alpha_{n0}\xi)\Big\}\cos\frac{n\pi}{b}y+\sum_m\frac{ac}{4m}$$

$$\times\Big\{F_{m0}\phi^{(1)}(\gamma_{m0}\zeta)+F'_{m0}\phi^{(2)}(\gamma_{m0}\zeta)\Big\}\sin\frac{m\pi}{a}x-\frac{a}{4}\sum_r\Big\{H_{r0}Q^{(1)}(\beta_{0r}\eta)$$

$$+H'_{0r}Q^{(2)}(\beta_{0r}\eta)\Big\}\cos\frac{r\pi}{c}z-\frac{a}{8}\Big\{H_{00}(1-2\xi)+H'_{00}\Big\}$$

$$m,\ n,\ r=1,\ 2,\ 3,\ 4,\cdots\cdots,$$

$$w=-\sum_m\sum_n\frac{c}{2}\Big[K_{mn}\Big\{Q^{(1)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\gamma_{mn}\zeta)\Big\}+K'_{mn}\Big\{Q^{(2)}(\gamma_{mn}\zeta)$$

$$+\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y+\sum_n\sum_r\frac{a}{2}\Big[E_{nr}\Big\{Q^{(1)}(\alpha_{nr}\xi)$$

$$-\frac{\mu+\lambda}{2\mu+\lambda}\frac{r^2a^2}{c^2\alpha_{nr}^2}P^{(1)}(\alpha_{nr}\xi)\Big\}+E'_{nr}\Big\{Q^{(2)}(\alpha_{nr}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{r^2a^2}{c^2\alpha_{nr}^2}P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]$$

$$\times\sin\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{c}z+\sum_m\sum_r\frac{b}{2}\Big[G_{mr}\Big\{Q^{(1)}(\beta_{mr}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{r^2b^2}{c^2\beta_{mr}^2}$$

$$\times P^{(1)}(\beta_{mr}\eta)\Big\}+G'_{mr}\Big\{Q^{(2)}(\beta_{mr}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{r^2b^2}{c^2\beta_{mr}^2}P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x$$

$$\times\sin\frac{r\pi}{c}z+\sum_n\sum_r\frac{a^3r}{4c\alpha_{nr}^2}\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Big\{A_{nr}P^{(1)}(\alpha_{nr}\xi)+A'_{nr}P^{(2)}(\alpha_{nr}\xi)\Big\}$$

$$\times\sin\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{c}z+\sum_m\sum_n\frac{c^3m}{4a\gamma_{mn}^2}\frac{\mu+\lambda}{2\mu+\lambda}\Big\{D_{mn}P^{(1)}(\gamma_{mn}\zeta)\qquad (9)$$

$$+D'_{mn}P^{(2)}(\gamma_{mn}\zeta)\Big\}\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y-\sum_n\sum_r\frac{a^3nr}{2bc\alpha_{nr}^2}\frac{\mu+\lambda}{2\mu+\lambda}$$

$$\times\Big\{J_{nr}P^{(1)}(\alpha_{nr}\xi)+J'_{nr}P^{(2)}(\alpha_{nr}\xi)\Big\}\sin\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{c}z$$

$$+\sum_m\sum_n\frac{b^3r}{4c\beta_{mr}^2}\frac{\mu+\lambda}{\mu(2\mu+\lambda)}\Big\{B_{mr}P^{(1)}(\beta_{mr}\eta)+B'_{mr}P^{(2)}(\beta_{mr}\eta)\Big\}\sin\frac{m\pi}{a}x$$

$$\times\sin\frac{r\pi}{c}z+\sum_m\sum_n\frac{c^3n}{4b\gamma_{mn}^2}\frac{\mu+\lambda}{2\mu+\lambda}\Big\{F_{mn}P^{(1)}(\gamma_{mn}\zeta)+F'_{mn}P^{(2)}(\gamma_{mn}\zeta)\Big\}$$

$$\times\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y+\sum_m\sum_r\frac{b^3mr}{2ac\beta_{mr}^2}\frac{\mu+\lambda}{2\mu+\lambda}\Big\{H_{mr}P^{(1)}(\beta_{mr}\eta)$$

$$+ H'_{mr} P^{(2)}(\beta_{mr}\eta)\Big\} \sin\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z\,,$$
$$m,\ n,\ r = 1,\ 2,\ 3,\ 4,\cdots\cdots,$$

The dilatation is, therefore, written in the form:

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = e &= \sum_m\sum_n \frac{\mu}{2\mu+\lambda}\gamma_{mn}\Big\{K_{mn}\phi^{(1)}(\gamma_{mn}\zeta) \\
&+ K'_{mn}\phi^{(2)}(\gamma_{mn}\zeta)\Big\} \sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y + \sum_n\sum_r \frac{\mu}{2\mu+\lambda}\frac{ar}{c}\Big\{E_{nr}Q^{(1)}(\alpha_{nr}\xi) \\
&+ E'_{nr}Q^{(2)}(\alpha_{nr}\xi)\Big\} \sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z + \sum_m\sum_r \frac{\mu}{2\mu+\lambda}\frac{br}{c}\Big\{G_{mr}Q^{(1)}(\beta_{mr}\eta) \\
&+ G'_{mr}Q^{(2)}(\beta_{mr}\eta)\Big\} \sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z - \sum_m\sum_r \frac{bm}{a}\frac{\mu}{2\mu+\lambda}\Big\{H_{mr}Q^{(1)}(\beta_{mr}\eta) \\
&+ H'_{mr}Q^{(2)}(\beta_{mr}\eta)\Big\} \sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z - \sum_n\sum_r \frac{an}{b}\frac{\mu}{2\mu+\lambda}\Big\{J_{nr}Q^{(1)}(\alpha_{nr}\xi) \\
&+ J'_{nr}Q^{(2)}(\alpha_{nr}\xi)\Big\} \sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z - \sum_m\sum_r \frac{b}{2\mu+\lambda}\Big\{B_{mr}Q^{(1)}(\beta_{mr}\eta) \\
&+ B'_{mr}Q^{(2)}(\beta_{mr}\eta)\Big\} \sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z - \sum_n\sum_r \frac{a}{2\mu+\lambda}\Big\{A_{nr}Q^{(1)}(\alpha_{nr}\xi) \\
&+ A'_{nr}Q^{(2)}(\alpha_{nr}\xi)\Big\} \sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z - \sum_n\sum_r \frac{c^2 n}{2b\gamma_{mn}}\frac{1}{2\mu+\lambda} \\
&\times \Big\{F_{mn}\phi^{(1)}(\gamma_{mn}\zeta) + F'_{mn}\phi^{(2)}(\gamma_{mn}\zeta)\Big\} \sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y - \sum_m\sum_n \frac{c^2 m}{2a\gamma_{mn}} \\
&\times \frac{1}{2\mu+\lambda}\Big\{D_{mn}\phi^{(1)}(\gamma_{mn}\zeta) + D'_{mn}\phi^{(2)}(\gamma_{mn}\zeta)\Big\} \sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z \\
&- \sum_n \frac{a}{4}\frac{1}{2\mu+\lambda}\Big\{A_{n0}Q^{(1)}(\alpha_{n0}\xi) + A'_{n0}Q^{(2)}(\alpha_{n0}\xi)\Big\} \sin\frac{n\pi}{b}y \\
&- \sum_m \frac{b}{4}\frac{1}{2\mu+\lambda}\Big\{B_{m0}Q^{(1)}(\beta_{m0}\eta) + B'_{m0}Q^{(2)}(\beta_{m0}\eta)\Big\} \sin\frac{m\pi}{a}x \\
&- \sum_m \frac{bm}{2a}\frac{\mu}{2\mu+\lambda}\Big\{H_{m0}Q^{(1)}(\beta_{m0}\eta) + H'_{m0}Q^{(2)}(\beta_{m0}\eta)\Big\} \sin\frac{m\pi}{a}x \\
&- \sum_n \frac{an}{2b}\frac{\mu}{2\mu+\lambda}\Big\{J_{n0}Q^{(1)}(\alpha_{n0}\xi) + J'_{n0}Q^{(2)}(\alpha_{n0}\xi)\Big\} \sin\frac{n\pi}{b}y.
\end{aligned} \tag{10}$$

Then, the relations (6) yield the components of stress as follows:

$$\begin{aligned}
\sigma_x &= 2\mu\frac{\partial u}{\partial x} + \lambda e \\
&= \sum_m\sum_n\Big[K_{mn}\Big\{\Big(\frac{\mu^2}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma_{mn}} + \frac{\lambda\mu}{2\mu+\lambda}\gamma_{mn}\Big)\phi^{(1)}(\gamma_{mn}\zeta) \\
&- \frac{\mu(\mu+\lambda)}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma^2_{mn}}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\} + K'_{mn}\Big\{\Big(\frac{\mu^2}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma^2_{mn}} + \frac{\lambda\mu}{2\mu+\lambda}\gamma_{mn}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\times \phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\,\frac{c^2m^2}{a^2\gamma_{mn}^2}\,\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y\\
&+\sum_m\sum_r\frac{br}{c}\Big[G_{mr}\Big\{\frac{\lambda\mu}{2\mu+\lambda}Q^{(1)}(\beta_{mr}\eta)-\frac{b^2m^2}{a^2\beta_{mr}^2}\,\frac{\mu(\lambda+\mu)}{2\mu+\lambda}P^{(1)}(\beta_{mr}\eta)\Big\}\\
&+G'_{mr}\Big\{\frac{\lambda\mu}{2\mu+\lambda}Q^{(2)}(\beta_{mr}\eta)-\frac{b^2m^2}{a^2\beta_{mr}^2}\,\frac{\mu(\lambda+\mu)}{2\mu+\lambda}P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\\
&\times\cos\frac{r\pi}{c}z+\sum_n\sum_r\frac{ra}{c}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{E_{nr}P^{(1)}(\alpha_{nr}\xi)+E'_{nr}P^{(2)}(\alpha_{nr}\xi)\Big\}\\
&\times\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_r\frac{bm}{a}\Big[H_{mr}\Big\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(1)}(\beta_{mr}\eta)\\
&+\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}\,\frac{b^2m^2}{a^2\beta_{mr}^2}P^{(1)}(\beta_{mr}\eta)\Big\}+H'_{mr}\Big\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(2)}(\beta_{mr}\eta)\\
&+\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}\,\frac{b^2m^2}{a^2\beta_{mr}^2}P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_r\frac{an}{b}\\
&\times\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{J_{nr}P^{(1)}(\alpha_{nr}\xi)+J'_{nr}P^{(2)}(\alpha_{nr}\xi)\Big\}\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_r\\
&\times\frac{b}{2}\Big[B_{mr}\Big\{\frac{\lambda}{2\mu+\lambda}Q^{(1)}(\beta_{mr}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{b^2m^2}{a^2\beta_{mr}^2}P^{(1)}(\beta_{mr}\eta)\Big\}+B'_{mr}\\
&\times\frac{\lambda}{2\mu+\lambda}Q^{(2)}(\beta_{mr}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{b^2m^2}{a^2\beta_{mr}^2}P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z\\
&-\sum_m\sum_n\frac{c^2n}{2b\gamma_{mn}}\Big[F_{mn}\Big\{\Big(\frac{\lambda}{2\mu+\lambda}+\frac{\lambda(\mu+\lambda)}{\mu(2\mu+\lambda)}\,\frac{c^2m^2}{a^2\gamma_{mn}^2}\Big)\phi^{(1)}(\gamma_{mn}\zeta)\\
&-\frac{c^2m^2}{a^2\gamma_{mn}^2}\,\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+F'_{mn}\Big\{\frac{\lambda}{2\mu+\lambda}+\frac{\lambda(\mu+\lambda)}{\mu(2\mu+\lambda)}\,\frac{c^2m^2}{a^2\gamma_{mn}^2}\Big)\\
&\times\phi^{(2)}(\gamma_{mn}\zeta)-\frac{c^2m^2}{a^2\gamma_{mn}^2}\,\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y\\
&-\sum_n\sum_r\frac{a}{2}\Big[A_{nr}\Big\{Q^{(1)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\alpha_{nr}\xi)\Big\}+A'_{nr}\Big\{Q^{(2)}(\alpha_{nr}\xi)\\
&+\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_n\frac{c^2m}{2a\gamma_{mn}}\Big[D_{mn}\\
&\times\Big\{\Big(\frac{4\mu+3\lambda}{2\mu+\lambda}-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{c^2m^2}{a^2\gamma_{mn}^2}\Big)\times\phi^{(1)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{c^2m^2}{a^2\gamma_{mn}^2}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}\\
&-D'_{mn}\Big\{\Big(\frac{4\mu+3\lambda}{2\mu+\lambda}-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{c^2m^2}{a^2\gamma_{mn}^2}\Big)\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{c^2m^2}{a^2\gamma_{mn}^2}\\
&\times\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\times\sin\frac{n\pi}{b}y-\sum_n\frac{a}{4}\Big[A_{n0}\Big\{Q^{(1)}(\alpha_{n0}\xi)\\
&+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\alpha_{n0}\xi)\Big\}+A'_{n0}\Big\{Q^{(2)}(\alpha_{n0}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\alpha_{n0}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\\
&-\sum_m\frac{bm}{2a}\Big[H_{m0}\Big\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(1)}(\beta_{m0}\eta)+\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}\times P^{(1)}(\beta_{m0}\eta)\Big\}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&+H'_{m0}\left\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(2)}(\beta_{m0}\eta)+\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}P^{(2)}(\beta_{m0}\eta)\right\}\Bigg]\sin\frac{m\pi}{a}x\\
&-\sum_{n}\frac{an}{2b}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\left\{J_{n0}P^{(1)}(\alpha_{n0}\xi)+J'_{n0}P^{(2)}(\alpha_{n0}\xi)\right\}\sin\frac{n\pi}{b}y\\
&-\sum_{m}\frac{b}{4}\Bigg[B_{m0}\left\{\frac{\lambda}{2\mu+\lambda}Q^{(1)}(\beta_{m0}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\beta_{m0}\eta)\right\}\\
&+B'_{m0}\left\{\frac{\lambda}{2\mu+\lambda}Q^{(2)}(\beta_{m0}\eta)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\beta_{m0}\eta)\right\}\Bigg]\sin\frac{m\pi}{a}x\,,
\end{aligned}
$$

$$
\begin{aligned}
\sigma_y&=2\mu\frac{\partial v}{\partial y}+\lambda e=\sum_{m}\sum_{n}\Bigg[K_{mn}\left\{\left(\frac{\mu^2}{2\mu+\lambda}\,\frac{c^2n^2}{b^2\gamma_{mn}}+\frac{\mu\lambda}{2\mu+\lambda}\gamma_{mn}\right)\phi^{(1)}(\gamma_{mn}\zeta)\right.\\
&-\left.\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\,\frac{c^2n^2}{b^2\gamma_{mn}}\psi^{(1)}(\gamma_{mn}\zeta)\right\}+K'_{mn}\left\{\left(\frac{\mu^2}{2\mu+\lambda}\,\frac{c^2n^2}{b^2\gamma_{mn}}+\frac{\mu\lambda}{2\mu+\lambda}\gamma_{mn}\right.\right.\\
&\times\left.\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\,\frac{c^2n^2}{b^2\gamma_{mn}}\psi^{(2)}(\gamma_{mn}\zeta)\right\}\Bigg]\sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}y\\
&-\sum_{n}\sum_{r}\frac{ar}{c}\Bigg[E_{nr}\left\{\frac{a^2n^2}{b^2\alpha_{nr}^2}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}P^{(1)}(\alpha_{nr}\xi)-\frac{\mu\lambda}{2\mu+\lambda}Q^{(1)}(\alpha_{nr}\xi)\right\}\\
&+E'_{nr}\left\{\frac{a^2n^2}{b^2\alpha_{nr}^2}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}P^{(2)}(\alpha_{nr}\xi)-\frac{\mu\lambda}{2\mu+\lambda}Q^{(2)}(\alpha_{nr}\xi)\right\}\Bigg]\sin\frac{n\pi}{b}y\\
&\times\cos\frac{r\pi}{c}z+\sum_{m}\sum_{r}\frac{rb}{c}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\times\left\{G_{mr}P^{(1)}(\beta_{mr}\eta)+G'_{mr}P^{(2)}(\beta_{mr}\eta)\right\}\\
&\times\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z+\sum_{n}\sum_{r}\frac{an}{b}\Bigg[J_{nr}\left\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(1)}(\alpha_{nr}\xi)\right.\\
&+\left.\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}\,\frac{a^2n^2}{b^2\alpha_{nr}^2}P^{(1)}(\alpha_{nr}\xi)\right\}+J'_{nr}\left\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(2)}(\alpha_{nr}\xi)\right.\\
&+\left.\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}\,\frac{a^2n^2}{b^2\alpha_{nr}^2}P^{(2)}(\alpha_{nr}\xi)\right\}\Bigg]\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z-\sum_{n}\sum_{r}\frac{am}{b}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\\
&\times\left\{H_{mr}P^{(1)}(\beta_{mr}\eta)+H'_{mr}P^{(2)}(\beta_{mr}\eta)\right\}\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_{n}\sum_{r}\frac{a}{2}\\
&\times\Bigg[A_{nr}\left\{\frac{\lambda}{2\mu+\lambda}Q^{(1)}(\alpha_{nr}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{a^2n^2}{b^2\alpha_{nr}^2}P^{(1)}(\alpha_{nr}\xi)\right\}+A'_{nr}\\
&\times\left\{\frac{\lambda}{2\mu+\lambda}Q^{(2)}(\alpha_{nr}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}\,\frac{a^2n^2}{b^2\alpha_{nr}^2}P^{(2)}(\alpha_{nr}\xi)\right\}\Bigg]\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z\\
&-\sum_{m}\sum_{r}\frac{c^2m}{2a\gamma_{mn}}\Bigg[D_{mn}\left\{\left(\frac{\lambda}{2\mu+\lambda}+\frac{\lambda(\mu+\lambda)}{\mu(2\mu+\lambda)}\,\frac{c^2n^2}{b^2\gamma_{mn}^2}\right)\phi^{(1)}(\gamma_{mn}\zeta)-\frac{c^2n^2}{b^2\gamma_{mn}^2}\right.\\
&\times\left.\frac{\mu+\lambda}{2\mu+\lambda}\psi^{(1)}(\gamma_{mn}\zeta)\right\}+D'_{mn}\left\{\left(\frac{\lambda}{2\mu+\lambda}+\frac{\lambda(\mu+\lambda)}{\mu(2\mu+\lambda)}\,\frac{c^2n^2}{b^2\gamma_{mn}^2}\right)\phi^{(2)}(\gamma_{mn}\zeta)\right.\\
&-\left.\frac{c^2n^2}{b^2\gamma_{mn}^2}\,\frac{\mu+\lambda}{2\mu+\lambda}\psi^{(2)}(\gamma_{mn}\zeta)\right\}\Bigg]\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y-\sum_{m}\sum_{r}\frac{b}{2}\\
&\times\Bigg[B_{mr}\left\{Q^{(1)}\beta_{mr}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\beta_{mr}\eta)\right\}+B'_{mr}\left\{Q^{(2)}(\beta_{mr}\eta)\right.
\end{aligned}
\tag{12}
$$

$$
\left.\begin{aligned}
&+\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_m\sum_n\frac{c^2m}{2b\gamma_{mn}}\Big[F_{mn}\\
&\times\Big\{\Big(\frac{4\mu+3\lambda}{2\mu+\lambda}-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma^2_{mn}}\Big)\phi^{(1)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma^2_{mn}}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}\\
&+F'_{mn}\Big\{\Big(\frac{4\mu+3\lambda}{2\mu+\lambda}-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n_2}{b^2\gamma^2_{mn}}\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma^2_{mn}}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}\Big]\\
&\times\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y-\sum_m\frac{b}{4}\Big[B_{m0}\Big\{Q^{(1)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\beta_{m0}\eta)\Big\}\\
&+B'_{m0}\Big\{Q^{(2)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\beta_{m0}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x-\sum_n\frac{an}{2b}\Big[J_{n0}\\
&\times\Big\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(1)}(\alpha_{n0}\xi)+\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}P^{(1)}(\alpha_{n0}\xi)\Big\}+J'_{n0}\Big\{\frac{\mu(3\mu+\lambda)}{2\mu+\lambda}Q^{(2)}(\alpha_{n0}\xi)\\
&+\frac{\lambda(\mu+\lambda)}{2\mu+\lambda}P^{(2)}(\alpha_{n0}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y-\sum_m\frac{bm}{2a}\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{H_{m0}P^{(1)}(\beta_{m0}\eta)\\
&+H'_{m0}P^{(2)}(\beta_{m0}\eta)\Big\}\sin\frac{m\pi}{a}x-\sum_n\frac{a}{4}\Big[A_{n0}\Big\{\frac{\lambda}{2\mu+\lambda}Q^{(1)}(\alpha_{n0}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times P^{(1)}(\alpha_{n0}\xi)\Big\}+A'_{n0}\Big\{\frac{\lambda}{2\mu+\lambda}Q^{(2)}(\alpha_{n0}\xi)-\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\alpha_{n0}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y,
\end{aligned}\right.
$$

$$
\left.\begin{aligned}
\sigma_z&=2\mu\frac{\partial w}{\partial z}+\lambda e=\sum_m\sum_n\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\gamma_{mn}\Big[K_{mn}\Big\{\phi^{(1)}(\gamma_{mn}\zeta)+\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}\\
&+K'_{mn}\Big\{\phi^{(2)}(\gamma_{mn}\zeta)+\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y+\sum_n\sum_r\frac{ar}{c}\\
&\times\Big[E_{nr}\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}Q^{(1)}(\alpha_{nr}\xi)-\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\frac{r^2a^2}{c^2\alpha^2_{nr}}P^{(1)}(\alpha_{nr}\xi)\Big\}+E'_{nr}\\
&\times\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}Q^{(2)}(\alpha_{nr}\xi)-\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\frac{r^2a^2}{c^2\alpha^2_{nr}}P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z\\
&+\sum_m\sum_r\frac{br}{c}\Big[G_{mr}\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}Q^{(1)}(\beta_{mr}\eta)-\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\\
&\times\frac{r^2b^2}{c^2\beta^2_{mr}}P^{(1)}(\beta_{mr}\eta)\Big\}+G'_{mr}\Big\{\frac{2\mu(\mu+\lambda)}{2\mu+\lambda}Q^{(2)}(\beta_{mr}\eta)-\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\frac{r^2b^2}{c^2\beta^2_{mr}}\\
&\times P^{(1)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z-\sum_n\sum_r\frac{a}{2}\Big[A_{nr}\Big\{\frac{\lambda}{2\mu+\lambda}Q^{(1)}(\alpha_{nr}\xi)\\
&-\frac{a^2r^2}{c^2}\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\alpha_{nr}\xi)\Big\}+A'_{nr}\Big\{\frac{\lambda}{2\mu+\lambda}Q^{(2)}(\alpha_{nr}\xi)-\frac{a^2r^2}{c^2}\frac{\mu+\lambda}{2\mu+\lambda}\\
&\times P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z+\sum_m\sum_n\frac{c^2m}{4a\gamma_{mn}}\Big[D_{mn}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\gamma_{mn}\zeta)\\
&-\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+D'_{mn}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\\
&\times\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y-\sum_m\sum_n\frac{an}{b}\Big[J_{nr}\Big\{\frac{\mu\lambda}{2\mu+\lambda}Q^{(1)}(\alpha_{nr}\xi)+\frac{r^2a^2}{c^2}
\end{aligned}\right\}\quad(13)
$$

$$\times \frac{\mu(\mu+\lambda)}{2\mu+\lambda} P^{(1)}(\alpha_{nr}\xi)\Big\} + J'_{nr}\Big\{\frac{\mu\lambda}{2\mu+\lambda} Q^{(2)}(\alpha_{nr}\xi) + \frac{r^2a^2}{c^2}\frac{\mu(\mu+\lambda)}{2\mu+\lambda}$$

$$\times P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z - \sum_m\sum_r \frac{b}{2}\Big[B_{mr}\Big\{\frac{\lambda}{2\mu+\lambda} Q^{(1)}(\beta_{mr}\eta)$$

$$+ \frac{b^2r^2}{c^2}\frac{\mu+\lambda}{2\mu+\lambda} P^{(1)}(\beta_{mr}\eta)\Big\} + B'_{mr}\Big\{\frac{\lambda}{2\mu+\lambda} Q^{(2)}(\beta_{mr}\eta) + \frac{b^2r^2}{c^2}\frac{\mu+\lambda}{2\mu+\lambda}$$

$$\times P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z + \sum_m\sum_n \frac{c^2n}{4b\gamma_{mn}}\Big[F_{mn}\Big\{\frac{\mu}{2\mu+\lambda}$$

$$\times \phi^{(1)}(\gamma_{mn}\zeta) - \frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\} - F'_{mn}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\gamma_{mn}\zeta) - \frac{\mu+\lambda}{2\mu+\lambda}$$

$$\times \Psi^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y - \sum_m\sum_r \frac{bm}{a}\Big[H_{mr}\Big\{\frac{\mu\lambda}{2\mu+\lambda}$$

$$\times Q^{(1)}(\beta_{mr}\eta) + \frac{r^2b^2}{c^2\beta_{mr}^2}\frac{\mu(\mu+\lambda)}{2\mu+\lambda} P^{(1)}(\beta_{mr}\eta)\Big\} + H'_{mr}\Big\{\frac{\mu\lambda}{2\mu+\lambda} Q^{(2)}(\beta_{mr}\eta)$$

$$+ \frac{b^2r^2}{c^2\beta_{mr}^2}\frac{\mu(\mu+\lambda)}{2\mu+\lambda} P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z - \sum_n \frac{a}{4}\frac{\lambda}{2\mu+\lambda}$$

$$\times \Big\{A_{n0}Q^{(1)}(\alpha_{n0}\xi) + A'_{n0}Q^{(2)}(\alpha_{n0}\xi)\Big\}\sin\frac{n\pi}{b}y - \sum_n \frac{an}{2b}\frac{\lambda}{2\mu+\lambda}\Big\{J_{n0}Q^{(1)}(\alpha_{n0}\xi)$$

$$+ J'_{n0}Q^{(2)}(\alpha_{n0}\xi)\Big\}\sin\frac{n\pi}{b}y - \sum_m \frac{bm}{2a}\frac{\mu\lambda}{2\mu+\lambda}\Big\{H_{m0}Q^{(1)}(\beta_{m0}\eta)$$

$$+ H'_{m0}Q^{(2)}(\beta_{m0}\eta)\Big\}\sin\frac{m\pi}{a}x - \sum_m \frac{b}{4}\frac{\lambda}{2\mu+\lambda}\Big\{B_{m0}Q^{(1)}(\beta_{m0}\eta)$$

$$+ B'_{m0}Q^{(2)}(\beta_{m0}\eta)\Big\}\sin\frac{m\pi}{a}x,$$

$$\tau_{yz} = \mu\Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\Big) = -\sum_m\sum_n \frac{cn}{b}\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{K_{mn}P^{(1)}(\gamma_{mn}\zeta)$$

$$+ K'_{mn}P^{(2)}(\gamma_{mn}\zeta)\Big\}\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y - \sum_m\sum_r \frac{\mu\beta_{mr}}{2}\Big[G_{mr}\Big\{\phi^{(1)}(\beta_{mr}\eta)$$

$$- \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{b^2r^2}{c^2\beta_{mr}^2}\Psi^{(1)}(\beta_{mr}\eta)\Big\} + G'_{mr}\Big\{\phi^{(2)}(\beta_{mr}\eta) - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{b^2r^2}{c^2\beta_{mr}^2}$$

$$\times \Psi^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z + \sum_n\sum_r \mu\frac{an}{2b}\Big[E_{nr}\Big\{Q^{(1)}(\alpha_{nr}\xi)$$

$$- \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{a^2r^2}{c^2\alpha_{nr}^2} P^{(1)}(\alpha_{nr}\xi)\Big\} + E'_{nr}\Big\{Q^{(2)}(\alpha_{nr}\xi) - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{a^2r^2}{c^2o_{nr}^2}$$

$$\times P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\cos\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{b}z - \sum_n\sum_r \mu\frac{br}{2c}\Big[J_{nr}\Big\{Q^{(1)}(\alpha_{nr}\xi)$$

$$- \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{n^2a^2}{b^2\alpha_{nr}^2} P^{(1)}(\alpha_{nr}\xi)\Big\} + J'_{nr}\Big\{Q^{(2)}(\alpha_{nr}\xi) - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{n^2a^2}{b^2\alpha_{nr}^2}$$

$$\times P^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\cos\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{c}z + \sum_m\sum_r \mu\frac{b^2mr}{2ac\beta_{mr}}\Big[H_{mr}\Big\{\frac{\mu}{2\mu+\lambda}$$

$$
\left.
\begin{aligned}
&\times \phi^{(1)}(\beta_{mr}\eta) + \frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(1)}(\beta_{mr}\eta)\Big\} + H'_{mr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\beta_{mr}\eta) \\
&+ \frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z + \sum_n\sum_r \mu\frac{a^3rn}{2bc}\frac{1}{\beta^2_{mr}} \\
&\times \frac{\mu+\lambda}{2\mu+\lambda}\Big\{A_{nr}P^{(1)}(\alpha_{nr}\xi) + A'_{nr}P^{(2)}(\alpha_{nr}\xi)\Big\}\cos\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{c}z + \sum_m\sum_n \\
&\times \frac{c^3mn}{2ab\gamma^2_{mn}}\frac{\mu+\lambda}{2\mu+\lambda}\Big\{D_{mn}P^{(1)}(\gamma_{mn}\zeta) + D'_{mn}P^{(2)}(\gamma_{mn}\zeta)\Big\}\sin\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y \\
&- \sum_m\sum_r \frac{b^2r}{4c\beta_{mr}}\Big[B_{mr}\Big\{\frac{2\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{mr}\eta) + \frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(1)}(\beta_{mr}\eta)\Big\} \\
&+ B'_{mr}\Big\{\frac{2\mu}{2\mu+\lambda}\phi^{(2)}(\beta_{mr}\eta) + \frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(2)}(\beta_{mr}\eta)\Big\}\Big]\sin\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z \\
&- \sum_m\sum_n \frac{c}{2}\Big[F_{mn}\Big\{Q^{(1)}(\gamma_{mn}\zeta) - \frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma^2_{mn}}P^{(1)}(\gamma_{mn}\zeta)\Big\} \\
&+ F'_{mn}\Big\{Q^{(2)}(\gamma_{mn}\zeta) - \frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma^2_{mn}} \times P^{(2)}(\gamma_{mn}\zeta)\Big\}\Big]\cos\frac{n\pi}{b}y\cdot\sin\frac{m\pi}{a}x \\
&- \sum_n \frac{c}{4}\Big\{F_{m0}Q^{(1)}(\gamma_{m0}\zeta) + F'_{m0}Q^{(2)}(\gamma_{m0}\zeta)\Big\}\sin\frac{n\pi}{b}y + \sum_r \frac{ar}{4c} \\
&\times \Big\{H_{0r}Q^{(1)}(\alpha_{0r}\xi) + H'_{0r}Q^{(2)}(\alpha_{0r}\xi)\Big\}\sin\frac{r\pi}{c}z\,,
\end{aligned}
\right\} \quad (14)
$$

$$
\left.
\begin{aligned}
\tau_{zx} &= \mu\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big) = -\sum_m\sum_n \frac{cm}{a}\frac{\mu(\mu+\lambda)}{2\mu+\lambda}\Big\{K_{mn}P^{(1)}(\gamma_{mn}\zeta) + K'_{mn} \\
&\times P^{(2)}(\gamma_{mn}\zeta)\Big\}\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y - \sum_n\sum_r \mu\frac{\alpha_{nr}}{2}\Big[E_{nr}\Big\{\phi^{(1)}(\alpha_{nr}\xi) \\
&- \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{a^2r^2}{c^2\alpha^2_{nr}}\Psi^{(1)}(\alpha_{nr}\xi)\Big\} + E'_{nr}\Big\{\phi^{(2)}(\alpha_{nr}\xi) - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{a^2r^2}{c^2\alpha^2_{nr}} \\
&\times \Psi^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{c}z + \sum_r\sum_m \mu\frac{bm}{2a}\Big[G_{mr}\Big\{Q^{(1)}(\beta_{mn}\eta) \\
&- \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{a^2r^2}{c^2\beta^2_{mr}}P^{(1)}(\beta_{mr}\eta)\Big\} + G'_{mr}\Big\{Q^{(2)}(\beta_{mr}\eta) - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{a^2r^2}{c^2\beta^2_{mr}} \\
&\times P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\cos\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z + \sum_n\sum_r \mu\frac{a^2nr}{2bc\alpha_{nr}}\Big[J_{nr}\Big\{\frac{\mu}{2\mu+\lambda} \\
&\times \phi^{(1)}(\alpha_{nr}\xi) + \frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(1)}(\alpha_{nr}\xi)\Big\} + J'_{nr}\Big\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\alpha_{nr}\xi) \\
&+ \frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(2)}(\alpha_{nr}\xi)\Big\}\Big]\sin\frac{n\pi}{b}y\cdot\sin\frac{r\pi}{c}z - \sum_m\sum_n \mu\frac{ar}{2c}\Big[H_{mr} \\
&\times \Big\{Q^{(1)}\beta_{mr}\eta) - \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{m^2b^2}{a^2\beta^2_{mr}}P^{(1)}(\beta_{mr}\eta)\Big\} + H'_{mr}\Big\{Q^{(2)}(\beta_{mr}\eta) \\
&- \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{m^2b^2}{a^2\beta^2_{mr}}P^{(2)}(\beta_{mr}\eta)\Big\}\Big]\cos\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z + \sum_m\sum_r \frac{b^3rm}{2ac\beta^2_{mr}}
\end{aligned}
\right\} \quad (15)
$$

$$\times \frac{\mu+\lambda}{2\mu+\lambda}\left\{B_{mr}P^{(1)}(\beta_{mr}\eta)+B'_{mr}P^{(2)}(\beta_{mr}\eta)\right\}\cos\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z$$
$$+\sum_m\sum_n\frac{c^3mn}{2ab\gamma_{mn}^2}\,\frac{\mu+\lambda}{2\mu+\lambda}\left\{F_{mn}P^{(1)}(\gamma_{mn}\zeta)+F'_{mn}P^{(2)}(\gamma_{mn}\zeta)\right\}$$
$$\times\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y-\sum_n\sum_r\frac{a^2r}{4c\alpha_{nr}}\left[A_{nr}\left\{\frac{2\mu}{2\mu+\lambda}\phi^{(1)}(\alpha_{nr}\xi)\right.\right.$$
$$\left.\left.+\frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(1)}(\alpha_{nr}\xi)\right\}+A'_{nr}\left\{\frac{2\mu}{2\mu+\lambda}\phi^{(2)}(\alpha_{nr}\xi)+\frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(2)}(\alpha_{nr}\xi)\right\}\right]$$
$$\times\sin\frac{n\pi}{b}y\ \sin\frac{r\pi}{c}z-\sum_m\sum_n\frac{c}{2}\left[D_{mn}\left\{Q^{(1)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(1)}(\gamma_{mn}\zeta)\right\}\right.$$
$$\left.+D'_{mn}\left\{Q^{(2)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}P^{(2)}(\gamma_{mn}\zeta)\right\}\right]\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y$$
$$-\sum_n\frac{c}{4}\left\{D_{0n}Q^{(1)}(\gamma_{0n}\zeta)+D'_{0n}Q^{(2)}(\gamma_{0n}\zeta)\right\}\sin\frac{n\pi}{b}y+\sum_r\frac{br}{4c}\left\{J_{0r}Q^{(1)}(\beta_{0r}\eta)\right.$$
$$\left.+J'_{0r}Q^{(2)}(\beta_{0r}\eta)\right\}\sin\frac{r\pi}{c}z\,,$$

$$\tau_{xy}=\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)=-\sum_m\sum_n\mu\frac{c^2}{ab}\frac{mn}{\gamma_{mn}}\left[K_{mn}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\gamma_{mn}\zeta)\right.\right.$$
$$\left.\left.+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}\zeta)\right\}+K'_{mn}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\gamma_{mn}\zeta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\gamma_{mn}\zeta)\right\}\right]$$
$$\times\cos\frac{m\pi}{a}x\times\cos\frac{n\pi}{b}y+\sum_m\sum_r\mu\frac{b^3mr}{2ca\beta_{mr}}\left[G_{mr}\left\{\frac{\lambda}{2\mu+\lambda}\phi^{(1)}(\beta_{mr}\eta)\right.\right.$$
$$\left.\left.-\frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(1)}(\beta_{mr}\eta)\right\}+G'_{mr}\left\{\frac{\lambda}{2\mu+\lambda}\phi^{(2)}(\beta_{mr}\eta)-\frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(2)}(\beta_{mr}\eta)\right\}\right]$$
$$\times\cos\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z+\sum_n\sum_r\mu\frac{a^3nr}{2cb\alpha_{nr}}\left[E_{nr}\left\{\frac{\lambda}{2\mu+\lambda}\phi^{(1)}(\alpha_{nr}\xi)\right.\right.$$
$$-\frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(1)}(\alpha_{nr}\xi)\Big\}+E'_{nr}\left\{\frac{\lambda}{2\mu+\lambda}\phi^{(2)}(\alpha_{nr}\xi)-\frac{2(\mu+\lambda)}{2\mu+\lambda}\right.$$
$$\left.\left.\times\Psi^{(2)}(\alpha_{nr}\xi)\right\}\right]\cos\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z+\sum_m\sum_n\frac{c^2m}{2a\gamma_{mn}}\left[F_{mn}\left\{1-\frac{\mu+\lambda}{2\mu+\lambda}\right.\right.$$
$$\left.\times\frac{c^2n^2}{b^2\gamma_{mn}^2}\right)\times\phi^{(1)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma_{mn}^2}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+F'_{mn}$$
$$\left.\times\left\{\left(1-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma_{mn}^2}\right)\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2n^2}{b^2\gamma_{mn}^2}\Psi^{(2)}(\gamma_{mn}\zeta)\right\}\right]$$
$$\times\cos\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y+\sum_m\sum_n\frac{c^2n}{2b\gamma_{mn}}\left[D_{mn}\left\{\left(1-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma_{mn}^2}\right)\right.\right.$$
$$\times\phi^{(1)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma_{mn}^2}\Psi^{(1)}(\gamma_{mn}\zeta)\Big\}+D'_{mn}\left\{\left(1-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma_{mn}^2}\right.\right.$$
$$\left.\left.\times\phi^{(2)}(\gamma_{mn}\zeta)-\frac{\mu+\lambda}{2\mu+\lambda}\frac{c^2m^2}{a^2\gamma_{mn}^2}\Psi^{(2)}(\gamma_{mn}\zeta)\right\}\right]\cos\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y$$

$$
\left.
\begin{aligned}
&+\sum_n\sum_r \frac{a^2n}{2b\alpha_{nr}}\Bigg[A_{nr}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\alpha_{nr}\xi)\right\}\\
&+A'_{nr}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\alpha_{nr}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\alpha_{nr}\xi)\right\}\Bigg]\cos\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y\\
&+\sum_m\sum_r \frac{b^2m}{2a\beta_{mr}}\Bigg[B_{mr}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{mr}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\beta_{mr}\eta)\right\}\\
&+B'_{mr}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\beta_{mr}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\beta_{mr}\eta)\right\}\Bigg]\cos\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z\\
&-\sum_m\sum_r\mu\frac{\beta_{mr}}{2}\left\{H_{mr}\left(1+\frac{b^2m^2}{a^2\beta^2_{mr}}\right)\phi^{(1)}(\beta_{mr}\eta)+H'_{mr}\left(1+\frac{b^2m^2}{a^2\beta^2_{mr}}\right)\right.\\
&\left.\times\phi^{(2)}(\beta_{mr}\eta)\right\}\cos\frac{m\pi}{a}x\times\cos\frac{r\pi}{c}z-\sum_n\sum_r\mu\frac{\alpha_{nr}}{2}\left\{J_{nr}\left(1+\frac{a^2n^2}{b^2\alpha^2_{nr}}\right)\right.\\
&\left.\times\phi^{(1)}(\alpha_{nr}\xi)+J'_{nr}\left(1+\frac{a^2n^2}{b^2\alpha^2_{nr}}\right)\phi^{(2)}(\alpha_{nr}\xi)\right\}\cos\frac{n\pi}{b}y\cdot\cos\frac{r\pi}{c}z\\
&+\sum_m\mu\frac{b}{4}\Bigg[B_{m0}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\beta_{m0}\eta)\right\}+B'_{m0}\\
&\times\left\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\beta_{m0}\eta)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\beta_{m0}\eta)\right\}\Bigg]\cos\frac{m\pi}{a}x+\sum_n\sum_r\frac{a}{4}\\
&\times\Bigg[A_{n0}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\alpha_{n0}\xi)+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\alpha_{n0}\xi)\right\}+A'_{n0}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(2)}(\alpha_{n0}\xi)\right.\\
&\left.+\frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(2)}(\alpha_{n0}\xi)\right\}\Bigg]\cos\frac{n\pi}{b}y-\sum_m\mu\frac{mb}{2a}\left\{H_{m0}\phi^{(1)}(\beta_{m0}\eta)\right.\\
&\left.+H'_{m0}\phi^{(2)}(\beta_{m0}\eta)\right\}\cos\frac{m\pi}{a}x-\sum_m\mu\frac{na}{2b}\left\{J_{n0}\phi^{(1)}(\alpha_{n0}\xi)+J'_{n0}\phi^{(2)}(\alpha_{n0}\xi)\right\}\\
&\times\cos\frac{n\pi}{b}y+\sum_n\mu\frac{c}{4}\left\{D_{0n}\phi^{(1)}(\gamma_{0n}\zeta)+D'_{0n}\phi^{(2)}(\gamma_{0n}\zeta)\right\}\cos\frac{n\pi}{b}y\\
&-\sum_r\mu\frac{rb}{4c}\left\{J_{0r}\phi^{(1)}(\beta_{0r}\eta)+J'_{0r}\phi^{(2)}(\beta_{0r}\eta)\right\}\cos\frac{r\pi}{c}z+\sum_m\mu\frac{c}{4}\\
&\times\left\{F_{m0}\phi^{(1)}(\gamma_{m0}\zeta)+F'_{m0}\phi^{(2)}(\gamma_{m0}\zeta)\right\}\cos\frac{m\pi}{a}x+\sum_r\mu\frac{ra}{4c}\left\{H_{0r}\phi^{(1)}(\alpha_{0r}\xi)\right.\\
&\left.+H'_{0r}\phi^{(2)}(\alpha_{0r}\xi)\right\}\cos\frac{r\pi}{c}z+\frac{1}{4}(H_{00}+J_{00}),
\end{aligned}
\right\}\quad(16)
$$

where

$$
\left.
\begin{aligned}
&\left.\begin{matrix}Q^{(1)}(\alpha_{nr}\xi)\\ Q^{(2)}(\alpha_{nr}\xi)\end{matrix}\right\}=\frac{\cosh\pi\alpha_{nr}(1-\xi)\pm\cosh\pi\alpha_{nr}\xi}{\cosh\pi\alpha_{nr}\pm1},\\
&\left.\begin{matrix}P^{(1)}(\alpha_{nr}\xi)\\ P^{(2)}(\alpha_{nr}\xi)\end{matrix}\right\}=\frac{\pi\alpha_{nr}\{\xi\sinh\pi\alpha_n(1-\xi)\pm(1-\xi)\sinh\pi\alpha_n\xi\}}{\cosh\pi\alpha_{nr}\pm1},\\
&\left.\begin{matrix}\phi^{(1)}(\alpha_{nr}\xi)\\ \phi^{(2)}(\alpha_{nr}\xi)\end{matrix}\right\}=\frac{\sinh\pi\alpha_{nr}(1-\xi)\mp\sinh\pi\alpha_{nr}\xi}{\cosh\pi\alpha_{nr}\pm1},
\end{aligned}
\right\}\quad(17)
$$

$$\left.\begin{array}{l}\Psi^{(1)}(\alpha_{nr}\xi)\\ \Psi^{(2)}(\alpha_{nr}\xi)\end{array}\right\} = \frac{\pi\alpha_{nr}\{\xi\cosh\pi\alpha_n(1-\xi)\mp(1-\xi)\cosh\pi\alpha_n\xi\}}{\cosh\pi\alpha_{nr}\pm1},$$

$$\alpha_n^2 = \frac{a^2}{b^2}n^2 + \frac{a^2}{c^2}r^2, \quad \xi = \frac{x}{a},$$

$$\left.\begin{array}{l}Q^{(1)}(\beta_{mr}\eta)\\ Q^{(2)}(\beta_{mr}\eta)\end{array}\right\} = \frac{\cosh\pi\beta_{mr}(1-\eta)\pm\cosh\pi\beta_{mr}\eta}{\cosh\pi\beta_{mr}\pm1},$$

$$\left.\begin{array}{l}P^{(1)}(\beta_{mr}\eta)\\ P^{(2)}(\beta_{mr}\eta)\end{array}\right\} = \frac{\pi\beta_{mr}\{\eta\sinh\pi\beta_{mr}(1-\eta)\pm(1-\eta)\sinh\pi\beta_{mr}\eta\}}{\cosh\pi\beta_{mr}\pm1},$$

$$\left.\begin{array}{l}\phi^{(1)}(\beta_{mr}\eta)\\ \phi^{(2)}(\beta_{mr}\eta)\end{array}\right\} = \frac{\sinh\pi\beta_{mr}(1-\eta)\mp\sinh\pi\beta_{mr}\eta}{\cosh\pi\beta_{mr}\pm1}, \tag{18}$$

$$\left.\begin{array}{l}\Psi^{(1)}(\beta_{mr}\eta)\\ \Psi^{(2)}(\beta_{mr}\eta)\end{array}\right\} = \frac{\pi\beta_{mr}\{\eta\cosh\pi\beta_{mr}(1-\eta)\mp(1-\eta)\cosh\pi\beta_{mr}\eta\}}{\cosh\pi\beta_{mr}\pm1},$$

$$\beta_{mr}^2 = \frac{b^2}{a^2}m^2 + \frac{b^2}{c^2}r^2, \quad \eta = \frac{y}{b},$$

$$\left.\begin{array}{l}Q^{(1)}(\gamma_{mn}\zeta)\\ Q^{(2)}(\gamma_{mn}\zeta)\end{array}\right\} = \frac{\cosh\pi\gamma_{mn}(1-\zeta)\pm\cosh\pi\gamma_{mn}\zeta}{\cosh\pi\gamma_{mn}\pm1},$$

$$\left.\begin{array}{l}P^{(1)}(\gamma_{mn}\zeta)\\ P^{(2)}(\gamma_{mn}\zeta)\end{array}\right\} = \frac{\pi\gamma_{mn}\{\zeta\sinh\pi\gamma_{mn}(1-\zeta)\pm(1-\zeta)\sinh\pi\gamma_{mn}\zeta\}}{\cosh\pi\gamma_{mn}\pm1},$$

$$\left.\begin{array}{l}\phi^{(1)}(\gamma_{mn}\zeta)\\ \phi^{(2)}(\gamma_{mn}\zeta)\end{array}\right\} = \frac{\sinh\pi\gamma_{mn}(1-\zeta)\mp\sinh\pi\gamma_{mn}\zeta}{\cosh\pi\gamma_{mn}\pm1}, \tag{19}$$

$$\left.\begin{array}{l}\Psi^{(1)}(\gamma_{mn}\zeta)\\ \Psi^{(2)}(\gamma_{mn}\zeta)\end{array}\right\} = \frac{\pi\gamma_{mn}\{\zeta\cosh\pi\gamma_{mn}(1-\zeta)\mp(1-\zeta)\cosh\pi\gamma_{mn}\zeta\}}{\cosh\pi\gamma_{mn}\pm1},$$

$$\gamma_{mn}^2 = \frac{c^2}{a^2}m^2 + \frac{c^2}{b^2}n^2, \quad \zeta = \frac{z}{c}.$$

Letting the suffix m, n, r involve zero respectively, eighteen unknown values A_{nr}, A'_{nr}, B_{mr}, B'_{mr}, D_{mn}, D'_{mn}, E_{nr}, E'_{nr}, F_{mn}, F'_{mn}, G_{mr}, G'_{mr}, H_{mr}, H'_{mr}, J_{nr}, J'_{nr}, K_{mn}, K'_{mn}, denote the finite Fourier transformations relating to both the components of stress and displacement at the boundary, as follows:

$$\left.\begin{array}{l}\pi A'_{n0}\\ \pi A_{n0}\end{array}\right\} = \int_0^c S_n[(\sigma_x)_{x=a}]dz \mp \int_0^c S_n[(\sigma_x)_{x=0}]dz, \tag{20}$$

$$\left.\begin{array}{l}\pi B'_{m0}\\ \pi B_{m0}\end{array}\right\} = \int_0^c S_m[(\sigma_y)_{y=b}]dz \mp \int_0^c S_m[(\sigma_y)_{y=0}]dz, \tag{21}$$

$$\left.\begin{array}{l}\pi A'_{nr}\\ \pi A_{nr}\end{array}\right\} = S_nC_r[(\sigma_x)_{x=a}] \mp S_nC_r[(\sigma_x)_{x=0}], \tag{22}$$

$$\left.\begin{array}{l}\pi B'_{mr}\\ \pi B_{mr}\end{array}\right\} = S_mC_r[(\sigma_y)_{y=b}] \mp S_mC_r[(\sigma_y)_{y=0}], \tag{23}$$

$$\left.\begin{matrix}\pi D'_{0n}\\ \pi D_{0n}\end{matrix}\right\} = \int_0^a S_n[(\tau_{zx})_{z=c}]dx \mp \int_0^a S_n[(\tau_{zx})_{z=0}]dx\,, \tag{24}$$

$$\left.\begin{matrix}\pi F'_{m0}\\ \pi F_{m0}\end{matrix}\right\} = \int_0^b S_m[(\tau_{yz})_{z=c}]dy \mp \int_0^b S_m[(\tau_{yz})_{z=0}]dy\,. \tag{25}$$

$$\left.\begin{matrix}\pi D'_{mn}\\ \pi D_{mn}\end{matrix}\right\} = C_m S_n[(\tau_{zx})_{z=c}] \mp C_m S_n[(\tau_{zx})_{z=0}]\,, \tag{26}$$

$$\left.\begin{matrix}\pi F'_{mn}\\ \pi F_{mn}\end{matrix}\right\} = S_m C_n[(\tau_{yz})_{z=c}] \mp S_m C_n[(\tau_{yz})_{z=0}]\,. \tag{27}$$

$$\left.\begin{matrix}H'_{00}\\ H_{00}\end{matrix}\right\} = \int(v_{x=a})dA_x \mp \int(v_{x=0})dA_x\,, \tag{28}$$

$$\left.\begin{matrix}J'_{00}\\ J_{00}\end{matrix}\right\} = \int(u_{y=b})dA_y \mp (u_{y=0})dA_y\,, \tag{29}$$

$$\left.\begin{matrix}H'_{m0}\\ H_{m0}\end{matrix}\right\} = \int_0^c C_m[u_{y=b}]dz \mp \int_0^c C_m[u_{y=0}]dz\,, \tag{30}$$

$$\left.\begin{matrix}J'_{n0}\\ J_{n0}\end{matrix}\right\} = \int_0^c C_n[v_{x=a}]dz \mp \int_0^c C_n[v_{x=0}]dz\,, \tag{31}$$

$$\left.\begin{matrix}H'_{0r}\\ H_{0r}\end{matrix}\right\} = \int_0^a C_r[u_{y=b}]dx \mp \int_0^a C_r[u_{y=0}]dx\,, \tag{32}$$

$$\left.\begin{matrix}J'_{0r}\\ J_{0r}\end{matrix}\right\} = \int_0^b C_r[v_{x=a}]dy \mp \int_0^b C_r[v_{x=0}]dy\,, \tag{33}$$

$$\left.\begin{matrix}H'_{mr}\\ H_{mr}\end{matrix}\right\} = C_m C_r[u_{y=b}] \mp C_m C_r[u_{y=0}]\,, \tag{34}$$

$$\left.\begin{matrix}J'_{nr}\\ J_{nr}\end{matrix}\right\} = C_n C_r[v_{x=a}] \mp C_n C_r[v_{x=0}]\,, \tag{35}$$

$$\left.\begin{matrix}E'_{nr}\\ E_{nr}\end{matrix}\right\} = S_n S_r[w_{x=a}] \mp S_n S_r[w_{x=0}]\,, \tag{36}$$

$$\left.\begin{matrix}G'_{mr}\\ G_{mr}\end{matrix}\right\} = S_m S_r[w_{y=b}] \mp S_m S_r[w_{y=0}]\,, \tag{37}$$

$$\left.\begin{matrix}K'_{mn}\\ K_{mn}\end{matrix}\right\} = S_m S_n[w_{z=c}] \mp S_m S_n[w_{z=0}]\,. \tag{38}$$

Boundary Conditions for a Special Case

To investigate the state of stress in a bent plate, in particular in the neighborhood of the simply supported edge, we will choose the case where the plate is supported at its opposite two edges and is hinged at the remainning edges. Set the Cartesian rectangular co-ordinate, which is refered to the plate, as Fig. 2.

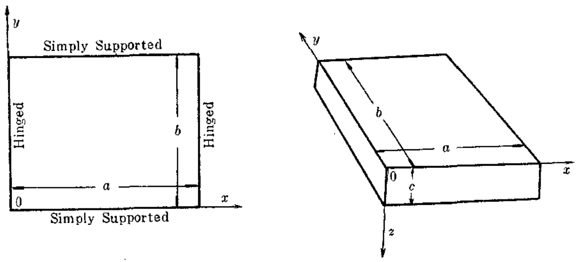

Fig. 2

The boundary conditions in this case, are satisfied by

$$w = 0 \qquad (x = 0\,, \quad x = a)\,, \tag{39}$$

$$v = 0 \qquad (x = 0\,, \quad x = a)\,, \tag{40}$$

$$\sigma_x = 0 \qquad (x = 0\,, \quad x = a)\,. \tag{41}$$

which correspond to that the edge-lines $x=0$ and $x=a$ are hinge, and

$$w = 0 \qquad (y = 0\,, \quad y = b)\,, \tag{42}$$

$$\sigma_y = 0 \qquad (y = 0\,, \quad y = b)\,, \tag{43}$$

$$\tau_{xy} = 0 \qquad (y = 0\,, \quad y = b)\,, \tag{44}$$

which coincide with the simply supported edges are on $y=0$ and $y=b$. The above conditions (39), (40), (41), (42), and (43) are identical with the relations

$$A_{nr} = A'_{nr} = 0\,, \quad B_{mr} = B'_{mr} = 0\,, \quad E_{nr} = E'_{nr} = 0\,,$$
$$G_{mr} = G'_{mr} = 0\,, \quad J_{nr} = J'_{nr} = 0\,,$$

where the suffix m, n, and r include zero in itself.

When the loads act normally to the surfaces made by $z=0$ and $z=c$, and distribute symmetrically with respect to the middle plane $z=c/2$, some remaining unknown values are eliminated, that is

$$D_{mn} = D'_{mn} = D_{0n} = D'_{0n} = 0\,, \quad F_{mn} = F'_{mn} = F_{m0} = F'_{m0} = 0\,, \quad K'_{mn} = 0\,,$$

and

$$H_{m0} = H'_{m0} = H_{00} = H'_{00} = 0\,.$$

Adding, hereon, one more restriction that the loads are also symmetrical with respect to both the center lines $x=a/2$, and $y=b/2$, then we find that

$$H'_{mr} = 0\,, \quad H_{0r} = H'_{0r} = 0\,.$$

After all, the components of displacement are written in the forms

$$\left.\begin{aligned} u &= \sum_m \sum_n \frac{c^2 m}{2a\gamma_{mn}} K_{mn} \left\{ -\frac{\mu}{2\mu+\lambda} \phi^{(1)}(\gamma_{mn}\zeta) + \frac{\mu+\lambda}{2\mu+\lambda} \Psi^{(1)}(\gamma_{mn}\zeta) \right\} \\ &\times \cos\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y + \sum_m \sum_r \frac{b}{2} H_{mr} \left\{ Q^{(1)}(\beta_{mr}\eta) \right. \\ &\left. - \frac{\mu+\lambda}{2\mu+\lambda} \frac{1}{\beta_{mr}^2} \left(\frac{mb}{a}\right)^2 P^{(1)}(\beta_{mr}\eta) \right\} \cos\frac{m\pi}{a}x \cdot \cos\frac{r\pi}{c}z , \end{aligned}\right\} \tag{45}$$

$$\left.\begin{aligned} v &= \sum_m \sum_n \frac{c^2 n}{2b\gamma_{mn}} K_{mn} \left\{ -\frac{\mu}{2\mu+\lambda} \phi^{(1)}(\gamma_{mn}\zeta) + \frac{\mu+\lambda}{2\mu+\lambda} \Psi^{(1)}(\gamma_{mn}\zeta) \right\} \\ &\times \sin\frac{m\pi}{a}x \cdot \cos\frac{n\pi}{b}y - \sum_m \sum_r \frac{b^2 m}{2a\beta_{mr}} H_{mr} \left\{ \frac{\mu}{2\mu+\lambda} \phi^{(1)}(\beta_{mr}\eta) \right. \\ &\left. + \frac{\mu+\lambda}{2\mu+\lambda} \Psi^{(1)}(\beta_{mr}\eta) \right\} \sin\frac{m\pi}{a}x \cdot \cos\frac{r\pi}{c}z , \end{aligned}\right\} \tag{46}$$

$$\left.\begin{aligned} w &= -\sum_m \sum_n \frac{c}{2} K_{mn} \left\{ Q^{(1)}(\gamma_{mn}\zeta) + \frac{\mu+\lambda}{2\mu+\lambda} P^{(1)}(\gamma_{mn}\zeta) \right\} \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y \\ &+ \sum_m \sum_r \frac{b^3 mr}{2ac\beta_{mr}^2} \frac{\mu+\lambda}{2\mu+\lambda} H_{mr} P^{(1)}(\beta_{mr}\eta) \sin\frac{m\pi}{a}x \cdot \sin\frac{r\pi}{c}z , \\ & m,\ n,\ r = 1,\ 3,\ 5,\ \cdots\cdots . \end{aligned}\right\} \tag{47}$$

The components of stress are expressed by

$$\left.\begin{aligned} \sigma_x &= \sum_m \sum_n \left[K_{mn} \left\{ \left(\frac{\mu^2}{2\mu+\lambda} \frac{c^2 m^2}{a^2 \gamma_{mn}^2} + \frac{\lambda\mu}{2\mu+\lambda} \gamma_{mn} \right) \phi^{(1)}(\gamma_{mn}\zeta) - \frac{\mu(\mu+\lambda)}{2\mu+\lambda} \frac{c^2 m^2}{a^2 \gamma_{mn}^2} \right. \right. \\ &\times \Psi^{(1)}(\gamma_{mn}\zeta) \Big\} \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y - \sum_m \sum_r \frac{bm}{a} H_{mr} \left\{ \frac{\mu(3\mu+\lambda)}{2\mu+\lambda} Q^{(1)}(\beta_{mr}\eta) \right. \\ &\left. + \frac{\lambda(\mu+\lambda)}{2\mu+\lambda} P^{(1)}(\beta_{mr}\eta) \right\} \sin\frac{m\pi}{a}x \cdot \cos\frac{n\pi}{b}y , \end{aligned}\right\} \tag{48}$$

$$\left.\begin{aligned} \sigma_y &= \sum \sum K_{mn} \left\{ \frac{\mu^2}{2\mu+\lambda} \frac{c^2 n^2}{b^2 \gamma_{mn}} + \frac{\mu\lambda}{2\mu+\lambda} \gamma_{mn} \right) \cdot \phi^{(1)}(\gamma_{mn}\zeta) \\ &- \frac{\mu(\mu+\lambda)}{2\mu+\lambda} \frac{c^2 n^2}{b^2 \gamma_{mn}} \Psi^{(1)}(\gamma_{mn}\zeta) \Big\} \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y \\ &- \sum_m \sum_r \frac{am}{b} \frac{\mu(\mu+\lambda)}{2\mu+\lambda} H_{mr} P^{(1)}(\beta_{mr}\eta) \sin\frac{m\pi}{a}x \cdot \cos\frac{r\pi}{c}z , \end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned} \sigma_z &= \sum_m \sum_n \frac{\mu(\mu+\lambda)}{2\mu+\lambda} \gamma_{mn} K_{mn} \left\{ \phi^{(1)}(\gamma_{mn}\zeta) + \Psi^{(1)}(\gamma_{mn}\zeta) \right\} \\ &\times \sin\frac{m\pi}{a}x \cdot \sin\frac{n\pi}{b}y - \sum_m \sum_r \frac{bm}{a} H_{mr} \left\{ \frac{\mu\lambda}{2\mu+\lambda} Q^{(1)}(\beta_{mr}\eta) \right. \\ &\left. + \frac{r^2 b^2}{c^2 \beta_{mr}^2} \frac{\mu(\mu+\lambda)}{2\mu+\lambda} P^{(1)}(\beta_{m}\eta) \right\} \sin\frac{m\pi}{a}x \cdot \cos\frac{r\pi}{c}z , \end{aligned}\right\} \tag{50}$$

$$\left.\begin{aligned}\tau_{yz} = & -\sum_m\sum_n \frac{cn}{b}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda} K_{mn} P^{(1)}(\gamma_{mn}\zeta)\sin\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y \\ & +\sum_m\sum_r \mu\frac{b^2mr}{2ac\beta_{mr}} H_{mr}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{mr}\eta)\right. \\ & \left. +\frac{2(\mu+\lambda)}{2\mu+\lambda}\psi^{(1)}(\beta_{mr}\eta)\right\}\sin\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z\,,\end{aligned}\right\} \quad (51)$$

$$\left.\begin{aligned}\tau_{zx} = & -\sum_m\sum_n \frac{cm}{a}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda} K_{mn} P^{(1)}(\gamma_{mn}\zeta)\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y \\ & -\sum_m\sum_r \mu\frac{ar}{2c} H_{mr}\left\{Q^{(1)}(\beta_{mr}\eta) - \frac{2(\mu+\lambda)}{2\mu+\lambda}\,\frac{m^2b^2}{a^2\beta_{mr}^2}P^{(1)}(\beta_{mr})\right\} \\ & \times\cos\frac{m\pi}{a}x\cdot\sin\frac{r\pi}{c}z\,,\end{aligned}\right\} \quad (52)$$

$$\left.\begin{aligned}\tau_{xy} = & -\sum_m\sum_n \mu\frac{c^2}{ab}\,\frac{mn}{\gamma_{mn}} K_{mn}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\gamma_{mn}\zeta)\right. \\ & \left. +\frac{\mu+\lambda}{2\mu+\lambda}\psi^{(1)}(\gamma_{mn}\zeta)\right\}\cos\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y \\ & -\sum_m\sum_r \mu\frac{\beta_{mr}}{2} H_{mr}\left(1+\frac{b^2m^2}{a^2\beta_{mr}^2}\right)\phi^{(1)}(\beta_{mr}\eta)\cos\frac{m\pi}{a}x\cdot\cos\frac{r\pi}{c}z\,,\end{aligned}\right\} \quad (53)$$

The remaining unknown values K_{mn} and H_{mr} should be determined to satisfy the boundary conditions at $z=0$, and $y=0$, respectively.

Determination of K_{mn} and H_{mn}

At first, to avoid the complexities of further evaluations, we will adopt the well-known assumption of "plane-conservation" for $(u)_{y=0}$ and $(u)_{y=b}$, then we may write as $H_{mr}=H_m/r^2$ in which r denotes odd integer. As for the usual plate, the results estimated through the above assumption, seem to be practically the same as those through the rigorous theory. Let the loads act normally to the surfaces $z=0$ and $z=c$, and be also expressed as

$$q_{xy} = q\sum_m\sum_n R_{mn}\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y\,. \quad (54)$$

Then the boundary conditions at $z=0$ and $z=c$, become

$$\sigma_z)_{z=0} = -q\sum_m\sum_n R_{mn}\sin\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y\,, \quad (55)$$

which by virtue of Eq. (50), yields

$$\mu K_{mn} = \frac{2\mu+\lambda}{\mu+\lambda} \frac{q \cdot R_{mn}}{\gamma_{mn}\{\phi^{(1)}(\gamma_{mn}0) + \Psi^{(1)}(\gamma_{mn}0)\}}$$
$$+ \frac{\mu H_m}{\mu+\lambda} \frac{nmb}{a} \left(\frac{c}{b}\right)^2 \frac{\pi/2\lambda\gamma_{mn} + \mu\phi^{(1)}(\gamma_{mn}0) + (\mu+\lambda)\Psi^{(1)}(\gamma_{mn}0)}{\gamma_{mn}^4\{\phi^{(1)}(\gamma_{mn}0) + \Psi^{(1)}(\gamma_{mn}0)\}} . \tag{56}$$

One more boundary condition is Eq. (44): τ_{xy} vanishs along the edge-line $y=0$ and $y=b$. Since the thickness of the plate is alwase considerably small comparing with the side-lengths a or b, so instead of Eq. (44) we can adopt the following relation

$$\int_0^c \tau_{xy}(c-2z)dz = 0 \qquad (y=0, \quad y=b). \tag{57}$$

This means that the resultant couples due to τ_{xy} vanishs along the edge-lines $y=0$ and $y=b$. Thus obtained result is written in the following equation

$$\sum_n \mu K_{mn} \frac{c^2}{ab} \frac{mn}{\gamma_{mn}^3} \left\{\frac{\pi}{2}\gamma_{mn} - \frac{3\mu+2\lambda}{2\mu+\lambda}\phi^{(1)}(\gamma_{mn}0) - \frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}0)\right\}$$
$$+ \sum_r \mu H_m \frac{\beta_{mr}}{2r^4}\left(1 + \frac{b^2m^2}{a^2\beta_{mr}^2}\right)\phi^{(1)}(\beta_{mr}0) = 0 . \tag{58}$$

Now, introducing new abbreviations

$$\begin{aligned}
\theta_{mn} &= \phi^{(1)}(\gamma_{mn}0) + \Psi^{(1)}(\gamma_{mn}0) , \\
\rho_{mn} &= \frac{\pi}{2}\frac{\lambda}{\mu+\lambda}\gamma_{mn} + \frac{\mu}{\mu+\lambda}\phi^{(1)}(\gamma_{mn}0) + \Psi^{(1)}(\gamma_{mn}0) , \\
\varphi_{mn} &= \frac{\pi}{2} - \frac{3\mu+2\lambda}{2\mu+\lambda}\phi^{(1)}(\gamma_{mn}0) - \frac{\mu+\lambda}{2\mu+\lambda}\Psi^{(1)}(\gamma_{mn}0) , \\
\omega_m &= \sum_r \frac{\beta_{mr}}{2r^4}\left(1 + \frac{b^2m^2}{a^2\beta_{mr}^2}\right)\phi^{(1)}(\beta_{mr}0) ,
\end{aligned} \tag{59}$$

and eliminating K_{mn} between Eqs. (56) and (58), we easily find that

$$\mu H_m = \frac{2\mu+\lambda}{\mu+\lambda} q \frac{-\dfrac{c^2}{ab}\sum_n \dfrac{nR_{mn}\cdot\varphi_{mn}}{\theta_{mn}\cdot\gamma_{mn}^4}}{\dfrac{\omega_m}{m} + \sum_n \left(\dfrac{c^4}{a^2b^2}\right)\dfrac{\rho_{mn}\cdot\varphi_{mn}}{\theta_{mn}\cdot\gamma_{mn}^7}} . \tag{60}$$

From Eq. (56) we have at once

$$\mu K_{mn} = \frac{2\mu+\lambda}{\mu+\lambda} q \left[\frac{R_{mn}}{\gamma_{mn}\cdot\theta_{mn}} + \frac{mn\rho_{mn}}{\gamma_m^4\theta_{mn}}\left(\frac{c^2}{ab}\right)\right.$$
$$\times \frac{-\sum_s \dfrac{R_{ms}\cdot s\cdot\varphi_{ms}}{\theta_{ms}\cdot\gamma_{ms}^4}}{\dfrac{\omega_m}{m} + \sum_s \dfrac{\rho_{ms}\cdot\varphi_{ms}\cdot s^2}{\theta_{ms}\cdot\gamma_{ms}^7}\left(\dfrac{c^2}{ab}\right)^2} . \tag{61}$$

Numerical Example

Let the loads q_{xy} be a sinusoidal shape shown as

$$q_{xy} = q \cdot \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} . \tag{62}$$

Then suffix m in those formulas ought to indicate number one only. Furthermore we make both the side-length a and b be equal, and also let

$$a = b = 10\,c , \qquad \lambda = 1.5\,\mu^* ,$$

where c denote the thickness of the plate. The plate now considered, is apparently belonging to "thin plate". After computing as above, we find necessary values for determinating K_{mn} and H_m, which are given in Table 1.

Table 1.

n	θ_{1n}	n	ρ_{1n}	n	φ_{1n}
1	0.007 028 4	1	0.009 181 6	1	−0.001 436 6
3	0.067 709 3	3	0.090 024	3	−0.011 174
5	0.216 235	5	0.298 40	5	−0.017 530
7	0.411 696	7	0.595 68	7	0.012 590
9	0.593 367	9	0.913 06	9	0.108 97
11	0.736 529	11	1.213 76	11	0.269 28
13	0.835 894	13	1.484 30	13	0.483 60
15	0.899 648	15	1.727 11	15	0.736 50
17	0.937 680	17	1.951 63	17	1.020 14
19	0.965 457	19	2.161 61	19	1.303 97
21	0.979 124	21	2.363 35	21	1.606 02

$\omega_1 = 5.2023$.

Now substituting the above results into Eq. (58), we have

$$\mu H_1 = -q \times 1.4308 ,$$

by which the values of K_{1n} are obtained in Table 2.

Table 2. Values of μK_{1n}

n	μK_{1n}	n	μK_{1n}
1	−1458.64	13	− 0.11
3	− 5.72	15	− 0.08
5	− 2.21	17	− 0.06
7	− 0.58	19	− 0.05
9	− 0.29	21	− 0.04
11	− 0.17		

* In this case Poisson's ratio of the plate is assumed to be 0.3.

To illustrate the variation in stress along the center line $x=a/2$, the shearing force S_y was calculated by means of the following equation

$$S_y=\int_0^c \tau_{yz}dz\,,$$

$$\left.\begin{aligned}S_y=&-\sum_m\sum_n\frac{cn}{b}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}K_{mn}\frac{2c}{\pi\gamma_{mn}}\left\{\phi^{(1)}(\gamma_{mn}0)+\Psi^{(1)}(\gamma_{mn}0)\right\}\\&\times\sin\frac{m\pi}{a}x\cdot\cos\frac{n\pi}{b}y+\sum_m\sum_r\mu H_m\frac{b_2}{a\pi}\,\frac{m}{\beta_{mr}}\,\frac{1}{r^2}\left\{\frac{\mu}{2\mu+\lambda}\phi^{(1)}(\beta_{mr}\eta)\right.\\&\left.+\frac{2(\mu+\lambda)}{2\mu+\lambda}\Psi^{(1)}(\beta_{mr}\eta)\right\}\sin\frac{m\pi}{a}x\,.\end{aligned}\right\}\quad(63)$$

that is vertical shear per unit of length acting on a cross-section of the plate normal to the y axis. In a same way, vertical shear per unit of length acting on a cross-section of the plate normal to the x axis, is denoted by S_x as follows:

$$\left.\begin{aligned}S_x=&\sum_m\sum_n\frac{cm}{a}\,\frac{\mu(\mu+\lambda)}{2\mu+\lambda}K_{mn}\frac{2c}{\pi\gamma_{mn}}\left\{\phi^{(1)}(\gamma_{mn}0)+\Psi^{(1)}(\gamma_{mn}0)\right\}\\&\times\cos\frac{m\pi}{a}x\cdot\sin\frac{n\pi}{b}y-\sum_m\sum_r\mu\frac{a}{\pi}H_m\frac{1}{r^2}\left\{Q^{(1)}(\beta_{mr}\eta)\right.\\&\left.-\frac{2(\mu+\lambda)}{2\mu+\lambda}\,\frac{m^2b^2}{a^2\beta_{mr}^2}P^{(1)}(\beta_{mr}\eta)\right\}\cos\frac{m\pi}{a}x\,.\end{aligned}\right\}\quad(64)$$

The values of S_y are shown in Fig. 3 by a real line, and another curve by dotted line presents the same force estimated according to the thin plate theory[6].

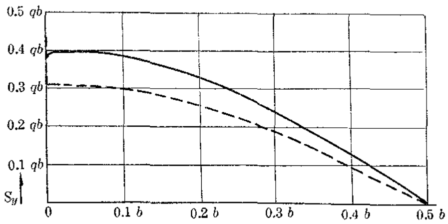

Fig. 3 S_y along the center line $x=\frac{1}{2}a$

The edge-reaction at $y=0$, in virtue of this theory, is written in

$$R_y=-0.3915\cdot qa\cdot\sin\frac{\pi x}{a}\,,$$

while the same reaction acquired by the thin plate theory, is given as

$$R_y=-0.4297\cdot qa\cdot\sin\frac{\pi x}{a}\,.$$

The variation of the edge-reactions along $x=0$ with y is shown in Fig. 4 in which the real line and the dotted line present the results according to Formula (64), and to the thin plate theory, respectively.

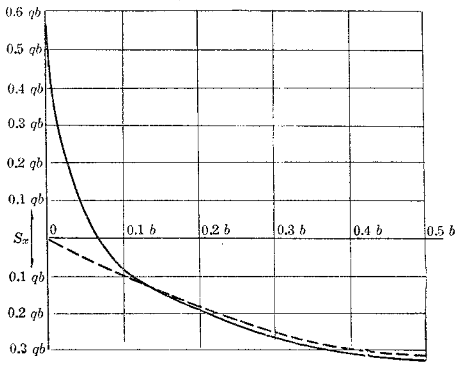

Fig. 4 S_x along the edge-line $x=0$

The prompt rise of the curved full line in the vicinity of the origin, is supposed to be the concentrated reaction at the corner of the plate, And the shearing force S_x and the bending moment M_x remain at the edge-line $y=0$, as follows:

$$S_x)_{y=0} = \int_0^c \tau_{zx}\, dz)_{y=0} = -0.5619 \cdot qb \cdot \sin \frac{\pi x}{a},$$

$$M_x)_{y=0} = \int_0^c \sigma_x (c-2z)\, dz)_{y=0} = 0.07560 \cdot qb^2 \cdot \sin \frac{\pi x}{a},$$

respectively. While S_x and M_x at the simply supported edge $y=0$ always vanish according to the thin plate theory.

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