Formulas of Frenet for a Vector Field in a Finsler Space

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Yukiyoshi Nagata

Abstract

T. K. Pan[1] demonstrated the generalized formulas of Frenet for a vector field in a subspace of a Riemannian space. This paper extends his investigation to a hypersurface and a subspace of a Finsler space.

1. Formulas of Frenet for a Vector Field in a Hypersurface.

Let F_{n-1} be a hypersurface given by the set of equations $x^\lambda = x^\lambda(u^1, u^2, \cdots, u^{n-1})$ $(\lambda = 1, \cdots, n)$ in a Finsler space F_n the fundamental quadratic form of which is $ds^2 = g_{\lambda\mu}(x, x')dx^\lambda dx^\mu$. F_{n-1} to which the element of support is tangential has the fundamental quadratic form $ds^2 = {}'g_{ab}du^a du^b$. Let v^λ be an arbitrary but fixed unit vector field defined at every point of F_{n-1} such that $v^\lambda = v^a B_a^{\ \lambda}$, ${}'g_{ab}v^a v^b = 1$. Let C: $u^a = u^a(s)$ $(a = 1, \cdots, n-1)$ be a curve on F_{n-1} and let N^λ be a unit vector normal to F_{n-1}. We define n vectors along C by the following equations:

$$\begin{aligned} &\eta_{(1)}{}^\lambda = v^\lambda, \qquad \eta_{(2)}{}^\lambda = {}_v k N^\lambda, \cdots, \\ &\eta_{(\gamma+1)}{}^\lambda = D\eta_{(\gamma)}{}^\lambda / ds \qquad (\gamma = 2, \cdots, n-1), \end{aligned} \tag{1.1}$$

where ${}_v k = g_{\lambda\mu} N^\lambda D v^\mu / ds$ and Dv^μ [2-3] denotes the absolute differential along C of the vector field v^μ at P of C. When $\eta_{(\beta)}{}^\lambda$ $(\beta = 1, \cdots, n)$ are linearly independent, the following n vectors $\sigma_{(p)}{}^\lambda$ $(p = 1, \cdots, n)$ which are expressed linearly with the components $\eta_{(\gamma)}{}^\lambda$ for $\gamma = 1, \cdots, p$ form a set of mutually orthogonal vectors:

$$\sigma_{(p)}{}^\lambda = \left(\frac{f_p}{f_{p-1}}\right)^{\frac{1}{2}} \eta_{(\gamma)}{}^\lambda F_p^\gamma \qquad (\gamma,\ \epsilon = 1, \cdots, p) \tag{1.2}$$

where

$$f_0 = 1, \quad f_1 = 1, \quad f_p = |f_\gamma^\epsilon|,$$

$$f_\gamma^\epsilon = f_\epsilon^\gamma = g_{\lambda\mu}\eta_{(\gamma)}{}^\lambda \eta_{(\epsilon)}{}^\mu, \quad f_\gamma^\epsilon F_\rho^\gamma = \delta_\rho^\epsilon.$$

Putting

$$\frac{D\sigma_{(q)\nu}}{ds}\sigma_{(p)}{}^\nu = \alpha_{qp} \qquad (p,\ q = 1, \cdots, n), \tag{1.3}$$

from $\sigma_{(q)\nu}\sigma_{(p)}{}^\nu = \delta_q^p$ we have

$$-\alpha_{qp} = \alpha_{pq}, \tag{1.4}$$

$$\frac{D\sigma_{(p)}{}^{\mu}}{ds} = \sum_{q} \alpha_{pq}\sigma_{(q)}{}^{\mu} \tag{1.5}$$

From (1.1) and (1.2) it follows that $D\sigma_{(p)}{}^{\mu}/ds$ is at most a linear expression in $\eta_{(1)}{}^{\mu}, \cdots, \eta_{(p+1)}{}^{\mu}$ and therefore in $\sigma_{(1)}{}^{\mu}, \cdots, \sigma_{(p+1)}{}^{\mu}$. Consequently, $\alpha_{kh}=0\ (k+1<h)$. Combining this result with (1.4), we have

$$\begin{aligned} \alpha_{p\,p+1} &= -\alpha_{p+1\,p} = {}_{v}K_{p} \\ \alpha_{pq} &= 0 \qquad (q \neq p\pm 1), \end{aligned} \tag{1.6}$$

where ${}_{v}K_{p}$ is defined by the first of these equations. Hence equations (1.5) are reduced to

$$\frac{D\sigma_{(p)}{}^{\mu}}{ds} = -{}_{v}K_{p-1}\sigma_{(p-1)}{}^{\mu} + {}_{v}K_{p}\sigma_{(p+1)}{}^{\mu}, \qquad (p=2, \cdots, n-1), \tag{1.7}$$

where ${}_{v}K_{p}$ for $p=1, \cdots, n-1$ are called, respectively, the associate curvatures of order $1, \cdots, n-1$ of the vector field v for the curve C. (1.7) may be considered as a generalization of the Frenet formulas for a curve and hold except the case $p=1$. And (1.7) apply to the case $p=n$ with the understanding that ${}_{v}K_{n}=0$. We call these the *formulas of Frenet of the second kind for v along C in F_n.*

In the following, we shall derive the formulas of Frenet of the first kind for v along C in F_n. We put

$$\begin{aligned} &\xi_{(1)}{}^{\lambda} = v^{\lambda}, \qquad \xi_{(2)}{}^{\lambda} = D\xi_{(1)}{}^{\lambda}/ds = {}_{v}Kw^{\lambda}, \cdots, \\ &\xi_{(\gamma+1)}{}^{\lambda} = D\xi_{(\gamma)}{}^{\lambda}/ds, \end{aligned} \tag{1.8}$$

where ${}_{v}K$ is the absolute curvature of v at P with respect to C and the sense of w is chosen in such a way as to make ${}_{v}K>0$. If these vectors $\xi_{(\alpha)}{}^{\lambda}$ $(\alpha=1, \cdots, n)$ are assumed to be linearly independent, the following linear combinations of them for $p=1, \cdots, n$ form a set of n mutually orthogonal vectors:

$$\mu_{(p)}{}^{\lambda} = \left(\frac{y_{p}}{y_{p-1}}\right)^{\frac{1}{2}} \xi_{(\gamma)}{}^{\lambda} Y^{\gamma}_{p} \qquad (\gamma,\ \epsilon = 1, \cdots, p) \tag{1.9}$$

where

$$y_0 = 1, \qquad y_p = |y^{\epsilon}_{\gamma}|, \qquad y^{\epsilon}_{\gamma} = y^{\gamma}_{\epsilon} = g_{\lambda\mu}\xi_{(\gamma)}{}^{\lambda}\xi_{(\epsilon)}{}^{\mu}, \qquad y^{\epsilon}_{\gamma}Y^{\gamma}_{\rho} = \delta^{\epsilon}_{\rho}.$$

And we have $\mu_{(1)}{}^{\lambda}=v^{\lambda}$, $\mu_{(2)}{}^{\lambda}=w^{\lambda}$.
Putting $(D\mu_{(h)\nu}/ds)\,\mu_{(k)}{}^{\nu}=\beta_{hk}$ $(h,\ k=1, \cdots, n)$, from $\mu_{(h)\nu}\mu_{(k)}{}^{\nu}=\delta^{k}_{h}$ we have

$$\beta_{kh} = -\beta_{hk} \tag{1.10}$$

$$\frac{D\mu_{(k)}{}^{\nu}}{ds} = \sum_{h} \beta_{kh}\mu_{(h)}{}^{\nu}. \tag{1.11}$$

(1.11) are reduced to

$$\frac{D\mu_{(k)}{}^{\nu}}{ds} = -{}_{v}L_{k-1}\mu_{(k-1)}{}^{\nu} + {}_{v}L_{k}\mu_{(k+1)}{}^{\nu} \qquad (k=2,\cdots,n-1) \tag{1.12}$$

where ${}_{v}L_{k}=\beta_{kk+1}=-\beta_{k+1k}$.

(1.12) apply to the case $k=1$ with the understanding that ${}_{v}L_{0}=0$ and ${}_{v}L_{1}={}_{v}K$. Also, we have (1.12) for $k=n$ with the understanding that ${}_{v}L_{n}=0$. ${}_{v}L_{k}$ $(k=1,\cdots,n-1)$ are called, respectively, the associate curvatures of order $1,\cdots,n-1$ of the vector field v for the curve C. We call (1.12) the *formulas of Frenet of the first kind for* v *along* C *in* F_n.

2. Extension.

We consider a subspace F_m $(m<n)$ given by $x^{\lambda}=x^{\lambda}(u^{1},\cdots,u^{m})$ $(\lambda=1,\cdots,n)$ in a Finsler space F_n. The element of support is tangential to F_m. Let N^{λ}_{p} $(p=m+1,\cdots,n)$ be $n-m$ mutually orthogonal unit vectors normal to F_m with respect to the metric of F_n. Let v^{λ} be an arbitrary but fixed unit vector field defined at every point of F_m such that $v^{\lambda}=v^{a}B^{\lambda}_{a}$, ${}'g_{ab}v^{a}v^{b}=1$ and C: $u^{a}=u^{a}(s)$ $(a=1,\cdots,m)$ be a curve on F_m.

We denote the absolute differential of v^{λ} with respect to C at P by Dv^{λ} and define the following vectors:

$$\begin{aligned} &\eta_{(1)}{}^{\lambda}=v^{\lambda}, \quad \eta_{(2)}{}^{\lambda}={}_{v}k_{1}N^{\lambda}_{q}, \cdots, \\ &\eta_{(\gamma+1)}{}^{\lambda}=\frac{D\eta_{(\gamma)}{}^{\lambda}}{ds} \qquad (\gamma=2,\cdots,n-1), \end{aligned} \tag{2.1}$$

where ${}_{v}k_{1}=g_{\lambda\mu}N^{\lambda}_{q}Dv^{\mu}/ds$.

If $\eta_{(\beta)}{}^{\lambda}$ $(\beta=1,\cdots,n)$ are linearly independent, the following linear combinations of them for $p=1,\cdots,n$ form a set of n mutually orthogonal vectors:

$$\sigma_{(p)}{}^{\lambda}=\left(\frac{f_{p}}{f_{p-1}}\right)^{\frac{1}{2}}\eta_{(\gamma)}{}^{\lambda}F^{\gamma}_{p} \qquad (\gamma,\ \epsilon=1,\cdots,p) \tag{2.2}$$

where

$$f_{0}=1, \quad f_{1}=1, \quad f_{p}=|f^{\epsilon}_{\gamma}|,$$

$$f^{\epsilon}_{\gamma}=f^{\gamma}_{\epsilon}=g_{\lambda\mu}\eta_{(\gamma)}{}^{\lambda}\eta_{(\epsilon)}{}^{\mu}, \quad f^{\epsilon}_{\gamma}F^{\gamma}_{\rho}=\delta^{\epsilon}_{\rho}.$$

Therefore putting $(D\sigma_{(q)\nu}/ds)\ \sigma_{(p)}{}^{\nu}=\alpha_{qp}$, we have

$$\frac{D\sigma_{(p)}{}^{\mu}}{ds}=-{}_{v}K_{p-1}\sigma_{(p-1)}{}^{\mu}+{}_{v}K_{p}\sigma_{(p+1)}{}^{\mu} \qquad (p=2,\cdots,n-1), \tag{2.3}$$

where ${}_{v}K_{p}=\alpha_{pp+1}=-\alpha_{p+1p}$.

We call (2. 3) the *formulas of Frenet of the second kind for v along C in F_n*. ${}_vK_p$ $(p=1, \cdots, n-1)$ are called, respectively, the associate curvatures of order $1, \cdots, n-1$ of the vector field v for the curve C. (2. 3) hold except the case $p=1$ and apply to the case $p=n$ with the understanding that ${}_vK_n=0$.

While, the formulas of Frenet of the first kind for v along C in F_n may be derived in the same way as is mentioned in 1.

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References

1) Pan, T. K.: Proc. Amer. Math. Soc. **8**, 294 (1957)
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