# On Complete Continuity of P. S. Uryson's Operator in Function Spaces

# On Complete Continnity of P. S. Uryson's Operator in Function Spaces

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**Abstract**

The purpose of this paper is to give conditions of both the continuity and compactness of Uryson's operator $\int K[s, t, \phi(t)]\, dt$ which acts in modulared function spaces.

**1. Introduction.** In non-linear integral equations, the complete continuity of an operator from which the equation is produced plays a very impotant role, for example, the existence of solutions or eigen-functions in the equations. (cf. M. A. Krasnosel'skii[3] and S. Yamamuro[13])

A sufficient condition of the complete continuity of Uryson's operator acting in the space $C$, as the totality of all continuous functions on a compact subset in Euclidean space, have been given by L. A. Ladyzhenskii[4].

In case the operator acts from the space $L_{p_1}$ $(p_1>1)$ to the space $L_{p_2}$ $(p_2>1)$, M. A. Krasnosel'skii and L. A. Ladyzhenskii have given some sufficient conditions of the complete continuity, but it seems that one result has a defect, so far as we see the fact described in [Amer. Math. Soc. Transl. Ser. 2, vol. 10, p 352].

In this paper, we will consider the operator acting in modulared function spaces with some restrictions, which was defined by H. Nakano[7], and we give some sufficient conditions for the complete continuity of the operator. (see Theorem 4 and 5)

**2. Preliminaries.** In this section, we will state an outline of modulared function spaces and fundamental definitions.

Let $\varDelta$ be a bounded subset in Euclidean space and mes $(\varDelta)=1$.

let $\boldsymbol{\Phi}(\xi, x)$ $(\xi \geqq 0, x \in \varDelta)$ be measurable on $\varDelta$ for each $\xi \geqq 0$ and non-decreasing convex function of $\xi \geqq 0$ for which satisfies:

1) $\boldsymbol{\Phi}(0, x)=0$ for all $x \in \varDelta$;
2) $\lim\limits_{\xi \to \alpha-0} \boldsymbol{\Phi}(\xi, x)=\boldsymbol{\Phi}(\alpha, x)$ for each $x \in \varDelta$;
3) $\lim\limits_{\xi \to +\infty} \boldsymbol{\Phi}(\xi, x)=+\infty$ for each $x \in \varDelta$;
4) for any $x \in \varDelta$, there exists $\alpha=\alpha(x)>0$ such that $\boldsymbol{\Phi}(\alpha, x)>+\infty$.

**The modulared function space** $\mathbf{L}_{\Phi}(\varDelta)$ is a totality of all measurable functions $\phi(x)$ on $\varDelta$ such that

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$$m(\alpha\phi)=\int_{\Delta}\Phi(\alpha|\phi(x)|, x)\,dx<+\infty \qquad \text{for some } \alpha>0.$$

When we define a **semi-order** (or partial order) in $\mathbf{L}_{\Phi}$ by the relation that $\phi\geqq\psi$ if and only if $\phi(x)\geqq\psi(x)$ except for a set of measure zero, the space $\mathbf{L}_{\Phi}$ is a supperuniversally continuous semiordered linear space*.

The above functional $m(\phi)$ on $\mathbf{L}_{\Phi}$ is called a modular on $\mathbf{L}_{\Phi}$ and satisfies the modular conditions[7]:

1) $0\leqq m(\phi)\leqq+\infty$ for all $\phi\in\mathbf{L}_{\Phi}$;
2) if $m(\xi\phi)=0$ for all $\xi\geqq 0$, then $\phi=0$
3) for any $\phi\in\mathbf{L}_{\Phi}$ there exists $\alpha>0$ such that $m(\alpha\phi)<+\infty$;
4) for every $\phi\in L_{\Phi}$, $m(\xi\phi)$ is a convex function of $\xi\geqq 0$;
5) $|\phi|\leqq|\psi|$ implies $m(\phi)\leqq m(\psi)$;
6) $|\phi|\cap|\psi|=0$ implies $m(\phi)+\psi)=m(\phi)+m(\psi)$;
7) $0\leqq\phi_{\lambda}\uparrow_{\lambda\in\Lambda}\phi$** implies $m(\phi)=\sup_{\lambda\in\Lambda} m(\phi_{\lambda})$.

Writing the left-derivative of $\Phi(\xi, x)$ at $\xi$ by $\varphi(\xi, x)$ with $\varphi(0, x)\equiv 0$, we have a measurable in $x$ and non-decreasing function $\varphi(\xi, x)$ in $\xi\geqq 0$. If we define an inverse function $\psi(\eta, x)$ of $\varphi(\xi, x)$ as $\eta=\varphi(\xi, x)$, such that it is non-decreasing function of $\eta\geqq 0$, $\psi(0, x)\equiv 0$ and

$$\psi(\eta-0, x)\leqq\xi\leqq\psi(\eta+0, x) \qquad \text{for } \eta=\varphi(\xi, x),$$
$$\varphi(\xi-0, x)\leqq\eta\leqq\varphi(\xi+0, x) \qquad \text{for } \xi=\psi(\eta, x),$$

then the function:

$$\Psi(\eta, x)=\int_0^{\eta}\psi(\eta, x)\,d\eta$$

is measurable in $x\in\Delta$ and satisfies the same conditions as $\Phi(\xi, x)$. Furtheremore, we have Young's inequality

$$\xi\eta\leqq\Phi(\xi, x)+\Psi(\eta, x)$$

for $\xi, \eta\geqq 0$ and $x\in\Delta$, with equality if one at least of the relations

$$\varphi(\xi-0, x)\leqq\eta\leqq\varphi(\xi+0, x), \quad \psi(\eta-0, x)\leqq\xi\leqq\psi(\eta+0, x)$$

is satisfied. By the function $\Psi(\eta, x)$, the space $\mathbf{L}_{\Psi}$ which is called a **conjugate space** of $\mathbf{L}_{\Psi}$ is defined, and further a modular $\overline{m}(\psi)$ on $\mathbf{L}_{\Psi}$, i. e.

$$\overline{m}(\psi)=\int_{\Delta}\Psi(|\psi(x)|, x)\,dx$$

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* A semi-ordered linear space $R$ is said to be supperuniversally continuous, if for any system $a_{\lambda}\geqq 0$ $(\lambda\in\Lambda)$ there exist countable $a_{\lambda_{\nu}}$ $(\lambda\in\Lambda)$ and $a\in R$ for which $a=\bigcap_{\nu=1}^{\infty}a_{\lambda_{\nu}}=\bigcap_{\lambda\in\Lambda}a_{\nu}$, where $\cap a_{\lambda}$ means a infimum of $a_{\lambda}$.

** $\phi_{\lambda}\uparrow_{\lambda\in\Lambda}\phi$ means that for any $\lambda$, $\mu\in\Lambda$ teere exists $\rho\in\Lambda$ such that $\phi_{\lambda}\cup\phi_{\mu}\leqq\phi_{\rho}$, and $\bigcap_{\mu\in\Lambda}\bigcup_{\lambda\geqq\mu}\phi_{\lambda}=\bigcup_{\mu\in\Lambda}\bigcap_{\lambda\geqq\mu}\phi_{\lambda}=\phi$, where $\bigcup_{\lambda\in\Lambda}\phi_{\lambda}$ is a supremum of $\phi_{\lambda}$.

is defined as follows:

$$\bar{m}(\psi) = \sup\left\{\int_\varDelta \psi(x)\,\phi(x)\,dx - m(\phi)\right\}$$

where $\bar{m}$ is called a **conjugate modular** of $m$.

In the space $\mathbf{L}_\Phi$, defining two kinds of norms:

$$|||\phi|||_\Phi = \inf_{m(\xi\phi)\leqq 1} \frac{1}{|\xi|}; \quad \|\phi\|_\Phi = \inf_{\xi>0} \frac{1+m(\xi\phi)}{\xi},$$

we have $|||\phi|||_\Phi \leqq \|\phi\|_\Phi \leqq 2\,|||\phi|||_\Phi$, and their norms are both monotone complete norms*, so $\mathbf{L}_\Phi$ is Banach space, because the above modular on $\mathbf{L}_\Phi$ is monotone complete*.

As examples of such spaces, we can denote the well-known following spaces.

Orlicz space[5,15]**, i. e., for a non-decreasing left-continuous function $\varphi(\xi)$ on $[0, \infty)$ with $\varphi(0)=0$, putting

$$\Phi(u) = \int_0^u \varphi(\xi)\,d\xi \qquad (u \geqq 0)$$

the totality of all measurable functions $\phi(x)$ on $\varDelta$ such that

$$\int_\varDelta \Phi(\alpha|\phi(x)|\,dx < +\infty \qquad \text{for some } \alpha > 0.$$

Space $\mathbf{L}_{p(x)}$, i. e., for a measurable function $p(x) \geqq 1$ $(x \in \varDelta)$, the totality of all measurable functions $\phi(x)$ on $\varDelta$ such that

$$\int_\varDelta \frac{1}{p(x)}|\alpha\phi(x)|^{p(x)}\,dx < +\infty \qquad \text{for some } \alpha > 0.$$

A modular $m(\phi)$ on $\mathbf{L}_\Phi$ is said to be upper bounded modular, if there exist $\alpha$, $\gamma > 1$ such that

$$\Phi(\alpha\xi, x) \leqq \gamma\Phi(\xi, x) \qquad \text{for all } \xi \geqq 0,\ x \in \varDelta,$$

And, $m$ is said to be lower bounded modular, if there exist $\alpha > \gamma > 1$ such that

$$\Phi(\alpha\xi, x) \geqq \gamma\Phi(\xi, x) \qquad \text{for all } \xi \geqq 0,\ x \in \varDelta.$$

If $m$ is lower (upper) bounded then its conjugate modular $\bar{m}$ is upper (lower) bounded.

$m$ is said to be bounded modular if it is upper and lower bounded modular. If $m$ is a bounded modular, then $\mathbf{L}_\Phi$ is reflexive as Banach space with the above norms[2], for instance, $\mathbf{L}_p$ $(p > 1)$ and Orlicz spaces defining by complementary Young's functions $\Phi(u)$ and $\Psi(v)$ for which satisfy both $(\varDelta^2)$-condition.

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* A norm $\|\phi\|$ is called to be monotone complete if $0 \leqq \phi_\nu \uparrow_{\nu=1}^{\infty}$ and $\sup_{\nu\geqq 1}\|\phi_\nu\| < +\infty$ implies the existence of an element $\phi$ such that $\phi_\nu \uparrow_{\nu=1}^{\infty} \phi$. A monotone completeness of a modular implies a monotone completeness of a norm, and a monotone completeness of a norm implies a completeness in ussal sense.[1,14]

** Orlicz spaces are modulared function spaces with constant modulars[7].

Throughout this paper we assume that the modulared function spaces $\mathbf{L}_{\Phi_i}$ and their conjugate spaces $\mathbf{L}_{\Psi_i}$ $(i=1, 2)$ have the bounded modulars, and the functions $\varphi_i(1, x)$, $\psi_i(1, x)$ are integrable on $\Delta$, where $\varphi_i$ and $\psi_i$ are the left-derivatives of $\boldsymbol{\Phi}_i$ and $\Psi_i$ respectively.

The integral operator :

$$\mathbf{A}\phi(s)=\int_{\Delta} K[s, t, \phi(t)]\, dt$$

is called the operator of P. S. Uryson, where the function $K[s, t, u]$ is defined for $(s, t)\in \Delta\times\Delta$ and for real number $u$.

In this paper, we will deal with the case which $K[s, t, u]$ is continuous in $u$ for fixed $(s, t)$ and measurable in the remainder of the variables for fixed $u$.

A subset F of Banach space E is called to be **compact** (**weakly compact**), if every infinite subset contains a subsequence converging (weak converging) in E.

An operator is called to be **bounded** if it transforms every bounded (in the norm) subset of Banach space $E_1$ into a set which is bounded (in the norm) in Banach space $E_2$.

An operator $\mathbf{A}$, acting from $E_1$ into $E_2$, is called to be **continuous at the point** $\phi_0\in E_1$ if, for every sequence $\{\phi_n\}$ converging to $\phi_0$, $\{\mathbf{A}\phi_n\}$ converges to $\mathbf{A}\phi_0$ in $E_2$. An operator is called to be **continuous** on E if it is continuous at each point of E.

An operator $\mathbf{A}$ is called to be **compact** if it transforms every bounded set into a compact set.

An operator $\mathbf{A}$ is called to be **completely continuous** if it is continuous and compact.

**3.** In this section, we will consider a sufficient condition of both the boundedness and continuity of Uryson's operator which acts from the space $\mathbf{L}_{\Phi_1}$ with a modular $m_1$ into the space $\mathbf{L}_{\Phi_2}$ with a modular $m_2$.

**Lemma 1.** *If $K[s, t, u]$ $(s, t\in\Delta, -\infty<u<+\infty)$ is measurable on $\Delta\times\Delta$ for fixed $u$ and continuous in $u$ for fixed $(s, t)$, then for any $a\leqq b$ there exists a bounded measurable function $h(s, t)$ on $\Delta\times\Delta$ such that*

$$\sup_{a\leqq u\leqq b}|K[s, t, u]|=|K[s, t, h(s, t)]| \qquad \textit{for each } s \textit{ and } t.$$

**Proof.** First, we shall show the measurability of the function

$$k(s, t)=\sup_{a\leqq u\leqq b}|K[s, t, u]|.$$

When we put, for any positive number $\alpha$,

$$E_\alpha=\{(s, t)\,;\, k(s, t)\leqq\alpha,\ F^r_{\alpha, n}=\{(s, t)\,;\, |K[s, t, u_r]|<\alpha+1/n\}$$

and
$$E_\alpha=\bigcap_{n=1}^{\infty}\bigcap_{u_r} F^r_{\alpha, n}{}^*,$$

* $\cap$ means the intersection of sets.

where $\{u_r\}$ is a totality of all rational numbers in the closed interval $[a, b]$ and $n$ is a natural number, we get a measurability of subset $F_\alpha$ of $\varDelta\times\varDelta$. Furthermore, we can see easily an equality $E_\alpha=F_\alpha$ so that $E_\alpha$ is a measurable subset of $\varDelta\times\varDelta$. Thus $k(s, t)$ is measurable on $\varDelta\times\varDelta$.

Next, we define the function $h(s, t)$ as, for each $(s, t)$, a maximum value of $u$'s for which hold the relations $k(s, t)=|K[s, t, u]|$.

For any $\beta$ $(a\leqq\beta\leqq b)$, putting

$$E_\beta=\{(s, t)\,;\ h(s, t)\leqq\beta\}$$

$$F_\alpha^n=\{(s, t)\,;\ \sup_{a\leqq\beta+1/n\leqq b}|K[s, t, u]|<\sup_{a\leqq u\leqq\beta}|K[s, t, u]|\}$$

and

$$F_\beta=\bigcap_{a\leqq\beta+1/n\leqq b}F_\beta^n$$

where $n$ is a natural number, we have also a measurable subset $F_\beta$ of $\varDelta\times\varDelta$ and an equality $E_\beta=F_\beta$, and hence $h(s, t)$ is measurable on $\varDelta\times\varDelta$. It is obvious that $h(s, t)$ is bounded on $\varDelta\times\varDelta$. We state the following:

**Theorem 1.** *Let $K[s, t, u]$ $(s, t\in\varDelta,\ -\infty<u<+\infty)$ be continuous in $u$ for fixed $s$ and $t$, and measurable on $\varDelta\times\varDelta$ for fixed $u$.*

*If it satisfies the following conditions:*

a) *for every bounded measurable function $h(s, t)$ on $\varDelta\times\varDelta$*

$$m_2\left(\int_\varDelta K[s, t,\ h(s, t)\,dt\right)<+\infty\,;$$

b) *for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\|\phi-\psi\|_{\Phi_1}<\delta$ implies*

$$m_2\left(\int_F\{K[s, t, \phi]-K[s, t, \psi]\}\,dt\right)<\varepsilon$$

*for* mes $(F)<\delta\,(F\subset\varDelta)$, *then Uryson's operator $\mathbf{A}\phi$ acts from $\mathbf{L}_{\Phi_1}$ into $\mathbf{L}_{\Phi_2}$, and is bounded and continuous.*

**Proof.** We prove at first that $\mathbf{A}\phi$ acts from $\mathbf{L}_{\Phi_1}$ into $\mathbf{L}_{\Phi_2}$ and is bounded. For any $\phi(t)\in\mathbf{L}_{\Phi_1}$, taking $\varepsilon=1$ in b) there exists $\delta=\delta(1)>0$ such that

$$m_2\left(\int_F\{K[s, t, \phi]-K[s, t, \psi]\}\,dt\right)<1$$

for $\|\phi-\psi\|_{\Phi_1}<\delta$ and mes $(F)<\delta$. Since we can select $\phi_i\in\mathbf{L}_{\Phi_1}$ $(i=0, 1, \cdots, k)$ such that $\phi_0=\phi$, $\|\phi_i-\phi_{i-1}\|_{\Phi_1}<\delta\,(i=1, 2, \cdots, k)$ and $\phi_k=0$, where $k=[\|\phi\|_{\Phi_1}/\delta]^*+1$, we have, by the convexity of $\Phi_2$,

$$m_2\left(\frac{1}{k+1}\int_E K[s, t, \phi]\,dt\right)$$

$$\leqq\sum_{i=0}^{k-1}\frac{1}{k+1}m_2\left(\int_F\{K[s, t, \phi_i]-K[s, t, \phi_{i+1}]\}\,dt\right)$$

$$+\frac{1}{k+1}m_2\left(\int_F K[s, t, 0]\,dt\right)\leqq\frac{k+A}{k+1}<1+A$$

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* $[x]$ is the symbol of Gauss.

where $A=m_2\left(\int_\varDelta |K[s,t,0]|\,dt\right)$, and

$$m_2\left(\int_F K[s,t,\phi]\,dt\right)\leqq B\cdot m_2\left(\frac{1}{k+1}\int_F K[s,t,\phi]\,dt\right)$$

where $B$ is only dependent on $k$, because $m_2$ is the upper bounded modular. Therefore, for a patition $\{\varDelta_1, \varDelta_2, \cdots, \varDelta_j\}$ of $\varDelta$, which satify mes $(\varDelta_i)<\delta$ for $i=1,2,\cdots,j$ where $j=[1/\delta]+1$, we have

$$\begin{aligned} m_2\left(\frac{1}{j}\int_F K[s,t,\phi]\,dt\right) &\leqq \sum_{i=1}^{j}\frac{1}{j}\, m_2\left(\int_{\varDelta_i} K[s,t,\phi]\,dt\right) \\ &\leqq B(1+A), \end{aligned}$$

and hence it follows that, by the upper boundedness of $m_2$,

$$\begin{aligned} m_2\left(\int_\varDelta K[s,t,\phi]\,dt\right) &\leqq C\cdot m_2\left(\frac{1}{j}\int_\varDelta K[s,t,\phi]\,dt\right) \\ &\leqq CB(1+A) \end{aligned}$$

where $C$ is only dependent on $j$. Thus, it is shown that $\mathbf{A}\phi(s)\in \mathbf{L}_{\Phi_2}$ and further $\|\phi\|_{\Phi_1}\leqq \gamma$ implies $m_2(\mathbf{A}\phi)\ <CB(1+A)$, that is, $\|\mathbf{A}\phi\|_{\Phi_2}<2\,CB(1+A)$*, where $k=[\gamma/\delta]+1$.

Next, we prove the continuity of the operator $\mathbf{A}\phi$.

If $\lim_{k\to\infty}\|\phi_k-\phi_0\|_{\Phi_1}=0$ ($\phi_k$, $\phi_0\in \mathbf{L}_{\Phi_1}$) then $\{|\phi_k-\phi_0|\}$ converge to 0 weakly and hence $\lim_{k\to\infty}\int_\varDelta |\phi_k(t)-\phi_0(t)|\,dt=0$. Accordingly, we can select a subsequence $\{\phi_n(t)\}$ converging to $\phi_0(t)$ for almost all $t$.

Since $\phi_0(t)$ is almost all finite on $\varDelta$, for any natural number $\nu$ there exist $M_\nu>0$ and a subset $\mathrm{E}_\nu\subset\varDelta$ such that mes$(\mathrm{E}_\nu)\geqq 1-1/\nu$ and $|\phi_0(t)|\leqq M_\nu$ for all $t\in \mathrm{E}_\nu$.

Furthermore, by Egoroff's theorem, for any $\varepsilon>0$ there exists a subset $\mathrm{X}\subset\varDelta$ such that

mes$(\varDelta-\mathrm{X})<\varepsilon$ and $\{\phi_n\}$ converge to $\phi_0$ uniformly on X.

Putting $\mathrm{X}_\nu=\mathrm{X}\cap \mathrm{E}_\nu$, we have mes $(\varDelta-\mathrm{X}_\nu)\leqq\varepsilon+1/\nu$ and for all of sufficiently large $n$,

$$\left|\int_{X_\nu} K[s,t,\phi_n(t)]\,dt\right| \ \leqq \int_{X_\nu}\sup_{|M_\nu-u|\leqq\varepsilon}|K[s,t,u]|\,dt<+\infty$$

for almost all $s$, beacause, by Lemma 1, the assumption a) implies

$$m_2\left(\int_\varDelta \sup_{|M_\nu-u|\leqq\varepsilon}|K[s,t,u]|\,dt\right)<+\infty,$$

and hence

* This is obtained from the fact that $m_2(x)\leqq 1$ implies $\|x\|_{\Phi_2}\leqq 1$.

** The step function $f(t)=1$ on $\varDelta$ belongs to the conjugate space $\mathbf{L}_{\Psi_1}$ of $\mathbf{L}_{\Phi_1}$, because

$$\int_\varDelta \Phi_1(\psi_1(1,x),x)\,dx+\int_\varDelta \Phi_1(1,x)\,dx=\int_\varDelta \psi_1(1,x)\,dx<+\infty.$$

$$\int_{\Delta}\sup_{|M_\nu-u|\leqq\varepsilon}|K[s,t,u]|\,dt\in\mathbf{L}_{\Phi_2}.$$

Therefore, by Lebesgue's theorem, we have

$$\lim_{n\to\infty}\int_{X_\nu}K[s,t,\phi_n]\,dt=\int_{X_\nu}K[s,t,\phi_0]\,dt$$

for all $\nu$ and almost all $s$, since $K[s,t,\phi_n]$ converge to $K[s,t,\phi_0]$ for almost all $t$, $s$, and consequently it follows that

$$\lim_{n\to\infty}\Phi_2\left(\left|\int_{X_\nu}\left\{K[s,t,\phi_n]-K[s,t,\phi_0]\right\}dt,s\right)=0$$

for almost all $s$.

And, we have for all of sufficiently large $n$

$$\Phi_2\left(\frac{1}{2}\left|\int_{X_\nu}\left\{K[s,t,\phi_n]-K[s,t,\phi_0]\right\}dt\right|,s\right)$$

$$\leqq\frac{1}{2}\Phi_2\left(\left|\int_{X_\nu}K[s,t,\phi_n]\,dt\right|,s\right)+\frac{1}{2}\Phi_2\left(\left|\int_{X_\nu}K[s,t,\phi_0]\,dt\right|,s\right)$$

$$\leqq\Phi_2\left(\int_{X_\nu}\sup_{|M_\nu-u|\leqq\varepsilon}|K[s,t,u]|\,dt,s,\right),$$

and the last term is integrable by a) and Lemma 1, so that, by Lebesgue's theorem,

$$\lim_{n\to\infty}\int_{\Delta}\Phi_2\left(\left|\int_{X_\nu}\left\{K[s,t,\phi_n]-K[s,t,\phi_0]\right\}dt\right|,s\right)ds=0,$$

because $m_2$ is upper bounded.

Now, for any $\varepsilon>0$, when we select $\nu$, $\varepsilon_1$ in the above as which satisfy $\varepsilon_1+1/\nu<\delta$ where $\delta=\delta(\varepsilon)$ is the number in the assumption b), there exists $n_0=n_0(\varepsilon)$ such that

$$\|\phi_n-\phi_0\|_{\Phi_1}<\delta\ \text{ and }\ \int_{\Delta}\Phi_2\left(\left|\int_{X_\nu}\left\{K[s,t,\phi_n]-K[s,t,\phi_0]\right\}dt\right|,s\right)ds<\varepsilon,$$

and consequently, it follows that, by the convexity and upper boundedness of $m_2$,

$$m_2(\mathbf{A}\phi_n(s)-\mathbf{A}\phi_0(s))<N\cdot\varepsilon$$

where $N$ is a constant for which satisfies

$$\Phi_2(2\xi,s)\leqq N\cdot\Phi_2(\xi,s)\qquad\text{for all }\xi\geqq0\text{ and }s.$$

This shows that $\{\mathbf{A}\phi_n\}$ converges to $\mathbf{A}\phi_0$ by the modular* and hence it follows that

$$\lim_{\to\infty n}\|\mathbf{A}\phi_n-\mathbf{A}\phi_0\|_{\Phi_2}=0.$$

If we suppose that $\lim_{k\to\infty}\|\phi_k-\phi_0\|_{\Phi_1}=0$ and

* If a modular $m$ is upper bounded, then $\lim_{n\to\infty}m(\xi(x_n-x))=0$ for all $\xi\geqq0$ is equivalent to $\lim_{n\to\infty}m(x_n-x)=0$, and that the modular convergence coincides with the norm convergence. (cf. H. Nakano[7])

$$(\sharp) \qquad \|\mathbf{A}\phi_k-\mathbf{A}\phi_0\|_{\Phi_2}\geqq\varepsilon \qquad \text{for some } \varepsilon>0 \text{ and } k=1,2,\cdots,$$

then we can find a subsequence $\{\phi_n(t)\}$ converging to $\phi_0(t)$ in almost all $t\in\varDelta$ and hence it follows, as is shown above, that

$$\lim_{n\to\infty}\|\mathbf{A}\phi_n-\mathbf{A}\phi_0\|_{\Phi_2}=0.$$

This is contradiction to ($\sharp$). Thus the operator is continuous.

**Remark.** In the operator of Hammerstein type, i.e.

$$\mathbf{H}\phi(s)=\int_\varDelta K(s,t)f(t,\phi(t))\,dt$$

it is known that the operator $\mathbf{H}\phi$ is continuous (moreover, it is compact) in Orlicz space $\mathbf{L}_\Phi^{*}$* if it satisfies the following conditions:

$$1) \qquad \int_\varDelta\Phi\left(\int_\varDelta\Psi_1\left(\left|K(s,t)\right|\right)dt\right)ds>+\infty\,;$$

$$2) \qquad |f(t,u)|\leqq a(t)+\Phi^{-1}(b\Phi(|u|))$$

where $a(t)\in\mathbf{L}_{\Phi_1}^*$, $b>0$ and $\Phi$, $\Phi_1$ and their complementary Young's functions $\Psi$, $\Psi_1$ satisfy the $(\varDelta_2)$-condition.[3,11,12,15]

Those conditions satisfy the condtions in Theorem 1, because the condition 2) implies the boundedness of the operator $\mathbf{f}$:

$$\mathbf{L}_\Phi^*\ni\phi(t)\to f(t,\phi(t)\in\mathbf{L}_{\Phi_1}^*,$$

and also the bounded set $\mathfrak{A}$ in $\mathbf{L}_{\Phi_1}^*$ is the absolutely equi-continuous integrals[8], since

$$\int_\varDelta f(x)\cdot\phi_1(f(x))\,dx\leqq M<+\infty \qquad \text{for all } f(x)\in\mathfrak{A},$$

where $\phi_1$ is a left-derivative of $\Psi_1$, consequently, the condition b) is satisfied.

**4.** In this section, we will consider the compactness of Uryson's operator. L. A. Ladyzhenskii[7] given a sufficient condition of the compactness of the operator acting in the space $\mathbf{C}$, which it is proved by use of Ascoli-Arzela's theorem. V. V. Nemyckii[9] shown a sufficient condition of the compactness of the operator in the space $\mathbf{C}$ and his proof is placed on the basis of Kolmogoroff's criteria concerning for a compactness of a set. Those conditions have been established under the assumption that $\varDelta$ is bounded closed set in $n$-dimensional Euclidian space $\mathbf{R}_n$ with Lebesgue measure.

We will give a theorem concerning for the compactness of the operator which acts in modlared function spaces defining on a bounded set in $\mathbf{R}_n$.

**Throem 2.** *Let the operator* $\mathbf{A}\phi$ *be the bounded operator which acts from the unit sphere* $\mathbf{S}_1$ *of* $\mathbf{L}_{\Phi_1}$ *into* $\mathbf{L}_{\Phi_2}$. *Further, if it satisfies the condition*

$$(\sharp\sharp) \qquad \int_\varDelta|K[x,t,\phi(t)]-K[s,t,\phi(t)]|\,dt\leqq f(s)\cdot p(h) \quad (\phi\in\mathbf{S}_1)$$

* $L_\Phi^*$ means the Orlicz space satisfying $(\varDelta_2)$-condition. cf. A. C. Zaanen[15]

*for* $\|x-s\| \leqq h$ *(*$\|x\|$ *is the usual norm in* $\mathbf{R}_n$*), where* $f(s) \in \mathbf{L}_{\Phi_2}$ *and* $p(h)$ *is some real function tending to zero as* $h \to 0$*, then* $\mathbf{A}\phi$ *is the compact operator from* $\mathbf{S}_1$ *into* $\mathbf{L}_{\Phi_2}$*.*

**Proof.** Putting

$$(\mathbf{A}\phi(s))^\delta = \frac{1}{V(\delta)} \int_{U(s,\delta)} \mathbf{A}\phi(x)\, dx$$

where $V(\delta)$ is the volume of $U(s, \delta)$ which is the sphere with the center $s$ and the radius $\delta$, we have, by (♯♯),

$$\begin{aligned} &\Phi_2(|\mathbf{A}\phi - (\mathbf{A}\phi)^\delta|, s) \\ &\leqq \Phi_2\Big(\frac{1}{V(\delta)} \int_U \Big| \int_\Delta \Big\{K[s, t, \phi] - K[x, t, \phi]\Big\} dt \,\Big| dx, s\Big) \\ &\leqq \Phi_2(f(s)\, p(\delta), s) \text{ for almost all } s \in \Delta \text{ and all } \phi \in \mathbf{S}_1, \end{aligned}$$

and the last term is integrable on $\Delta$.

On the other hand, we have obviously

$$\lim_{\delta \to 0} \Phi_2(f(s)\, p(\delta), s) = 0 \qquad \text{for almost all } s \in \Delta.$$

Therefore, we have

$$\lim_{\delta \to 0} m_2(\mathbf{A}\phi - (\mathbf{A}\phi)^\delta) = 0 \qquad \text{uniformly on } \mathbf{S}_1,$$

i. e., for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\|\mathbf{A}\phi - (\mathbf{A}\phi)^\delta\|_{\Phi_2} < \varepsilon \qquad \text{for all } \phi \in \mathbf{S}_1.$$

Accordingly, if it is shown that $\{(\mathbf{A}\phi)^\delta\}$ $(\phi \in \mathbf{S}_1)$ is compact set in $\mathbf{L}_{\Phi_2}$ then the compactness of the operator $\mathbf{A}\phi$ is obvious.

Since $\mathbf{L}_{\Phi_2}$ is reflexive as Banach space, the boundedness of $\{\mathbf{A}\phi\}$ $(\phi \in \mathbf{S}_1)$ implies the weak compactness of $\{\mathbf{A}\phi\}$ $(\phi \in \mathbf{S}_1)$. Therefore, for any infinite sequence in $\{\mathbf{A}\phi\}$ $(\phi \in \mathbf{S}_1)$ we can find a subsequence, such that for every $\delta > 0$

$$\lim_{n \to \infty} (\mathbf{A}\phi_n(s))^\delta = (\psi_0(s))^{\delta *} \qquad \text{for almost all } s \in \Delta,$$

where $\psi_0(s)$, $(\psi_0(s))^\delta \in \mathbf{L}_{\Phi_2}$**. Also, we have

$$\begin{aligned} |(\mathbf{A}\phi_n(s))^\delta - (\psi_0(s))^\delta| &\leqq \frac{1}{V(\delta)} \|\chi_\Delta\|_{\Psi_2} \cdot \|\mathbf{A}\phi_n - \psi_0\|_{\Phi_2} \\ &\leqq \frac{1}{V(\delta)} \|\chi_\Delta\|_{\Psi_2} \{M + \|\psi_0\|_{\Phi_2}\} \end{aligned}$$

for almost all $s \in \Delta$, where $M = \sup_{\phi \in \mathbf{S}_1} \|\mathbf{A}\phi\|_{\Phi_2}$.

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* Weakly convergent sequence $\{A\phi_n\}$ is the requirement, because all step functions belong to $L_{\Psi_2}$.

** It follows that $\left|\int_{U(s,\delta)} \psi_0(x)\, dx\right| \leqq \|\chi_\Delta\|_{\Psi_2} \|\psi_0\|_{\Phi_2}$ and all step functions on $\Delta$ belong to the spaces $L_{\Phi_2}$ and $L_{\Psi_2}$.

Therefore, we have

$$\lim_{n\to\infty} \|(\mathbf{A}\phi_n(s))^{\delta}-(\psi_0(s))^{\delta}\|_{\Phi_2}=0,$$

namely, $\{(\mathbf{A}\phi)^{\delta}\}$ $(\phi\in\mathbf{S}_1)$ is compact in $\mathbf{L}_{\Phi_2}$ by the definition.

**Theorem 3.** *When $\mathbf{L}_{\Phi_i}$ $(i=1,2)$ are Orlicz spaces satisfying the $(\Delta_2)$-condition, we can replace the condition (##) by the waeker conditions: for almost all s*

$$(\sharp_{\sharp\sharp})\qquad \lim_{\substack{x\to s\\ x,s\in\Delta}} \int_{\Delta} |K[x,t,\phi]-K[s,t,\phi]|\,dt=0 \qquad \text{uniformly on } \mathbf{S}_1;$$

$$(\sharp\sharp_{\sharp\sharp})\qquad \sup_{\phi\in\mathbf{S}_1} |\mathbf{A}\phi(s)|=f(s)\in\mathbf{L}_{\Phi_2}.$$

**Proof.** Since we know easily

$$\begin{aligned}\Phi_2(|\mathbf{A}\phi-(\mathbf{A}\phi)^{\delta}|) &\leqq C\left\{\Phi_2(|\mathbf{A}\phi|+\Phi_2\left(\frac{1}{V(\delta)}\left|\int_{\Delta}\int_{U(s,\delta)}K[x,t,\phi]\,dx\,dt\right|\right)\right\}\\ &\leqq C\left\{\Phi_2(|f(s)|)+\Phi_2\left(\frac{1}{V(\delta)}\|\chi_U\|_{\Phi_2}\cdot\|\mathbf{A}\phi\|_{\Phi_2}\right)\right\}\\ &\leqq C\left\{\Phi_2(|f(s)|+\Phi_2(\alpha\cdot M)\right\}\in\mathbf{L}_{\Phi_2}\end{aligned}$$

where $\alpha=\lim_{\xi\to\infty}\Psi_2^{-1}(\xi)/\xi<+\infty$*, $M=\sup_{\phi\in\mathbf{S}_1}\|\mathbf{A}\phi\|_{\Phi_2}$ and $C$ is some constant, the theorem is proved by the same method as the proof of Theorem 2.

**Lemma 2.** *If $\mathbf{L}_{\Phi}(\Delta,\mu)$ is a modulared function space, defining on a bounded set $\Delta$ in $\mathbf{R}_n$, with the upper bounded modular, then for any $\varphi\in\mathbf{L}_{\Phi}$, we have*

$$\lim_{||h||\to\phi}\|\varphi(x+h)-\varphi(x)\|_{\Phi}=0$$

*where $\varphi(x+h)=0$ if $x+h\in\Delta$ and $\|h\|$ is the usual norm in $\mathbf{R}_n$.*

**Proof.** For any $\varepsilon>0$ and $\varphi\in\mathbf{L}_{\Phi}$, there exists a closed subset G of $\Delta$ such that $\|\varphi-\varphi_G\|<\varepsilon$ where

$$\varphi_G(t)=\begin{cases}\varphi(t) & \text{if } t\in G\\ 0 & \text{if } t\notin G\end{cases}$$

and $\varphi_G\in\mathbf{L}_{\Phi}$.

Therefore, we will prove the lemma for a function on G.

(i) Putting, for $x\in G$

$$\varphi_n(x)=\begin{cases} n & \text{if } \varphi(x)\geqq n\\ \varphi(x) & \text{if } -n\leqq\varphi(x)\leqq n\\ -n & \text{fi } \varphi(x)\leqq -n,\end{cases}$$

we have $\lim_{n\to\infty}|\varphi_n(x)-\varphi(x)|=0$ for almost all $x\in G$ and $|\varphi_n(x)-\varphi(x)|\leqq 2|\varphi(x)|$,

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* $\lim_{\xi\to\infty}\Psi_2^{-1}(\xi)/\xi<+\infty$ is equivalent to $\lim_{\eta\to\infty}\eta/\Psi_2(\eta)=0$ and $\|\chi_U\|_{\Phi_2}\leqq 1/\Psi_2(1/V(\delta))$.

therefore, it follows that

$\lim_{n\to\infty} \|\varphi_n-\varphi\|_\Phi=0$, i. e., for any $\varepsilon>0$, there exists $n_0=n_0(\varepsilon)>0$ such that

$$\|\varphi_{n_0}-\varphi\|_\Phi<\varepsilon \text{ and } |\varphi_{n_0}(x)| \text{ is bounded on G.}$$

(ii) Let $f(x)$ is bounded on G, i. e. $|f(x)|\leqq M$ on G. For any $\varepsilon>0$ and $\sigma>0$, there exists a continuous function $g(x)$ on G such that $|g(x)|\leqq M$ on G and

$$\mu(\{x;\ |f(x)-g(x)|\geqq\sigma\}<\varepsilon.$$

This statement is proved by the same method as the proof of Borel's theorem which is stated for $G=[0, 1]$ (cf. I. P. Natanson[8]).

Namely, for such nutural number $l$ as $M/l<\sigma$, putting

$$E_i=\{x;\ (i-1)M/l\leqq f(x)\leqq iM/l \text{ and } x\in G\}\quad (i=1-l, 2-l, \cdots, l-1)$$

and
$$E_l=\{x;\ (l-1)M/l\leqq f(x)\leqq M\},$$

we get a partition $\{E_i\}$ $(i=1-l, 2-l, \cdots, l)$ of G.

Since $E_i$ are Lebesgue measurable sets, we can select closed sets $F_i$ such that

$$\mu(F_i)>\mu(E_i)-\varepsilon/2l \quad\text{and}\quad F_i\subset E_i.$$

Defining a continuous function $g_1(x)$ on $F=\bigcup_{i=1-l}^{l} F_i$ such as

$$g_1(x)=iM/l \quad \text{if } x\in F_i \quad (i=1-l, 2-l, \cdots, l),$$

we have $|f(x)-g_1(x)|\leqq M/l<\sigma$ for $x\in F$.

Further, we get a continuous function $g(x)$ on G such that it is a extension of $g_1(x)$ on G, for which satisfies

$$|g(x)|\leqq M \quad\text{and}\quad g(x)=0 \quad\text{if } x\in G-F.$$

The function $g(x)$ is the requirement.

(iii) By (ii), there exists a sequence $\{g_n(x)\}$ of continuous functions on G such that it converges in measure on G. Therefore we have, by Lebesgue's theorem,

$$\lim_{k\to\infty}\int_\Delta \Phi_2(|f(x)-g_{n_k}(x)|, x)\,dx=0$$

for some subsequence of $\{g_n(x)\}$. Accordingly, we have

$$\lim_{k\to\infty}\|f-g_{n_k}\|_\Phi=0,$$

and hence there exists a continuous function $g(x)$ on G such that

$$\|f-g\|_\Phi<\varepsilon.$$

(iv) If we assume that $f(x+h)=g(x+h)=0$ for $x+h\notin G$, then we have, for enough small $\|h\|$, $\|g(x+h)-g(x)\|_\Phi<\varepsilon$ and $\|f(x+h)-g(x+h)\|_\Phi<\varepsilon$, which implies the required fact, i. e.

$$\|\varphi(x+h)-\varphi(x)\|_\Phi<5\varepsilon.$$

**Remark.** Suppose $\Delta$ is a bounded set in $\mathbf{R}_n$. Let $\Phi_i, \Psi_i$ $(i=1, 2)$ be Young's functions satisfying the $(\Delta_2)$-condition.

If
$$\int_\Delta\int_\Delta \Psi(R(s, t))\, ds\, dt < +\infty$$
where $\Psi \equiv \Phi_2[\Psi_1]$, then the linear operator $\int_\Delta R(s, t)\phi(t)\, dt$ satisfies the conditions in Theorem 3, and hence the operator is a compact operator from $\mathbf{S}_1 \subset \mathbf{L}^*_{\Phi_1}$ into $\mathbf{L}^*_{\Phi_2}$. (cf. A. C. Zaanen[15], Krasnoselskii and Ya. B. Rutitskii ; Dokl. Akad. Nauk SSSR (n. s) 85 (1952), 33–36. Russian)
Because, by Lemma 2, we have
$$\lim_{\|n\|,\ \|k\|\to 0}\int_\Delta\int_\Delta \Psi(|R(s+h, t+k)-R(s, t)|)\, ds\, dt=0$$
and hence
$$\lim_{\|h\|\to 0}\int_\Delta \Psi_1(|R(s+h, t)-R(s, t)|\, dt=0 \qquad \text{for almost all } s\in\Delta.$$
And, we have also
$$\int_\Delta |R(s+h, t)-R(s, t)|\cdot|\phi(t)|\, dt \leqq \|R(s+h, t)-R(s, t)\|_{\Psi_1} \qquad \text{for } \phi\in\mathbf{S}.$$
Namely, the assumptions of Theorem 3 are satisfied.

**5.** Combined the results in the section 3 with those in the section 4, we get the conditions of the complete continuity of the operator.

**Theorem 4.** *Let $\mathbf{L}_{\Phi_i}$ $(i=1, 2)$ are modulared function spaces with the bounded modulars, defining on a bounded set $\Delta$ in $\mathrm{R}_n$. Let $K[s, t, u]$ be continuous in $u$ $(-\infty<u<+\infty)$ for fixed $(s, t)$ and measurable on $\Delta\times\Delta$ for fixed $u$ satisfying the follywing conditions* :

a) *for every bounded measurable function $h(s, t)$ on $\Delta\times\Delta$*
$$m_2\left(\left|\int_\Delta K[s, t, h(s, t)]\, dt\right|, s\right) < +\infty\,;$$

b) *for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that $\|\phi-\psi\|_{\Phi_1}<\delta$ implies*
$$m_2\left(\left|\int_F \Big\{K[s, t, \phi]-K[s, t, \psi]\Big\}\, dt\right|, s\right) < \varepsilon$$
*for* $\mathrm{mes}(\mathrm{F})<\delta$ $(\mathrm{F}\subset\Delta)$ ;

c) $$\int_\Delta \Big|K[x, t, \phi(t)]-K[s, t, \phi(t)]\Big|\, dt \leqq f(s)\, p(h) \quad (\phi\in\mathbf{S}_1)$$
*for $\|x-s\|<h$, where $f(s)\in\mathbf{L}_{\Phi_2}$ and $p(h)$ tends to zero as $h\to 0$, then the operator $\mathbf{A}\phi(s)$ acts from $\mathbf{S}_1\subset\mathbf{L}_{\Phi_1}$ into $\mathbf{L}_{\Phi_2}$ and is completely continuous.*

**Theorem 5.** *In Theorem 4, if $\mathbf{L}_{\Phi_i}$ are Orlicz spaces, the condition* c) *is replaced by the following condition* :

c′) *for any bounded set $\mathfrak{A}$ in $\mathbf{L}_{\Phi_1}$ and almost all $s\in\Delta$,*

$$\lim_{x\to s}\int_{\Delta}\Big|K[x,t,\phi(t)]-K[s,t,\phi(t)]\Big|dt=0 \text{ uniformly on } \mathfrak{A}$$

*and*

$$\sup_{\phi\in\mathfrak{A}}|\mathbf{A}\phi(s)|=f(s)\in\mathbf{L}_{\Phi}.$$

**Remark.** Under the assumptions in the *remark of the section* 3, we obtain that *the operator of Hammerstein type* $\mathbf{H}\phi(s)$ acts in Orlicz space $\mathbf{L}_{\Phi}^{*}$ and is completely continuous in the unit sphere $\mathbf{S}_1$ of $\mathbf{L}_{\Phi}^{*}$. Since it is shown that the operator acts in $\mathbf{L}_{\Phi}^{*}$ and is continuous, it is sufficient to show the compactness of the operator.

Putting

$$K[s,t,u]=R(s,t)f(t,u)$$

we have for any $\phi\in\mathbf{S}_1$

$$\int_{\Delta}\Big|K[x,t,\phi]-K[s,t,\phi]\Big|dt=\int_{\Delta}\Big|R(x,t)-R(s,t)\Big|\ \Big|f(t,\phi)\Big|dt$$
$$\leqq\|R(x,t)-R(s,t)\|_{\Psi_1}\ \|f(t,\phi)\|_{\Psi_1}\leqq\|R(x,t)-R(s,t)\|_{\Psi_1}\cdot M$$

and

$\|R(s,t)\|_{\Psi_1}\in\mathbf{L}_{\Phi}^{*}$ from the assumption 1), where $\sup_{\varphi\in S_1}\|f(t,\phi)\|_{\Psi_1}=M<\infty$,

because the operator $\mathbf{f}$ : $\mathbf{L}_{\Phi}^{*}\in\phi(t)\to f(t,\phi)\in\mathbf{L}_{\Phi_1}^{*}$ is bounded[12]. Therefore, on the assumptions 1), 2) and Lemma 2, we will know that the conditions ($\sharp$) and ($\sharp\sharp$) in Theorem 3 are satisfied, namely the operator $\mathbf{H}\phi(s)$ is compact.

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