Orthogonal Projection of the Space X of Univoque Functions

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Abstract

Concerning the spatial construction of $\boldsymbol{X}$, the space of univoque functions $x(\xi)$ $(\xi \in \Xi)$, based on the reaxilization principle, orthogonal projection is defined in an explicit way and some problems are solved and detailed.

1. Introduction

The space $\boldsymbol{X}$ is posited as the aggregation of functions

$$x(\xi)\,(\xi \in \Xi)\,,$$

Ξ being a metric space provided with a normal measure $\tilde{\mu}$**, by which the product $(x|y)$ is defined as

$$(x|y) = \underset{\xi \in \Xi}{\mathfrak{S}} x(\xi)\overline{y(\xi)}\,\mu_\xi\,.$$

The norm $\|x\|$ is counted by the formula

$$\|x\|^2 = (x|x)\,,$$

but, in our theory, vectors x are not always restricted to be of finite norm. $x(\xi)$ is a univoque function of the variable ξ, which is complex-valued, but no more restrictions are given at all. In a previous paper[1] it has been demonstrated that: *if a subspace $\boldsymbol{Y}$ in $\boldsymbol{X}$ is a vector space by complex coefficients, then it is a span of the vectors*

$$(\rho_\lambda)_{\lambda \in \Lambda}$$

which are uniquely determined for $\boldsymbol{Y}$ such that the supports Ξ_λ of ρ_λ are mutually disjoint. This destination is called the *reaxilization*. The family (ρ_λ) is called the *natural basis* of $\boldsymbol{Y}$ and is generally denoted by

$$B(\boldsymbol{Y})\,.$$

To make an analysis by means of an integral of the form

$$F = \tilde{\mu}(f) = \mathfrak{S} f(\xi)\,\mu_\xi$$

we conform it to the rule that F is regarded as effective when and only when it

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** $\Xi \supset \Gamma \triangleright \tilde{\mu}\Gamma = \underset{\xi \in \Gamma}{\mathfrak{S}} \mu_\xi = \mu \cdot \mathfrak{n}(\Gamma)$; $\mu_\xi = \mu =$ infinitesimal.

is absolutely convergent, i.e.

$$\mathfrak{S}\,|f(\xi)|\cdot|\mu_\xi|$$

is convergent. If to indicate this rule specially, "*Riemann law*" may be the most pertinent name, because he is Riemann who showed for the first time that a not absolutely convergent series of real terms may be counted to take an arbitrarily chosen value if a proper rearrangement of terms is adopted. In addition, when

$$\mathfrak{S}f(\xi)\,\mu_\xi = \circledcirc\ (=\text{empty null})$$

and

$$\mu_\xi = \mu > \circledcirc\ ,$$

then it is thought equivalent to

$$\mathfrak{S}f(\xi) = 0\ .$$

We may have an explicit course of analysis when we adopt the formula

$$P_Y x = \mathop{\mathfrak{S}}_{\lambda\in\Lambda} \frac{(x|\rho_\lambda)}{\|\rho_\lambda\|^2}\,\rho_\lambda \tag{1, 1}$$

on condition that

$$(\rho_\lambda)_{\lambda\in\Lambda} = B(\boldsymbol{Y})\,,$$

to define (*orthogonal*) *projection* of a vector $x \in X$ on a (vector) subspace $\boldsymbol{Y}$ in $\boldsymbol{X}$. If the operator P_Y is found effective for all $x \in \boldsymbol{X}$, $\boldsymbol{Y}$ is said to be a *projective subspace*. When the space $\langle\!\langle y \rangle\!\rangle$ (which is generated by a single vector y) is projective, y is called a *projective vector*. Thus classified, linear operators are naturally found to be relative to the projection. As for the case $\|x\| = \infty$, the problem is settled of itself afterwards.

In a vector space $\boldsymbol{H}$ generated by an enumerable family of vectors, the theory of Hilbert space offers several results obtained to build up certain models of spatial construction, which are applicable to our analysis if we make any modification for parts of enumerable generation.

2. Projection and Orthogonal Supplements.

When a vector y is projective, by definition it is demanded that

$$\begin{aligned} P_y \partial_\xi^* &= \frac{(\partial_\xi|y)}{\|y\|^2}\,y = \frac{\overline{y(\xi)}\,\mu_\xi}{\|y\|^2}\,y = \frac{\overline{y(\xi)}\,\mu_\xi}{\mathfrak{S}|y(\eta)|^2\mu_\eta}\cdot y\ \ (\mu_\xi = \mu_\eta = \mu) \\ &= \frac{\overline{y(\xi)}}{\mathfrak{S}|y(\eta)|^2}\,y\ . \end{aligned} \tag{2, 1}$$

So, if $y \neq 0$, we may have

* ∂_ξ is the characteristic function of the single point set $\{\partial_\xi\}$.

$$0<\mathfrak{S}|y(\eta)|^2<\infty \tag{2, 2}$$

because then there must exist at least one ξ for which

$$P_y\partial_\xi\neq 0^*$$

in order that

$$\mathfrak{S}y(\xi)P_y\partial_\xi=\mathfrak{S}y(\xi)\frac{\overline{y(\xi)}\mu}{\|y\|^2}y=\frac{\|y\|^2}{\|y\|^2}y=y\neq 0$$

and because it should not be that

$$\mathfrak{S}|y(\eta)|^2=0$$

in order that $P_y\partial_\xi\neq\infty$. We see directly from (2, 2) that such values that $y(\eta)\neq 0$ make at most an enumerable set, so that we may write as

$$P_y\partial_\xi=\frac{\overline{y(\xi)}}{\sum_k|y(\eta_k)|^2}y\,. \tag{2, 3}$$

In case $y=0$, we have naturally

$$P_y\partial_\xi=0\quad\text{for all}\quad \xi\in\Xi\,.$$

To define a projective subspace there may be another way from that shown in Introduction. If

$$B(\boldsymbol{Y})=(\rho_\lambda)_{\lambda\in\Lambda}$$

and each ρ_λ is a projective vector, then $\boldsymbol{Y}$ is a projective subspace. By (2, 1) we then have

$$\begin{aligned}P_Yx&=\mathfrak{S}_\lambda P_{\rho_\lambda}x=\mathfrak{S}_\lambda\mathfrak{S}_\xi x(\xi)P_{\rho_\lambda}\partial_\xi=\mathfrak{S}\mathfrak{S}x(\xi)\mu\cdot\frac{\overline{\rho_\lambda(\xi)}}{\|\rho_\lambda\|^2}\rho_\lambda\\&=\mathfrak{S}_{\lambda\in\Lambda}\Big(\mathfrak{S}_\xi x(\xi)\frac{\overline{\rho_\lambda(\xi)}}{\|\rho_\lambda\|^2}\mu\Big)\rho_\lambda=\mathfrak{S}_\lambda\frac{\mathfrak{S}x(\xi)\overline{\rho_\lambda(\xi)}\mu_\xi}{\|\rho_\lambda\|^2}\rho_\lambda\\&=\mathfrak{S}\frac{(x|\rho_\lambda)}{\|\rho_\lambda\|^2}\rho_\lambda\end{aligned}$$

which gives us an induction of (1, 1) from (2, 1).

Now, in view of (2, 3) we may write as

$$\frac{(x|\rho_\lambda)}{\|\rho_\lambda\|^2}=\frac{\sum_k x(\xi_{\lambda k})\rho_\lambda(\xi_{\lambda k})}{\sum_k|\rho_\lambda(\xi_{\lambda k})|^2} \tag{2, 4}$$

where $\Xi_\lambda=(\xi_{\lambda 1},\xi_{\lambda 2},\cdots)$ $(\lambda\in\Lambda)$ are respectively the supports of ρ_λ. When $\boldsymbol{Y}$ is projective (2, 4) is demanded to be finite, so that in view of (2, 2) which gives us

$$0<\sum_k|\rho_\lambda(\xi_{\lambda k})|^2<\infty\,,$$

* This induces that $\mathfrak{S}|y(\eta)|^2<\infty$.

the sum

$$\sum x(\xi_{\lambda k})\rho_\lambda(\xi_{\lambda k})$$

must be convergent. Then, by the Riemann law

$$\sum |x(\xi_{\lambda k})|\cdot|\rho_\lambda(\xi_{\lambda k})|$$

is convergent for all $x \in \boldsymbol{X}$. Besides, by Hölder's inequality

$$\sum |x(\xi_{\lambda k})|\cdot|\rho_\lambda(\xi_{\lambda k})| \geqslant \sqrt{\sum |x(\xi_{\lambda k})|^2}\sqrt{\sum |\rho_\lambda(\xi_{\lambda k})|^2}\,.$$

So, it must be that

$$\sum |x(\xi_{\lambda k})|^2 < \infty \qquad \text{for all} \quad x \in \boldsymbol{X},$$

whereas, from our standpoint, we may take a vector x such that

$$x(\xi_{\lambda k}) = \frac{1}{\rho_\lambda(\xi_{\lambda k})}$$

whenever

$$\rho_\lambda(\xi_{\lambda k}) \neq 0\,.$$

Therefore, we may conclude:

Proposition 1. *In order that a vector subspace $\boldsymbol{Y}$ may be projective, it is necessary and sufficient that each vector $\rho_\lambda(\lambda \in \Lambda)$ of $B(\boldsymbol{Y})$ has its support Ξ_λ as a finite set.*

Corollary. *In order that a vector y may be projective, it is necessary and sufficient that the points η for which*

$$y(\eta) \neq 0$$

make at most a finite set.

When $\boldsymbol{Y}$ is a projective subspace, let us take

$$z = x - P_Y x$$

for any vector $x \in \boldsymbol{X}$, then we have

$$\begin{aligned}(z|P_Y x) &= \mathfrak{S}_\lambda \mathfrak{S}_\xi \Big(x(\xi) - \frac{(x|\rho_\lambda)}{\|\rho_\lambda\|^2}\rho_\lambda(\xi)\Big)\overline{\frac{(x|\rho_\lambda)}{\|\rho_\lambda\|^2}}\,\overline{\rho_\lambda(\xi)}\,\mu_\xi \\ &= \mathfrak{S}\Big(\frac{\overline{(x|\rho_\lambda)}}{\|\rho_\lambda\|^2}(x|\rho_\lambda) - \frac{|(x|\rho_\lambda)|^2}{\|\rho_\lambda\|^4}\|\rho_\lambda\|^2\Big) \\ &= \mathfrak{S}_\lambda\Big(\frac{|(x|\rho_\lambda)|^2}{\|\rho_\lambda\|^2} - \frac{|(x|\rho_\lambda)|^2}{\|\rho_\lambda\|^2}\Big)\end{aligned}$$

so that

$$z \perp P_Y x\,.$$

Hence the aggregation

$$\boldsymbol{Y}^\perp = \{z :\ x \in \boldsymbol{X} \text{ and } z = x - P_Y x\}$$

coincides with the *orthogonal* (or *projective*) *supplement* of $\boldsymbol{Y}$ i.r.t. the product $(\,|\,)$.

Proposition 2. *If $\boldsymbol{Y}$ is a projective subspace, its orthogonal supplement $\boldsymbol{Y}^{\perp}$ is projective, too.*

In effect, as $\boldsymbol{Y}$ is a vector subspace, so is $\boldsymbol{Y}^{\perp}$. Let it be that

$$B(\boldsymbol{Y})=(\rho_{\lambda})_{\lambda\in\Lambda} \text{ and } B(\boldsymbol{Y}^{\perp})=(\overset{*}{\rho}_{\mu})_{\mu\in M}$$

and let the supports of ρ_{λ} and $\overset{*}{\rho}_{\mu}$ be denoted by

$$\overset{*}{\Xi}_{\lambda} \text{ and } \overset{*}{\Xi}_{\mu}$$

respectively. Then we see that

$$(\forall\lambda)\,(\Xi_{\lambda}\cap\overset{*}{\Xi}_{\mu}=\text{void})\rhd(\exists\xi)\,(\overset{*}{\rho}_{\mu}=\partial_{\xi})$$

which means

$$\overset{*}{\Xi}_{\mu}=\{\xi\}$$

and

$$\Xi_{\lambda}\cap\overset{*}{\Xi}_{\mu}\neq\text{void}\rhd\overset{*}{\Xi}_{\mu}\subseteq\Xi_{\lambda}\,,$$

because

$$\xi\in\Xi_{\lambda}\rhd(\text{the support of } \partial_{\xi}-P_{Y}\partial_{\xi}\subseteq\Xi_{\lambda})\,.$$

Consequently all $\overset{*}{\Xi}_{\mu}(\mu\in\boldsymbol{M})$ are finite sets.

The cases $\|y\|=\infty$ or $\|\rho_{\lambda}\|=\infty$, are left out of what have been dealt with hitherto, but they may be treated in a simple way. In effect, such vectors may not be allowed to be projective ones on the point that their supports are infinite sets. So, they should be classified in the genre of non-projectivity.

3. Projective Operators

A linear operator L is understood as

$$Lx\in\boldsymbol{X} \text{ for each } x\in\boldsymbol{X},$$

i.e.

$$L(\boldsymbol{X})\subseteq\boldsymbol{X}.$$

If Lx is found to be a projective vector whenever x is so, then L is said to be a projective operator.

When L is a projective operator and

$$L\partial_{\xi}=\delta_{\xi}(\xi\in\Xi)\,,$$

we have

$$\bigvee_{\xi\in\Xi}\langle\!\langle\delta_{\xi}\rangle\!\rangle\subseteq L(\boldsymbol{X})=\boldsymbol{R}_{L}$$

($\boldsymbol{R}_L$ being the range of L). Hence, if

$$B(\boldsymbol{R}_L)=(\rho_\lambda)_{\lambda\in\Lambda}$$

and

$$\Xi_\lambda=\text{the support of } \rho_\lambda\,,$$

it must be that

$$(\forall\lambda)\,(\lambda\in\Lambda)\,(\exists\eta_\lambda)\,(\eta_\lambda\in\Xi \text{ and } \Xi_\lambda\subseteq\text{the support of } \delta_{\eta_\lambda})\,.$$

Besides, as each ∂_ξ is a projective vector, by the corollary of Proposition 1, the support of δ_{η_λ} is at most a finite set. Therefore it follows that Ξ_λ is a finite set, so that ρ_λ is a projective vector. Thus we have:

Proposition 3. *When L is a projective operator, its range $\boldsymbol{R}_L$ is a projective subspace of $\boldsymbol{X}$.*

Let I denote identity, i.e.

$$Ix=x \quad \text{for all} \quad x\in\boldsymbol{X}\,,$$

then it is evident that

$$P_Y+P_{Y^\perp}=I\,.$$

4. Proper Spaces and L-Span

Let it be denoted as

$$\widetilde{\boldsymbol{Z}}_\omega=\{x:\ x\in\boldsymbol{X} \text{ and } Lx=\omega x\}$$

and

$$\widetilde{\boldsymbol{Z}}_L=\bigvee_{\omega\in\Omega}\widetilde{\boldsymbol{Z}}_\omega$$

where Ω is the spectrum of L; then $\widetilde{\boldsymbol{Z}}_\omega$ is the proper space of ω; $\widetilde{\boldsymbol{Z}}_L$ shall be called the *spectral span* with respect to L or simply *L-span*. Naturally we have

$$\widetilde{\boldsymbol{Z}}_L\subseteq\boldsymbol{X}$$

but the equality is not always promised. If $\widetilde{\boldsymbol{Z}}_\omega$ is projective and such that

$$\left.\begin{aligned}&B(\widetilde{\boldsymbol{Z}}_\omega)=(\sigma_\kappa)_{\kappa\in K}\,;\ L(\boldsymbol{X})=\boldsymbol{R}_L\,;\\ &\text{and}\\ &B(\boldsymbol{R}_L)=(\rho_\lambda)_{\lambda\in\Lambda}\,;\end{aligned}\right\}\tag{4, 1}$$

then $L\sigma_\kappa$ is expressed in the form

$$\begin{aligned}L\sigma_\kappa&=\mathfrak{S}_{\lambda\in\Lambda_\kappa}\sigma_\kappa(\lambda)\rho_\lambda\\ &=\omega\sigma_\kappa\end{aligned}$$

where

$$\Lambda_\kappa=\{\lambda:\ \sigma_\kappa(\lambda)\neq 0\}\,.\tag{4, 2}$$

So, we have the representation

$$\sigma_\kappa = \underset{\lambda \in \Lambda_\kappa}{\mathfrak{S}} \sigma'_\kappa(\lambda) \rho_\lambda$$

with

$$\sigma'_\kappa(\lambda) = \frac{\sigma_\kappa(\lambda)}{\omega}.$$

This means that

$$\tilde{\boldsymbol{Z}}_\omega \subseteq \boldsymbol{R}_L$$

and moreover that

$$\bigcup_{\lambda \in \Lambda_\kappa} \Xi_\lambda = \text{the support of } \sigma_\kappa$$

Ξ_λ being the support of ρ_λ. By the reaxilization law Ξ_λ are mutually disjoint and since $\tilde{\boldsymbol{Z}}_\omega$ is projective the support of σ_κ is a finite set. It follows, therefore, that both Λ_κ and Ξ_λ are finite sets. Thus we have:

Proposition 4. *When $\tilde{\boldsymbol{Z}}_\omega$ is a projective subspace, on the notations* (4, 1) *and* (4, 2) *both Λ_κ and Ξ_λ (the support of ρ_λ) are finite sets on condition that $\lambda \in \Lambda_\kappa$.*

When

$$L(\boldsymbol{Y}) = \boldsymbol{Y}$$

$\boldsymbol{Y}$ may be said to be *precisely invariant* under L, but we will then simply say: $\boldsymbol{Y}$ *is L-invariant.* Since

$$z \in \tilde{\boldsymbol{Z}}_\omega \bowtie \omega z \in \tilde{\boldsymbol{Z}}_\omega$$

$\tilde{\boldsymbol{Z}}_\omega$ is L-invariant in any case, the following formula (4, 3) seems to be possibly verified in some way. Neverthless, in view that we may not be infatuated with a bulky volume of arguments, we will pass by here simply positing it as an axiom, i.e.:

Axiom 1. *L be a linear operator of $\boldsymbol{X}$ and $\boldsymbol{Y}_\mu$ ($\mu \in \boldsymbol{M}$) vector subspaces in $\boldsymbol{X}$*, then

$$L(\bigvee_{\mu \in M} \boldsymbol{Y}_\mu) = \bigvee_{\mu \in M} L(\boldsymbol{Y}_\mu). \tag{4, 3}$$

In using this formula we have:

Proposition 5. *For any linear operator L, the L-span is L-invariant.*

By a similar way to the proof of Proposition 4 we can prove:

Proposition 6. *When $\tilde{\boldsymbol{Z}}_\omega$ are projective for each $\omega \in \Omega$, the L-span $\tilde{\boldsymbol{Z}}_L$ is projective, too.*

5. Hilbertian Interpretation

If

$$B(\boldsymbol{Y}) = (y_\kappa)_{\kappa \in K}$$

and

$$\|y_\kappa\| < \infty \quad \text{for each} \quad \kappa \in \boldsymbol{K},$$

the product of two vectors

$$x = \mathfrak{S} x(\kappa) y_\kappa \quad \text{and} \quad z = \mathfrak{S} z(\kappa) y_\kappa$$

is proved to be written in the form

$$(x|z) = \mathfrak{S} x(\kappa) \overline{z(\kappa)} \|y_\kappa\|^2 . \tag{5, 1}$$

And if $\boldsymbol{Y}$ is invariant under L, i.e.

$$L(\boldsymbol{Y}) \subseteq \boldsymbol{Y}$$

and L is a normal operator of $\boldsymbol{X}$, L may be evidently thought as a normal operator of $\boldsymbol{Y}$ w.r.t. the formula (5, 1).

In this section we take the special case where $\boldsymbol{Y}$ is a projective subspace, invariant under L and $\boldsymbol{K}$ is an enumerable set. Then, if

$$B(\boldsymbol{Y}) = (y_k)_{k=1,2,\cdots}$$

it may be written as

$$Ly_k = \sum_{j=1}^{m_k} \phi(y_k, y_{k_j}) y_{k_j}$$

with the condition that

$$0 < m_k < \infty \ (k = 1, 2, \cdots) .$$

Moreover, $\|y_k\|^2$ is representable in the form

$$\|y_k\|^2 = c_k^2 \mu$$

with the condition that

$$0 < c_k < \infty \ (k = 1, 2, \cdots) .$$

Now, let it be posited so that

$$\dot{y}_k = y_k / c_k \sqrt{\mu} \ ;$$

$$\Psi(k, k_j) = \phi(y_k, y_{k_j}) \frac{c_{k_j}}{c_k} \ ;$$

and

$$\dot{\boldsymbol{Y}} = \bigvee_{k=1}^{\infty} \langle\!\langle \dot{y}_k \rangle\!\rangle \ ;$$

and for any two vectors

$$\dot{x} = \sum x'(k) \dot{y}_k \quad \text{and} \quad \dot{z} = \sum z'(k) \dot{y}_k$$

let it be denoted as

$$[\dot{x}|\dot{z}] = \sum x'(k)\overline{z'(k)}\,.$$

Then we have:

$$L\dot{y}_k = L\left(\frac{y_k}{c_k\sqrt{\mu}}\right) = \sum_{j=1}^{m_k}\phi(y_k, y_{k_j})\frac{y_{k_j}}{c_k\sqrt{\mu}}$$

$$= \sum_{j=1}^{m_k}\Psi(k, k_j)\frac{c_k}{c_{k_j}}\cdot\frac{y_{k_j}}{c_k\sqrt{\mu}}$$

$$= \sum_{j=1}^{m_k}\Psi(k, k_j)\dot{y}_{k_j}$$

i.e.

$$L\dot{y}_k = \sum_{j=1}^{m_k}\Psi(k, k_j)\dot{y}_{k_j}\,. \tag{5, 3}$$

If we introduce a correspondence between the spaces $\boldsymbol{Y}$ and $\dot{\boldsymbol{Y}}$ such that

$$\boldsymbol{Y}\in x \longleftrightarrow \dot{x}\in\dot{\boldsymbol{Y}} \tag{5, 4}$$

$$x = \sum x(k)y_k \quad \text{and} \quad \dot{x} = \sum x'(k)\dot{y}_k\,,$$

by the relation

$$x'(k) = c_k x(k)\,, \tag{5, 5}$$

then

$$(x|z) = \sum x(k)\overline{z(k)}\,\|y_k\|^2 = \sum x'(k)\overline{z'(k)}\frac{\|y_k\|^2}{c_k^2}$$

$$= \sum x'(k)\overline{z'(k)}\mu = [\dot{x}|\dot{z}]\mu\,,$$

i.e.

$$(x|z) = [\dot{x}|\dot{z}]\mu\,,$$

so that

$$\|x\|^2 = |[x]|^2\mu$$

on condition that $|[\dot{x}]|^2 = [\dot{x}|\dot{x}]$. In addition, we have

$$(Lx|z) = (x|\bar{L}z) \rhd (L\dot{x}|\dot{z}) = (\dot{x}|\bar{L}\dot{z})$$

$\bar{L}$ being the adjoint of L w.r.t. $(\ |\ )$.

$\dot{\boldsymbol{Y}}$ will be called the *Hilbertian interpretation* of $\boldsymbol{Y}$ with respect to the operator L. It is remarkable that

$$|[\dot{y}_k]| = 1 \quad \text{for all} \quad k = 1, 2, \cdots$$

and when L is a normal operator of $\boldsymbol{X}$, so is on $\dot{\boldsymbol{Y}}$. It is greatly different from the others that

$$\|y_k\|^2$$

is an infinitesimal quantity when $\boldsymbol{Y}$ is a projective subspace. But, by means of the Hilbertian interpretation, the topological structure of $\boldsymbol{Y}$ is made to be an ordinary object for the theory of Hilbert space.

When $\tilde{\boldsymbol{Z}}_L$ is projective, by Proposition 2 $\tilde{\boldsymbol{Z}}_L^{\perp}$ is projective, too, so that if $\tilde{\boldsymbol{Z}}_L^{\perp} \neq$void

$$(\exists y)\,(y \in \tilde{\boldsymbol{Z}}_L^{\perp} \text{ and } y \text{ is projective}).$$

Then, for the space

$$\boldsymbol{Y} = \bigvee_{n=0}^{\infty} \langle\!\langle L^n y \rangle\!\rangle \tag{5, 6}$$

$B(\boldsymbol{Y})$ is an enumerable set when $\boldsymbol{L}$ is projective, because then each $L^n y$ are projective vectors so that all of their supports are finite sets ; hence $\boldsymbol{Y}$ has its Hilbertian interpretation $\dot{\boldsymbol{Y}}$ effective. As above remarked, a normal operator L of $\boldsymbol{X}$ is thought so on $\dot{\boldsymbol{Y}}$, too. Now, if L is proved to have at least one proper vector

$$\dot{\rho} = \sum \rho'(k)\dot{y}_k$$

and

$$L\dot{\rho} = \omega\dot{\rho}\,,$$

then by (5, 3) and (5, 2)

$$\sum_k \Psi(k, 1)\rho'(k) = \omega\rho'(1)$$

i.e.

$$\sum_k \phi(y_k, y_1)\frac{\rho'(k)}{c_k} = \omega\frac{\rho'(1)}{c_1}$$

so that

$$\sum_k \phi(y_k, y_1)\rho(k) = \omega\rho(1)$$

where

$$\rho'(k) = c_k\rho(k) \ \ (k = 1, 2, \cdots).$$

Hence, for the vector

$$\rho = \sum \rho(k)y_k$$

we have

$$L\rho = \omega\rho\,. \tag{5, 7}$$

Since by (5, 5) and (5, 4) ρ is the corresponding vector to $\dot{\rho}$,

$$\rho \in \boldsymbol{Y}. \tag{5, 8}$$

In addition

$$\rho \in \tilde{\boldsymbol{Z}}_L \tag{5, 9}$$

because by (5, 7) ρ is a proper vector of L in $\boldsymbol{X}$.

On the other hand:

Proposition 7. *If L is a normal operator and $\widetilde{Z}_L$ is projective, then we have*: (i) $L(\widetilde{Z}_L)\subseteq\widetilde{Z}_L$, (ii) $L(\widetilde{Z}_L^{\perp})\subseteq\widetilde{Z}_L^{\perp}$, (iii) $\overline{L}(\widetilde{Z}_L)\subseteq\widetilde{Z}_L$, *and* (iv) $\overline{L}(\widetilde{Z}_L^{\perp})\subseteq\widetilde{Z}_L^{\perp}$.

(i) is involved in Proposition 6, but it is remarkable that this relation can be proved without Axiom (4, 3), whereas its inversive relation may not be so. Let ρ be a proper vector which belongs to the proper value ω, then

$$L\overline{L}\rho=\overline{L}L\rho=\overline{L}(\omega\rho)=\omega\overline{L}\rho\ ,$$

hence $\overline{L}\rho$ is also a proper vector which belongs to ω. Since $\widetilde{Z}_L$ is the span of proper vectors, (iii) is directly gained from this. Next, let $z\in\widetilde{Z}_L$ and $y\in\widetilde{Z}_L^{\perp}$, then as $\overline{L}z\in\widetilde{Z}_L$

$$(z|Ly)=(\overline{L}z|y)=0$$

i.e.

$$Ly\in\widetilde{Z}_L^{\perp},$$

which verifies (ii). And, by the same ρ

$$(\rho|\overline{L}y)=(L\rho|y)=\omega(\rho|y)\rhd\overline{L}y\in\widetilde{Z}_L^{\perp}\ \text{for any}\ y\in\widetilde{Z}_L^{\perp}$$

so that (iv) is verified.

By (ii) we see that the subspace Y defined by (5, 6) is included in $\widetilde{Z}_L^{\perp}$, then (5, 8) is contradictory to (5, 9). So we may consequently have

$$\widetilde{Z}_L^{\perp}=\text{void}$$

that means

$$\widetilde{Z}_L=X.$$

Finally, it is to be remarked that after the above-stated reasoning the following problem is left over: whether is L completely continuous[2] (on $\dot{Y}$) or not ?.

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