Theory of Description on a Set-function Restricted within a Euclidian Space

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Abstract

In this theory a euclidian space is characterized as an *omnium*, which may historically be detailed by descriptions but eternally left as an unfinished whole. *Residual description* is defined as a final one induced descriptively and explained in connection with set-function analysis. The last section is appropriated to the classification of set-functions, which will give rudiments to our analysis.

I. Introduction

The present author has devoted his works of the recent ten years to the establishment of a renovated system of measure theoretical notions and axioms in a euclidian space of finite dimension. His first aim has been to square the whole analysis with what has been taught on the classical euclidian geometry. The set theory instigated by G. Cantor and rapidly developed by successive authors has sometimes been, and really is, found impertinent to be directly connected with the euclidian geometry. The axiom of choice and results of the theory of ordinal numbers are specially avoided in our theory.

B. Cavalieri once posited a line which consists of points to be measured as of zero size, but being forced by unexpected criticism he had to change the assumption. In our theory, points are assumed as spatial positions provided with respective *point-occupations* [1] and a line is defined as a set of such elements, which simply are called *points*. However, a point may not be a strictly concrete element to intuition, though it may be believed that the definition is fair and appropriate. Such is one of the descriptions about a point, which is reduced from many points of view. All the sentences and formulas appearing in our analytical proceedings are regarded as *descriptions*.

Descriptions may induce any decision, but at times may limp into undecidable states of conclusion. But, if some subsidiary aspect is found to establish an adequate course of reasoning, an undecidable thing may be turned to be a decidable one. Such will be regarded as an artificial logical completion, but it should be an addition by which our reason finds a believable way to reach a spatial resolution of what may be called euclidian construction of the space. Such is not originally a given conception, but may be reached by a certain accumulation of succsesful modifications. Such may be said an ultimate object in which our geometry may dwell, though it may not be completed within human history.

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Let it be called an *omnium*. The aim of our works is to dig it.

To suppose a space to consist of everywhere homogeneous and similar parts has generally been held in the rudiments on a euclidian space. So, a measure which measures all points as of equi-measure will stand as a fundamental one in a euclidian space. Such a one is called a *normal measure*. In comparison with a normal measure, are found the ways not only to reconstruct an integral but also to classify set-functions in a euclidian space, which may be in specific relations of descriptions.

When a point P in a (euclidian) space $\boldsymbol{E}$ is to be specified by a property $\mathfrak{p}$, we will write

$$P \subset \mathfrak{p}$$

if P has $\mathfrak{p}$, and

$$P \not\subset \mathfrak{p} \text{ or } P \sqsubset \mathfrak{p}$$

if P has not $\mathfrak{p}$. When the set

$$\{P: P \subset \mathfrak{p}\} = \boldsymbol{E}(\mathfrak{p})$$

is recognized as determinate*, and

$$\boldsymbol{C}\boldsymbol{E}(\mathfrak{p}) = \boldsymbol{E} - \boldsymbol{E}(\mathfrak{p}) = \{P: P \sqsubset \mathfrak{p}\},$$

then $\mathfrak{p}$ will be said to be *descriptive* in $\boldsymbol{E}$. With a view to specify a determinate set, H. Poincaré intended to establish the notion of *predicativity*. This notion has recently been developed considerably in connection with the notion of *recursivity*.[2] However, such a course of logic has not been thought wholly resonant with what has been estimted in our theory. As for above-shown descriptivity, emphasis will not be laid on the build-up of any complete system of categories, though it may make a step to start on our consideration with specification by any property.

2. Residual Description

When a family of sets $(M_\iota)_{\iota \in I}$ is given and the set of indices I is a simply ordered set, we write

$$\widetilde{M}_x = \bigcup_{\iota \leq x} M_\iota$$

and

$$\widetilde{M} = \bigcup \widetilde{M}_x.$$

Let L be a given linear operator, then if

$$\overline{\lim_{x \in I}} |L(\widetilde{M} - \widetilde{M}_x)| \neq 0$$

* This means it is porved that $\boldsymbol{E}(\mathfrak{p})$ and $\boldsymbol{C}\boldsymbol{E}(\mathfrak{p})$ are a priori measurable, including the case of infinite value.

we say "*L has a (non-vanishing) residual (description) of summation on the family* $(\boldsymbol{M}_\iota)$ "or "*the L-residual (of summation) on* $(\boldsymbol{M}_\iota)$ *does not vanish*".

When L is an a priori measure $\widetilde{m}$, the analysis is put in a clearer aspect if it is proceeded in connection with *de Morgan formula,* say:

$$\widetilde{\boldsymbol{M}}-\bigcup\widetilde{\boldsymbol{M}}_\chi=\bigcap(\widetilde{\boldsymbol{M}}-\widetilde{\boldsymbol{M}}_\chi).$$

If $\widetilde{\boldsymbol{M}}$ is a bounded set we have

$$\widetilde{m}(\widetilde{\boldsymbol{M}}-\bigcup\widetilde{\boldsymbol{M}}_\chi)=\widetilde{m}\bigcap(\widetilde{\boldsymbol{M}}-\widetilde{\boldsymbol{M}}_\chi)$$

$$=\lim\ \widetilde{m}(\boldsymbol{M}-\boldsymbol{M}_\chi).^*$$

Then, as

$$\widetilde{\boldsymbol{M}}-\bigcup\widetilde{\boldsymbol{M}}_\chi=\widetilde{\boldsymbol{M}}-\widetilde{\boldsymbol{M}}=\text{void}$$

the left hand must vanish, so we see the $\widetilde{m}$-residual on $(\boldsymbol{M}_\iota)$ must vanish.

Let it be defined such that

$$\widetilde{\boldsymbol{R}}_\chi=\bigcap_{\iota\leqq\chi}\boldsymbol{M}_\iota \quad\text{and}\quad \boldsymbol{R}=\bigcap\widetilde{\boldsymbol{R}}_\chi$$

then we have

$$\boldsymbol{R}=\bigcap\boldsymbol{M}_\iota.$$

On this construction, if

$$\overline{\lim}\,|L(\widetilde{\boldsymbol{R}}_\chi-\boldsymbol{R})|\neq 0$$

we say "*L has a (non-vanishing) residual (description) of interseciton on the family* $(\boldsymbol{M}_\iota)$" or "*the L-residual (of intersection) on* $(\boldsymbol{M}_\iota)$ *does not vanish*"・ Even if two families $(\boldsymbol{M}_\iota)$ and $(\boldsymbol{N}_\iota)$ satisfy the relation

$$\bigcap\boldsymbol{M}_\iota=\bigcap\boldsymbol{N}_\iota=\boldsymbol{R},$$

it may not always be observed that both of L- residuals on $(\boldsymbol{M}_\iota)$ and on $(\boldsymbol{N}_\iota)$ vanish or do not vanish at the same time. This being so, we distinguish a family $(\boldsymbol{M}_\iota)$ by denoting it as

$$\varphi:(\boldsymbol{M}_\iota)$$

and introduce the notion of *L-atmosphere* $)\boldsymbol{R}($$_\varphi$ which is defined such as

$$|L()\boldsymbol{R}()_\varphi|=\overline{\lim}'|L(\widetilde{\boldsymbol{R}}_\iota-\boldsymbol{R}\ |,$$

The atmosphere $)\boldsymbol{R}($$_\varphi$ may coincide with the state $\bigcap(\widetilde{\boldsymbol{R}}_\iota-\boldsymbol{R})$, but is not strictly the same with the set $\bigcap(\widetilde{\boldsymbol{R}}_\iota-\boldsymbol{R})$, which is a void set. It is easily seen that any $\widetilde{m}$-residual of intersection vanishes on condition that $\boldsymbol{R}$ is a bounded

* This may be, as it is, almost a mere decision, but in some way is demonstrated in the theory of a priori measure.

set (w. r. t. $\widetilde{m}$). When L-residual of intersection vanishes, we say "*the L-atmosphere* $(\mathrm{]}\boldsymbol{R}\mathrm{[})_{\varphi}$ *vanishes*". It is notable that the notion of atmosphere is referred to a new category which is induced by the process of residual description, and is added to our logical system in order to reach a clearer aspect of the omnium.

3. Truncate Description

As mentioned in the previous section, the vanish of $\widetilde{m}$-residuals (of summation and of intersection) cannot generally be ascertained except when $(\boldsymbol{M}_{\iota})$ is a bounded family. We define four subsets of any set $\boldsymbol{M}$ such as:

(i) $\acute{S}^{r}(\boldsymbol{M})=\{P: P\in\boldsymbol{M},\ |P|<r\}$ (ii) $S_{r}(\boldsymbol{M})=\{P: P\in\boldsymbol{M},\ |P|>r\}$;

(iii) $S^{\frown r}(\boldsymbol{M})=\{P: P\in\boldsymbol{M},\ |P|\leqq r\}$ (iv) $S_{\underset{\smile}{r}}(\boldsymbol{M})=\{P: P\in\boldsymbol{M},\ |P|\geqq r\}$,

$|P|$ being the distance of P from the origin O, and call (i) *(initial) section,* (ii) *final section*, (iii) *closed (initial) section* and (iv) *closed final section of* $\boldsymbol{M}$ *of radius* r respectively. Using these notations we may write

$$\lim_{r\to\infty} f(S^{r}(\boldsymbol{M}))=f(\boldsymbol{M}-(\mathrm{]}\infty\mathrm{[}))$$

and

$$\lim_{r\to\infty} f(S_{r}(\boldsymbol{M}))=f(\boldsymbol{M}\cap(\mathrm{]}\infty\mathrm{[})),$$

f being a given set-function (, real-valued).

If
$$f(\boldsymbol{M}\cap(\mathrm{]}\infty\mathrm{[}))=0$$

we say "*f is of a truncate description on $\boldsymbol{M}$*" or "*the set $\boldsymbol{M}$ is a truncate set with respect to f*", and if

$$f(\boldsymbol{M}\cap(\mathrm{]}\infty\mathrm{[}))\neq 0$$

"*f has a residual on $\boldsymbol{M}$*" or "*the set $\boldsymbol{M}$ has a residual with respect to f*". The atmosphere of this case is called the *radial atmosphere,* in distinction. It is important that analysis performed in a bounded domain may be extended in terms of any limiting process only when a truncate set is dealt with.

In view of vanish of a residual, it seems convenient if we define a *positive (set-) function* by the following descriptions: (i) for any set $\boldsymbol{M}$, $f(\boldsymbol{M})\geqq 0$; (ii) if $\boldsymbol{M}_1\supset\boldsymbol{M}_2\supset\cdots$ and $\lim f(\boldsymbol{M}_k)\geqq\delta$, then exists a subset $\boldsymbol{S}$ of $\cap\boldsymbol{M}_k$ such that $f(\boldsymbol{S})>\frac{\delta}{2}$. The condition (ii) may be thought as a *generalization of Archimedes' proposition.* * A function φ is called a *negative function,* when $-\varphi$ is a positive function.

4. Classification of Real-valued Set-functions

If a unit of length is fixed in a euclidian space $\boldsymbol{E}$, any two a priori measures

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* This is that for any positive real numbers a and b there is a positive integer n such that $(n\text{-}1)\,b<a\leqq nb$.

$\widetilde{m}_1$ and $\widetilde{m}_2$ have the same value for any set $\boldsymbol{M}$ in $\boldsymbol{E}$, i. e.

$$\widetilde{m}_1 \boldsymbol{M} = \widetilde{m}_2 \boldsymbol{M}.$$

This means that

$$\underset{P\in \boldsymbol{M}}{\mathfrak{S}} \mu_1(P) = \underset{P\in \boldsymbol{M}}{\mathfrak{S}} \mu_2(P)$$

$\mu_k(P)$ being point dimension of P ($k=1,2$). However, it must be noted that respective inversion numbers $\mathfrak{n}_1(\boldsymbol{M})$ and $\mathfrak{n}_2(\boldsymbol{M})$ are not always equal, because $\mu_1(P)$ and $\mu_2(P)$ are not always equal. If both $\widetilde{m}_1$ and $\widetilde{m}_2$ are normal a priori measures (i. e. equi-measuring homogeneous measures), then we have

$$\widetilde{m}_1(\boldsymbol{M}) = \mu_1 \cdot \mathfrak{n}_1(\boldsymbol{M}) = \mu_2 \cdot \mathfrak{n}_2(\boldsymbol{M}) = \widetilde{m}_2(\boldsymbol{M})$$

so that

$$\mu_1 / \mu_2 = \mathfrak{n}_2(\boldsymbol{M}) / \mathfrak{n}_2(\boldsymbol{M}) \cdot$$

An application $\widetilde{\gamma}$ is defined such as

$$\widetilde{\gamma}(\boldsymbol{M}) = \underset{P\in \boldsymbol{M}}{\mathfrak{S}} \gamma_P . \tag{4, 1}$$

γ_P being point mass applied to P. Therefore, it corresponds to a Stieltjes-integral on $\boldsymbol{M}$ of the function 1. In this definition, the assignment

$$P \in \boldsymbol{M} \tag{4, 2}$$

must be related to a particular construction

$$\boldsymbol{M} = \underset{P}{\mathfrak{S}} P \tag{4, 3}$$

so that it may be related to an a priori measure $\widetilde{m}$ which is defined as

$$\widetilde{m} \boldsymbol{M} = \underset{P}{\mathfrak{S}} \mu_P$$

in terms of the assignment (4, 2). This measure $\widetilde{m}$ is called the *carrier* of $\widetilde{\gamma}$. Unless a particular construction is demanded as non-homogeneous, we will not adopt a carrier which is not normal. When $\widetilde{\gamma}$ has a normal measure as its carrier, then we say $\widetilde{\gamma}$ has a *normal carrier*.

In our system, integral of a point function $f(P)$ may be reduced to an application, because if we set such as

$$f(P)\mu_P = \gamma_P$$

in terms of point dimension μ_P of an a priori measure $\widetilde{m}$, then we have

$$\widetilde{\gamma}(\boldsymbol{M}) = \underset{P\in \boldsymbol{M}}{\mathfrak{S}} \gamma_P = \underset{P\in \boldsymbol{M}}{\mathfrak{S}} f(P)\mu_P.$$

It is specially notable that any set $\boldsymbol{M}$ cannot be conceived unless the description (4, 3) is given in liaison with the assignment (4, 2). So, we may say that no set of points is conceivable without being given its reconstruction in proportion to a certain a priori measure. As for the application $\widetilde{\gamma}$, it may be distinct from other set-functions only in that it is given in the form (4, 1). We generally think it to be destined that any application may be represented as a sum of a positive application and a negative application. When a set $\boldsymbol{M}$ is bounded with respect to a positive application $\widetilde{\gamma}$, we may conclude by the generalized Archimedes' proposition and the destination process [3] that $\boldsymbol{M}$ is $\widetilde{\gamma}$-measurable.

A (real-valued) set-functon $f(\boldsymbol{M})$ which satisfies the relation

$$f(\boldsymbol{M}_1 \cup \boldsymbol{M}_2) = f(\boldsymbol{M}_1) + f(\boldsymbol{M}_2) - f(\boldsymbol{M}_1 \cap \boldsymbol{M}_2)$$

but is not necessrily promised to be representable in the form

$$f(\boldsymbol{M}) = \mathfrak{S} f_P$$

is called an *ultra (set-) function.* It is easily seen that any ultra function implies a case of non-vanishing residual. Value of residual with respect to an ultra function varies corresponding to the limiting behavior of basic family of sets, and, in effect, cannot always be invariant independently of the choice of basic family. Such a probabilistic impilcity about residual description makes an important property specific to an ultra function.

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