Empiricist Viewpoints and Set-theoretical Analysis

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Abstract

Empiricist stand toward set-theoretical analysis is introduced to the effect that the use of ordinal numbers is to be limited to the 2nd class at most. The family of Borel sets is thereby necessarily given some revisions. Through some reconstructive problems, notions of *theoretical noise* and *reflective effect* of axioms newly come to be involved in arguments. In the final part, the method of trans-induction is again taken up and is given some detailed discussions.

1. Reachability

In practical observations, the number of samples is naturally expected to be limited to finiteness. Hence it cannot be extended beyond enumerability. So, if we insist on this practical viewpoint, all limiting processes should be ascertained by enumerable steps. For instance, we will hereby assert:

Destination R_1. *On a simple-ordered set of indices I, if the formula*

$$\lim_{\iota \in I} J_\iota = J$$

is assertively posited, a certain subsequence of indices

$$(\iota_k)_{k=1,2,\cdots}$$

must exist in I such that

$$\lim_{k} J_{\iota_k} = J .$$

This may be stated thus: *the limit J is reachable by* (J_ι). As is well-known, the 3rd class ordinal number Ω (say, that of the set of all ordinal numbers up through the 2nd class) cannot be reached by any enumerable stepping of numbers of the 2nd class. Therefore, the concept Ω must be suppressed if Destination R_1 is to be demanded. Thus we may start the empiricist theory in which are to be avoided all the ordinal numbers of higher classes than the 2nd one.

In the empiricist view of inspection, the construction of an integral may not be more complicated than what is stated in the following:

Destination R_2. *For any ensemble of disjoint sets* $(N_k)_{k=1,2,\cdots}$ $(\subset M)$, *if it always effect that*

$$f(\tilde{N}) = \sum_{k=1}^{\infty} f(N_k)$$

with

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$$\tilde{N} = \cup N_k ,$$

then we have

$$f(N) = \underset{P \in N}{\mathfrak{S}} f(P) \text{ for any } N \subseteq M .$$

This is, as it is, a definition of the integral

$$\mathfrak{S} f(P)^{1)}. \tag{1. 1}$$

But, it has never been from the first evident that we could consider the concept (1. 1) to be enclosed with such a simple constitution. It is just a destination engendered by the empiricist process of inspection. Whatever destination emerges, all the limiting processes occured in it cannot be managed beyond the destination R_1 and all dividing processes cannot be managed beyond the destination R_2, as far as the empiricist viewpoint is to be conformed.

On account of the above-stated situation, the destination R_1 (resp. R_2) may be thought as a sort of reflector for the empiricist process of limiting (resp. dividing). In effect, we are to encounter with more cases with similar properties, so that we may have the theory of *reflections* on empiricist analysis, in which Destinations R_1 and R_2 will be posited as Reflections R_1 and R_2 respectively.

2. Rough Destination

Borel sets are defined as the sets which belong to the smallest family $\mathfrak{B}$ of subsets of the given set (the finite dimensional Euclidian space, in our case) satisfying the following three conditions:

(a) *every closed set belongs to* $\mathfrak{B}$;

(b) *if* $M_n \in \mathfrak{B}$ *for* $n=1, 2, \cdots$, *then* $\cup M_n \in \mathfrak{B}$;

(c) *if* $M_n \in \mathfrak{B}$ *for* $n=1, 2, \cdots$, *then* $\cap M_n \in \mathfrak{B}$.

In the classic theory, making use of ordinal numbers, Borel sets are classified in classes $\mathfrak{B}_\alpha$, where $\alpha < \Omega$, in the following manner:

1. the class $\mathfrak{B}_0$ is the family of all closed sets;

2. for $\alpha = \lambda + n > 0$, where λ is a limit ordinal and n is a non-negative integer, the class $\mathfrak{B}_\alpha$ is the family of all sets of the form

$$\bigcap_{k=1}^{\infty} M_k \text{ or } \bigcup_{k=1}^{\infty} M_k$$

according to whether n is even or odd, and sets $M_1, M_2, \cdots$ belong to classes of indices less than α.

From the empiricist viewpoint, Borel sets cannot be accepted as making up a factual family, because $\mathfrak{B}_\alpha$ must be laid upon the succession of ordinal numbers up through the 2nd class which are not reachable to Ω by any enumerable stepping. However, the number Ω may herein be regarded as a symbolical bound of processes devised over the 1st and 2nd class ordinal numbers, though they are not to make up a factual total. Moreover, the multifariousness thus symbolically bounded by Ω may not be counted as beyond the density of continuum $\mathfrak{c}$. Hence we may have

$$\bar{\Omega} \leqslant \mathfrak{c} \tag{2. 1}$$

in the sense of symbolical use. On the other hand, it is clearly verified that

$$\bar{\bar{\mathfrak{B}}}_0 \leqslant \mathfrak{c}\,.$$

These being so, it may be permitted with no contradiction to the empiricist view that

$$\bar{\bar{\mathfrak{B}}} \leqslant \mathfrak{c}\,. \tag{2. 2}$$

(2. 1) and (2. 2) may be regarded as destinations in connection with the insertion of the symbolical bound Ω. We adopt and specify such a process as "*principle of a rough destination*". It is notable that reflections and rough destinations alike are the effects of the devices assumed for residual parts of descriptions.

3. Noises

When, for any element x, one and only one of the following two cases, (i) $x \in M$, and (ii) $x \notin M$, is promised to occur, the set M is *descriptive*. Then, if the family of Borel sets is to be admitted as the infimum of families which satisfy the three conditions (a) through (c) cited in Section 2, it must be regarded as descriptive. In this case, for a set M, whether we can really know M to be a Borel set or not is essentially beside the question. However, it may not be contradicted that such a promise of descriptivity is too abstract and too hypothetical. In effect, as has already been referred to in Section 2, if a set M is to be recognized as a Borel set when, and only when, there exists a 1st or 2nd class ordinal number α such that

$$M \in \mathfrak{B}_\alpha\,,$$

then the family of Borel sets $\mathfrak{B}$ may not be managed without the rough destination formula

$$\mathfrak{B} = \bigcup_{\alpha < \Omega} \mathfrak{B}_\alpha\,.$$

In this view, if we take the empiricist stand, the family $\mathfrak{B}$ may not be regarded as descriptive ; hence the question of whether a set M is a Borel set or not will give a noise for our recognition. Thus we see that a *theoretical noise* is engendered in accordance with the situation in which we intend to manage a construction of objects.

As has so far been stated, an empiricist cannot reach the density of continuum by any stepping of ordinal numbers. But, if he takes a real axis, he may not insist on any noise about descriptivity of the set of real numbers, the sum of which gives the density of continuum.

In classic texts, it is stated that the theorem of well-ordering is deduced from the axiom of choice. But, an empiricist does not look at the situation in this way. As he gives up using of ordinal numbers beyond the 2nd class, no deducing connection can be found between the well-ordering and the axiom of choice. Thus we see that any noise about the well-ordering is engendered from the endless succession of ordinal numbers, and not from the axiom of choice itself.

4. Trans-induction

In the previous paper[2], the author introduced the method of trans-induction and applied it to inductions of some important propositions. But, afterwards, he has come to feel discontented with the exposition therein given by him. So, in this place, he intends to give some detailed discussions about the trans-inductive mode to reconstruct the design of the induction*.

When a property $\mathfrak{p}$ relates to a set Y, the following two cases are distinguished:

(i) that Y satisfies $\mathfrak{p}$ exactly means that every element of Y satisfies $\mathfrak{p}$;

(ii) $\mathfrak{p}$ depends on some constructive relations between Y and its elements.

In either case, if $\mathfrak{p}$ is satisfied by Y, we write

$$Y \subset \mathfrak{p} \text{ or } \mathfrak{p} \supset Y. \tag{4.1}$$

When (i) is the case, this means that

$$(\forall x \in Y)(x \subset \mathfrak{p}).$$

Moreover, after (4. 1) occurs, there are distinguished two cases:

(1) Y is yet *extensible* w.r.t. $\mathfrak{p}$, i.e. there exists another set Z such that

$$Y \subset Z \text{ and } Z \subset \mathfrak{p}; \tag{4.2}$$

(2) Y is *inextensible* w.r.t. $\mathfrak{p}$, i.e. there is no such set Z that (4. 2).

For an inextensible case we will conveniently involve the case where $Y \not\subset \mathfrak{p}$.

If the implication

$$(Y\text{: extensible w.r.t. } \mathfrak{p}) \triangleright (\exists y)(y \notin Y,\ Y \cup \{y\} \subset \mathfrak{p}) \tag{4.3}$$

is promised, it gives a hopeful light toward the trans-inductive mode[2]. But this is not sufficient, because an inextensible set may not always be reached by the only property (4. 3), that is to say: if we, by virtue of (4. 3), have an increasing sequence of sets

$$Y_1 \subset Y_2 \subset \cdots$$

with

$$Y_k \subset \mathfrak{p} \text{ for all } k = 1, 2, \cdots,$$

it is not always assured that the set

$$\tilde{Y} = \cup Y_k$$

* Propositions previously demonstrated by means of trans-induction may still be held unchanged.

may be made inextensible. For an argument of this situation, abstract treatment seems difficult, so let us turn our eyes to an explicit case in the following.

If

$$Y \subset \mathfrak{p} \triangleright (Y \text{ is a basal system}),$$

$\mathfrak{p}$ is called a *basal property*. Let us denote the span of a set Y (the smallest complete vector space which involves Y) by $\vee_Y$. For a given basal property $\mathfrak{p}$ and for a given set Z, we intend to seek for such a system Y that $Y \subset \mathfrak{p}$ and $\vee_Y = \vee_Z$ (i.e. Y is an equivalent basal set to Z). In this case, we can succesfully take the following situation which is called a *trans-inductive mode*: if $Y \subset \vee_Z$, $Y \subset \mathfrak{p}$ and $y \in \vee_Z - \vee_Y$, then we may, by a practically exact procedure, find such an element y' that

$$\left.\begin{aligned} &y' \in \vee_Z - \vee_Y \\ \text{and} \quad & \\ &Y' \equiv \{y'\} \cup Y \subset \mathfrak{p} \text{ and } y \in \vee_{Y'}. \end{aligned}\right\} \tag{4. 4}$$

If a trans-inductive mode with respective to $\mathfrak{p}$ holds in the space $\vee_Z$, the process (4. 4) is to be found possible unless Y is inextensible. Hence, the consummation of the process (4. 4) must hereupon be found as the existence of an inextensible set, which may be expressed as the limit of a sequence of extensible sets. This is a *trans-induction* (say, of *progressive type*).

If by the transfinite induction, the processes (4. 4) are to be laid upon a successive disposition of indices which are ordinal numbers. Then, on suppressing the property of well-ordering of the indication*, there may be left only the increasing state of indices, so that we may have

$$(\exists I)(I \text{ is a simple-ordered set})(\forall \iota, \kappa \in I)(Y_\iota \subset \vee_Z \;\&\; Y_\iota \subset \mathfrak{p})(\iota \prec \kappa \triangleright Y_\iota \subset Y_\kappa). \tag{4. 5}$$

In this case, the trans-induction will effect the result that the set

$$\tilde{Y} = \cup Y_\iota \tag{4. 6}$$

is inextensible.

In the above-stated situation, we may not neglect the point that, if a noise is promised to the proceeding of ordinal numbers, then it may be natural to expect that some noise shall be promised to the leaping from (4. 4) to the existence of an indication I with property (4. 6). On this problem, it will be specially notable that, there is no essential obstruction for the ensemble of sets (Y_ι) to be reachable (in the empiricist view) to an inextensible set $\tilde{Y}$. Besides, if no factual reason is found to prevent the existence of I, no noise shall be really destined for I.

5. Trans-inductive Infimum

About a result of trans-induction on a basal case, it may not always be assured

* This means the set of indices.

that the set $\widetilde{Y}$ of (4.6) satisfies $\mathfrak{p}$. A property $\mathfrak{p}$ is called an *integrant property* if $\mathfrak{p}$ satisfies the following conditions: (1) $\mathfrak{p}$ *is regressive, i.e. whenever* $Y^{(1)} \subset Z^{(1)}$, $Y^{(1)} \subset \mathfrak{p}$ *and* $Z^{(2)} \subset Z^{(1)}$, *we have* $Y^{(1)} \cap Z^{(2)} \subset \mathfrak{p}$; (2) *for any simple-ordered set I, if* $Y_\iota \subset Y_\kappa$ *for* $\iota \prec \kappa \in I$, *and* $(\forall \iota \in I)(Y_\iota \subset \mathfrak{p})$, *then* $\cup Y_\iota \subset \mathfrak{p}$. When $\mathfrak{p}_0$ is an integrant property, the basal property $\mathfrak{p}$ defined by

$$\mathfrak{p} = (\mathfrak{p}_0 \ \& \ \text{basal}), \tag{5.1}$$

gives us the expectation that there may exist an equivalent basis $\hat{Y}$ to $\widetilde{Y}$ of (4.6) with respect to $\mathfrak{p}$ such that

$$\hat{Y} \subset \mathfrak{p}.$$

By $\mathfrak{p}$ we will indicate, in the sequel, a property defined by (5.1).

It is readily seen from the definition that

$$\hat{Y} \subset \mathfrak{p}_0,$$

if

$$\hat{Y} \subset \widetilde{Y}. \tag{5.2}$$

As the problem really occurs only when $\widetilde{Y}$ is a superbasis, $\hat{Y}$ will possibly be expected in the relation (5.2). When $\widetilde{Y}$ is a superbasis, there may be found a family of superbases $(\hat{Y}_\lambda)_{\lambda \in \Lambda}$ with a simple-ordered indication Λ such that

$$\hat{Y}_\lambda \supset \hat{Y}_{\lambda'} \text{ whenever } \lambda \prec \lambda' \, (\in \Lambda).$$

Then, on replacing the family (Y_ι) shown in (4.5) by the family $(\widetilde{Y}_\iota)$ defined as

$$\widetilde{Y}_\iota = \hat{Y} \cap Y_\iota$$

with

$$\hat{Y} = \cap \hat{Y}_\lambda, \tag{5.3}$$

we have

$$\widetilde{Y}_\iota \subset \widetilde{Y}_\kappa \subseteq \hat{Y} \subset \hat{Y}_\mu \subset \hat{Y}_\lambda \tag{5.4}$$

whenever $\iota \prec \kappa \, (\in I)$ and $\lambda \prec \mu \, (\in \Lambda)$. In this case, it is readily seen that

$$\hat{Y} = \cup \widetilde{Y}_\iota. \tag{5.5}$$

If $\hat{Y}$ is a superbasis again, starting from $\hat{Y}$ instead of $\widetilde{Y}$, a similar process will possibly be taken. So we adopt this situation as a trans-inductive mode and demand the conclusion that $\hat{Y}$ is a basis of $\vee_{\hat{Y}}$. This is also a trans-induction (say, of *regressive type*). The basal set $\hat{Y}$ hereby resulted, will be called a *trans-inductive infimum.*

$\hat{Y}$ is not an usual infimum, because it is assigned basalness to be provided for its existence. This may be thought as a modality which consists of inference of the formulas (5.3) through (5.5) and the conclusion such destined for the objects as couched in the above, and which may be established in the behavior that any noise is to be suppressed unless it is a factual one to obstruct the induction.

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References

1) Mem. Muroran Inst. Tech. 4 (2), 495–496 (1963).
2) Mem. Muroran Inst. Tech. 4 (3), 804–813 (1964).