Contribution to Engineering Geometry

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Abstract

The purpose of this paper is to present a method through which is to be found the area of the portion of a circular cylindrical surface bounded by the intersection of a circular cylinder and a circular cone which are placed in a special positional relation. Here it is shown that this problem is to be solved by applying integral calculus to engineering geometry.

1. Engineering Geometrical Consideration

The case is here considered in which the vertex of a circular cone is on a right-circular cylindrical surface and the axis of the right-circular cylinder and the axis of the circular cone do not intersect but are perpendicular to each other. Let the right-circular cylinder be positioned with its base plane paralleled to the frontal plane. The subscript F is used to denote the front view of a point.[1] In the present paper the top view is excluded.

Let the axis of the right-circular cylinder be 0_1–0_2, its radius be r, the vertex of the circular cone be A, the half of the vertical angle be θ, and the angle made by the axis of the circular cone and the horizontal plane be η. When plane $\mathfrak{D}$ perpendicular to the frontal plane and containing the axis AD of the circular cone and plane $\mathfrak{J}$ determined by the point A and the axis 0_1–0_2 are established, their frontal views appear as the chord $A_F D_F$ and the diameter $A_F I_F$ of the circle O_F respectively in Fig. 1. In the same figure, the triangle $A_F E_F F_F$ is the frontal view of the right-circular cone with its axis AD. A plane which passes through the vertex A and makes the angle $\alpha\,(0 \leqq \alpha \leqq \theta)$ with the plane $\mathfrak{D}$ will generally intersect the circular conical surface along its two straight-line elements, viz., generatrices AG_1 and AG_2. The intersections of these elements with the right-circular cylindrical surface are the points H_1 and H_2 on the intersection of the circular conical surface and the right-circular cylindrical surface. Since the right-circular cylindrical surface is developable, it is developed on its tangent plane through the point D. At the same time the intersection of the circular conical surface and the right-circular cylindrical surface is also developed on this tangent plane. This intersection shall be called a *developed curve* in the following.

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2. Parametric Equations of Developed Curve, Area on Right-circular Cylindrical Surface.

Let the generatrix through the point D on the right-circular cylinder be x-axis on the tangent plane at D and for the x-axis the positive sense be to the frontal plane. As y-axis, we take a line through the point D and perpendicular to the x-axis on the above-mentioned tangent plane. In the following, for the sake of brevity, we use the same letter to denote a point on the sectional curve and its corresponding point on the developed curve.

We consider first the case when $0<\theta\leqq(\pi/2)-\eta$, $0<\eta<\pi/2$. To find the coordinates of the point H_1 on the developed curve, we use auxiliary elevation of the intersection of the circular cone and a plane which is perpendicular to the axis AD and passes through the line segment H_1H_2 on the right-circular cylindrical surface. Observing Fig. 1, we are to obtain the following equations as those of the developed curve in the first quadrant of the tangent plane at D:

$$(1)\quad \begin{aligned} x &= 2r\cos(\alpha+\eta)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,, \\ y &= 2\alpha r \qquad (0\leqq\alpha\leqq\theta)\,. \end{aligned}$$

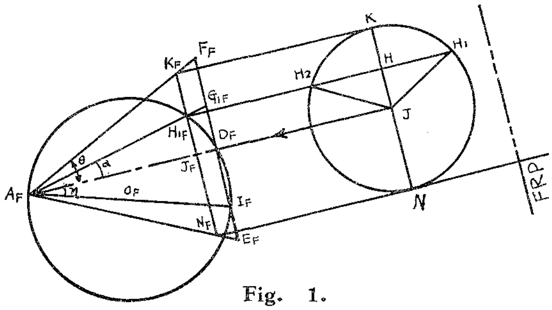

Fig. 1.

Also, we have the following expressions for the developed curve in the second quadrant:

$$(2)\quad \begin{aligned} x &= -2r\cos(\alpha+\eta)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,, \\ y &= 2\alpha r \qquad (0\leqq\alpha\leqq\theta)\,. \end{aligned}$$

Next, we shall find equations of the developed curve in the third quadrant. In this case, though the angle α is generated by a clockwise rotation as is shown in Fig. 2, we consider it as being positive for convenience' sake. When $0\leqq\alpha\leqq\eta$, from Fig. 2 the coordinates of the point H_2 on the developed curve in the third quadrant are

$$(3)\quad \begin{aligned} x &= -2r\cos(\eta-\alpha)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,, \\ y &= -2\alpha r\,. \end{aligned}$$

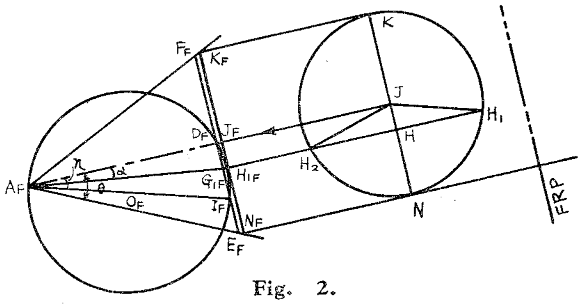

Fig. 2.

When $\eta \leqq \alpha \leqq \theta$, from Fig. 3 the coordinates of the point H_2 on the developed curve in the third quadrant are

$$(4) \qquad \begin{aligned} x &= -2r\cos(\eta-\alpha)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,, \\ y &= -2\alpha r\,. \end{aligned}$$

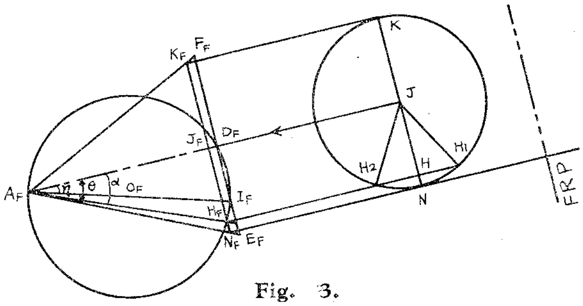

Fig. 3.

Hence we have the following equations for the developed curve in the third quadrant:

$$(5) \qquad \begin{aligned} x &= -2r\cos(\eta-\alpha)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,, \\ y &= -2\alpha r \qquad\qquad (0 \leqq \alpha \leqq \theta)\,. \end{aligned}$$

The expressions for the developed curve in the fourth quadrant are

$$(6) \qquad \begin{aligned} x &= 2r\cos(\eta-\alpha)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,, \\ y &= -2\alpha r \qquad\qquad (0 \leqq \alpha \leqq \theta)\,. \end{aligned}$$

Now, since the area of the portion of the right-circular cylindrical surface bounded by the intersection is equal to the area bounded by the developed curve on the tangent plane, we shall find the expression for the latter instead of the former. If we denote by S_1 the area bounded by the curve (1) and the coordinate axes, then

$$S_1 = 4r^2\int_0^\theta \cos(\alpha+\eta)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,d\alpha\,,$$

and if we denote by S_2 the area bounded by the curve (6) and the coordinate axes, then

$$S_2 = 4r^2 \int_0^\theta \cos(\eta-\alpha)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,d\alpha\,.$$

Hence we have

$$\begin{aligned} S_1+S_2 &= 4r^2\int_0^\theta \left\{\cos(\alpha+\eta)+\cos(\eta-\alpha)\right\}\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,d\alpha \\ &= 4r^2\cos\eta\sec\theta\left[\sin\alpha\sqrt{\sin^2\theta-\sin^2\alpha}+\sin^2\theta\,\mathrm{Sin}^{-1}\left(\frac{\sin\alpha}{\sin\theta}\right)\right]_0^\theta \\ &= 2\pi r^2\cos\eta\sin\theta\tan\theta\,. \end{aligned}$$

Consequently the area S to be found is given by

$$S = 2(S_1+S_2) = 4\pi r^2\cos\eta\sin\theta\tan\theta\,.$$

Finally, considering the case when $(\pi/2)-\eta<\theta<\pi/2$, $0<\eta<\pi/2$, we obtain the following result:

$$\begin{aligned} S &= 8r^2\left\{\int_0^{\frac{\pi}{2}-\eta}\cos(\alpha+\eta)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,d\alpha \right. \\ &\qquad \left. +\int_0^\theta\cos(\eta-\alpha)\cos\alpha\sqrt{\tan^2\theta-\tan^2\alpha}\,d\alpha\right\} \\ &= 4r^2\left[\sec\theta\sqrt{\sin^2\theta-\cos^2\eta}+\cos\eta\sec\theta\sin^2\theta\left\{\frac{\pi}{2}+\mathrm{Sin}^{-1}\left(\frac{\cos\eta}{\sin\theta}\right)\right\}\right. \\ &\qquad \left.+\sin\eta\cos\theta\left\{\log(\cos\theta)-\log\left(\sin\eta+\sqrt{\sin^2\eta-\cos^2\theta}\right)\right\}\right]. \end{aligned}$$

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References

1) H. W. Shupe, P. E. Machovina: A Manual of Engineering Geometry and Graphics for Students and Draftsmen, McGraw Hill (1956).

2) J. R. Britton: Calculus, Rinehart & Company (1958).