On Cut-Approach

On Cut-Approach

Yoshio Kinokuniya*

Abstract

A pair of mutually *cut-conjugate* properties, which are to be exclusively distinguished on any set of points, are firstly defined. By these properties a fundamental theorem is induced and then some important investigations are developed over ultra-functions of a set.

1. Set Function

In this paper, by a *set function* we mean a non-negative additive function of a set; i.e. if f is a set function, for any two sets $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ of points in a finite-dimensional euclidian space $\boldsymbol{E}$, it always effects that

$$0 \leqslant f(\boldsymbol{M}_1 \cup \boldsymbol{M}_2) = f(\boldsymbol{M}_1) + f(\boldsymbol{M}_2) - f(\boldsymbol{M}_1 \cap \boldsymbol{M}_2) .$$

If a set function f may, for any subset $\boldsymbol{F} \subseteq \boldsymbol{M}$, be expressed in the form

$$f(\boldsymbol{F}) = \underset{P \in F}{\mathfrak{S}} f(P) \tag{1. 1}$$

f is an *application* in $\boldsymbol{M}$; or, if not an application, an *ultra-function*.

In the previous paper[1] the present author stated about the empiricist view of analysis, through which set functions may be observed in new ways. In the empiricist theory of analysis, if the relation (1. 1) were to be true, it must, for any enumerable partition $(\boldsymbol{M}_k)_{k=1,2,\dots}$ of $\boldsymbol{M}$, be that

$$f(\boldsymbol{M}) = \Sigma f(\boldsymbol{M}_k) ,$$

which directly means that f is completely additive in $\boldsymbol{M}$. Therefore, that a set function f is an ultra-function is equivalent to it that f is not everywhere completely additive (in $\boldsymbol{M}$). In the following, a reconstructive study of ultra-functions with some results will be stated.

2. Principle of Cut-Approach

For a given property $\mathfrak{p}$ to be tested for a set, if

$$(\forall \boldsymbol{M},\ \boldsymbol{N} \subseteq \boldsymbol{E})\ ((\boldsymbol{M} \supseteq \boldsymbol{N}\ \&\ \boldsymbol{N} \subset \mathfrak{p}) \triangleright \boldsymbol{M} \subset \mathfrak{p})$$

$\mathfrak{p}$ is said to be *progressive*, and if

$$(\forall \boldsymbol{M},\ \boldsymbol{N} \subseteq \boldsymbol{E})\ ((\boldsymbol{M} \supseteq \boldsymbol{N}\ \&\ \boldsymbol{M} \subset \mathfrak{p}) \triangleright \boldsymbol{N} \subset \mathfrak{p}) ,$$

to be *regressive*. If it is, for any set $\boldsymbol{M}$ in $\boldsymbol{E}$, necessary that

---

* 紀国谷芳雄

$$\boldsymbol{M}\subset\mathfrak{p}\vee\boldsymbol{M}\subset\mathfrak{q}$$

and if the relations $\boldsymbol{M}\subset\mathfrak{p}$ and $\boldsymbol{M}\subset\mathfrak{q}$ cannot simultaneously be observed, then $\mathfrak{p}$ and $\mathfrak{q}$ are said to be mutually *cut-conjugate*. Then the following fact is directly obtained by the definition:

Proposition 2.1. *If $\mathfrak{p}$ and $\mathfrak{q}$ are cut-conjugate properties and $\mathfrak{p}$ is regressive (resp. progressive), then $\mathfrak{q}$ is a progressive (resp. regressive) property.*

If $\mathfrak{p}$ is regressive and $\mathfrak{q}$ is progressive, making up a pair of cut-conjugate properties, and if

$$\boldsymbol{B}_1\subset\mathfrak{p},\ \boldsymbol{B}_1\subset\boldsymbol{M} \text{ and } \boldsymbol{M}\subset\mathfrak{q},$$

then there may exist two sequences of sets $(\boldsymbol{B}_k)$ and $(\boldsymbol{A}_k)$ such that

$$\boldsymbol{B}_1\subseteq\boldsymbol{B}_2\subseteq\cdots\subseteq\boldsymbol{A}_2\subseteq\boldsymbol{A}_1\subseteq\boldsymbol{M},$$
$$\boldsymbol{B}_k\subset\mathfrak{p} \text{ and } \boldsymbol{A}_k\subset\mathfrak{q}\quad (k=1,2,\cdots).$$

In this case, if it is not that

$$\boldsymbol{B}=\bigcup\boldsymbol{B}_k=\bigcap\boldsymbol{A}_k=\boldsymbol{A}$$

we have

$$\boldsymbol{A}-\boldsymbol{B}\neq\text{void}. \tag{2.1}$$

When $\boldsymbol{B}\subset\mathfrak{q}$, we may choose such that

$$\boldsymbol{A}_2=\boldsymbol{A}_3=\cdots=\boldsymbol{B}$$

so that $\boldsymbol{A}=\boldsymbol{B}$. When $\boldsymbol{A}\subset\mathfrak{p}$, we may choose such that

$$\boldsymbol{B}_2=\boldsymbol{B}_3=\cdots=\boldsymbol{A}$$

so that $\boldsymbol{A}=\boldsymbol{B}$. Therefore, we may, for the case of (2.1), suppose that

$$\boldsymbol{B}\subset\mathfrak{p} \text{ and } \boldsymbol{A}\subset\mathfrak{q}. \tag{2.2}$$

In case of (2.2), a set can be inserted between $\boldsymbol{A}$ and $\boldsymbol{B}$ and it, of course, has one property of $\mathfrak{p}$ and $\mathfrak{q}$. This being so, a trans-inductive mode may be defined to reach a construction such that

$$(\forall\iota\in\boldsymbol{I})\ (\boldsymbol{B}_\iota\subset\mathfrak{p})\ (\boldsymbol{I} \text{ is a simple-ordered set of indices}),$$
$$(\forall\iota,\ \kappa\in\boldsymbol{I})\ (\iota<\kappa\triangleright\boldsymbol{B}_\iota\subseteq\boldsymbol{B}_\kappa),$$
$$(\forall\lambda\in\Lambda)\ (\boldsymbol{A}_\lambda\subset\mathfrak{q})\ (\Lambda \text{ is a simple-ordered set of indices}),$$
$$(\forall\lambda,\ \mu\in\Lambda)\ (\lambda<\mu\triangleright\boldsymbol{A}_\lambda\supseteq\boldsymbol{A}_\mu),$$

and if $\boldsymbol{B}=\bigcup\boldsymbol{B}_\iota$ and $\boldsymbol{A}=\bigcap\boldsymbol{A}_\lambda$, then we have one of the cases

$$\text{(i) } \boldsymbol{B}=\boldsymbol{A}, \text{ or (ii) } \boldsymbol{B}\subset\mathfrak{p},\ \boldsymbol{A}\subset\mathfrak{q} \text{ and } (\forall\boldsymbol{G})\ (\boldsymbol{B}\subset\boldsymbol{G}\triangleright\boldsymbol{G}\subset\mathfrak{q}).$$

The case (ii) can readily be turned into the case (i). In addition, in the empiricist theory,there must exist enumerable subsets of indices (ι_k) and (λ_k) such that

$$\boldsymbol{B}=\bigcup\boldsymbol{B}_{\iota_k} \text{ and } \boldsymbol{A}=\bigcap\boldsymbol{A}_{\lambda_k}.$$

Thus we have:

Proposition 2.2. (*Principle of Cut-Approach*). *If* $\mathfrak{p}$ *and* $\mathfrak{q}$ *are cut-cojugate properties and* $\mathfrak{p}$ *is regressive, and if* $\boldsymbol{M}\subset\mathfrak{q}$, *then there exist two enumerable sequences of subsets* $(\boldsymbol{B}_k)$ *and* $(\boldsymbol{A}_k)$ *such that*

$$\boldsymbol{B}_1\subseteq\boldsymbol{B}_2\subseteq\cdots\subseteq\boldsymbol{A}_2\subseteq\boldsymbol{A}_1\subseteq\boldsymbol{M},$$

$$\boldsymbol{B}_k\subset\mathfrak{p} \text{ and } \boldsymbol{A}_k\subset\mathfrak{q} \quad (k=1,2,\cdots),$$

and on denoting as $\boldsymbol{B}=\cup\boldsymbol{B}_k$ *and* $\boldsymbol{A}=\cap\boldsymbol{A}_k$, *we have* $\boldsymbol{A}=\boldsymbol{B}$.

The set $\boldsymbol{B}$ obtained in the above, is called a $\mathfrak{p}$-*cut* of the set $\boldsymbol{M}$, and then the sequences $(\boldsymbol{B}_k)$ is called a *cut-approach from below* and $(\boldsymbol{A}_k)$ a *cut-approach from above* with respect to $\mathfrak{p}$ (or $\mathfrak{q}$).

3. Cut-Amosphere

When

$$0<c<f(\boldsymbol{M})$$

let us define $\mathfrak{p}$ and $\mathfrak{q}$ such that

$$(\boldsymbol{F}\subset\mathfrak{p})\equiv(f(\boldsymbol{F})\leqslant c) \text{ and } (\boldsymbol{F}\subset\mathfrak{q})\equiv(f(\boldsymbol{F})>c),$$

then $\mathfrak{p}$ and $\mathfrak{q}$ are apparently cut-conjugate and $\mathfrak{p}$ (resp. $\mathfrak{q}$) is a regressive (resp. progressive) property. So, applying the principle of cut-approach, we may have cut-approaches $(\boldsymbol{B}_k)$ and $(\boldsymbol{A}_k)$ such that

$$\boldsymbol{B}_1\subseteq\boldsymbol{B}_2\subseteq\cdots\subseteq\boldsymbol{A}_2\subseteq\boldsymbol{A}_1\subseteq\boldsymbol{M}$$

$$f(\boldsymbol{B}_k)\leqslant f(\boldsymbol{B}_{k+1})\leqslant c<f(\boldsymbol{A}_{k+1})\leqslant f(\boldsymbol{A}_k) \quad (k=1,2,\cdots) \tag{3.1}$$

and if $\boldsymbol{B}=\cup\boldsymbol{B}_k$ and $\boldsymbol{A}=\cap\boldsymbol{A}_k$, we can expect that $\boldsymbol{B}=\boldsymbol{A}$. In this case, if $\lim f(\boldsymbol{B}_k)=\beta$ and $\lim f(\boldsymbol{A}_k)=\alpha$, we have

$$\beta\leqslant c\leqslant\alpha \tag{3.2}$$

and inversely, if c is an arbitrary value found in the relation (3.2), the relation (3.1) is to hold on for the same sequences $(\boldsymbol{B}_k)$ and $(\boldsymbol{A}_k)$. Therefore, when $\beta\neq\alpha$, the state observed through the limiting procedure

$$(\boldsymbol{A}_k-\boldsymbol{B}_j)\ (k,\ j\to\infty) \tag{3.3}$$

must give an atmospheric state in point that the limiting value of $f(\boldsymbol{A}_k-\boldsymbol{B}_j)$ is to be counted as equal to the positive number $\alpha-\beta$ despite of the fact

$$\lim(\boldsymbol{A}_k-\boldsymbol{B}_j)=\text{void}.$$

So the state indicated by (3.3) is called a *cut-atmosphere* when $\beta\neq\alpha$.

If ◎ $<f(P)<$ ⟁ ($\equiv$ infinitesimal) for every point P in $\boldsymbol{M}$, f is said to be *powdery* in $\boldsymbol{M}$. If we demand $\beta=\alpha$ in (3.2), it must be that the sequences $(\boldsymbol{B}_k)$ and $(\boldsymbol{A}_k)$ are dexterously chosen to satisfy the condition. For this purpose, it might be helpful to suppose that, in case of a powdery function f, the mass value

$f(\boldsymbol{M})>0$ should at any rate be thought as an accumulation of infinitesimal quatities $f(P)$ $(P\in\boldsymbol{M})$. Then, if $f(\boldsymbol{N})>f(\boldsymbol{F})$, by transferring points from $\boldsymbol{N}$ to $\boldsymbol{F}$, the difference of f-value might be decreased until it vanishes. Such a principle of transferring may be asserted as a generalization of the archimedian principle in arithmetic. In logical view of the matter, this principle may serve as a modal mediation between the independent descriptions: (1) $(\forall P\in\boldsymbol{M})$ $(f(P)=0)$, and (2) $(\forall \boldsymbol{F}\subseteq\boldsymbol{M})$ $(f(\boldsymbol{F})\geqslant 0)$ & $f(\boldsymbol{M})>0$. In effect, as long as (1) and (2) are the only premises, it seems almost impossible to induce any of the following facts which are very naturally expected: (i) if $f(\boldsymbol{M})=\infty$, there exists a subset $\boldsymbol{F}$ of $\boldsymbol{M}$ such that

$$0<f(\boldsymbol{F})<\infty\,;$$

(ii) in case of $0<f(\boldsymbol{M})<\infty$, for any natural integer n, there exists a partition $\{\boldsymbol{M}_1, \boldsymbol{M}_2, \cdots, \boldsymbol{M}_n\}$ of $\boldsymbol{M}$ such that

$$f(\boldsymbol{M}_1)=f(\boldsymbol{M}_2)=\cdots=f(\boldsymbol{M}_n)=\frac{f(\boldsymbol{M})}{n}\,.$$

However, if we apply the above-mentioned principle or transferring, the state may turn out to be very hopeful. Consequently, we might expect the following proposition to be assertively obtained, except the accuracy of reasoning.

Proposition 3.1. (*Principle of Continuous Accumulation*). *If f is a powdery set function and*

$$0<c<f(\boldsymbol{M})\,,$$

there exists a subset $\boldsymbol{C}$ of $\boldsymbol{M}$ such that

$$f(\boldsymbol{C})=c\,. \tag{3.4}$$

The subset $\boldsymbol{C}$ satisfying (3.4) is called a *c-cut* of the set $\boldsymbol{M}$ with respect to f.

4. Cut-Probabilism

By Proposition 3.1 we insisted that, for any powdery set function, we may have at least one c-cut for any intermediate value of c for any set $\boldsymbol{M}$. However, this fact may not mean that any powdery set function is completely additive. For instance, let us take real numbers all to be equi-probable to occur in the real axis and define the aleatory variable x as the occurrence of a real number x in this probabilistic construction, then the set function $\pi(\boldsymbol{M})$ defined by

$$\pi(\boldsymbol{M})=\text{Prob. }(x\in\boldsymbol{M})$$

cannot be completely additive, while for any value c such that $0<c<1$ a set $\boldsymbol{M}$ may be made existent to satisfy the relation

$$\pi(\boldsymbol{M})=c\,.$$

In effect, if $\boldsymbol{M}$ is the sum of the intervals

$$(n,\ n+c)\ (n=0,\ \pm 1,\ \pm 2, \cdots)$$

it may naturally be admitted that $\pi(\boldsymbol{M})=c$.

If f is a powdery set function and if

$$\boldsymbol{M}_1 \subset \boldsymbol{M}_2 \subset \cdots \subset \boldsymbol{M} \text{ and } \cup \boldsymbol{M}_k = \boldsymbol{M},$$

then, on a c-cut C of the set $\boldsymbol{M}$, we are tempted to suppose that

$$\lim \frac{f(\boldsymbol{M}_k \cap \boldsymbol{C})}{f(\boldsymbol{M}_k)} = \frac{c}{f(\boldsymbol{M})}.$$

But, in point of fact, this relation is not always effected especially depending on whether

$$\cap (\boldsymbol{C}-\boldsymbol{M}_k) = \text{void}.$$

or not. Therefore, if we insist on any asymptotic approach, we shall work at the residual part $\cap(\boldsymbol{C}-\boldsymbol{M}_k)$, and in effect we find a light in this part, accompanied by a new mode of reconstruction.

If, for any subset $\boldsymbol{F}$ of $\boldsymbol{M}$, the relation

$$\lim_k \frac{f(\boldsymbol{F}-\boldsymbol{M}_k)}{f(\boldsymbol{M}-\boldsymbol{M}_k)} = \lim_k \left(\lim_j \frac{f(\boldsymbol{F} \cap (\boldsymbol{M}_j - \boldsymbol{M}_k))}{f(\boldsymbol{M}_j - \boldsymbol{M}_k)} \right)$$

is reckoned as true, $\boldsymbol{M}$ is said to be *regressively cut-probabilistic* in respect to f. When f is a powdery set function, defined for any set $\boldsymbol{F}$ such that $(\exists k)$ $(\boldsymbol{F} \subseteq \boldsymbol{M}_k)$, then, for any number b such that

$$\lim f(\boldsymbol{M}_k) = \beta < b,$$

f may be extended through the additional definition

$$\overset{*}{f}(\boldsymbol{F}) = \lim_k f(\boldsymbol{F} \cap \boldsymbol{M}_k) + (b-\beta) \lim_k \left(\lim_j \frac{f(\boldsymbol{F} \cap (\boldsymbol{M}_j - \boldsymbol{M}_k))}{f(\boldsymbol{M}_j - \boldsymbol{M}_k)} \right)$$

$\boldsymbol{F}$ being an arbitrary subset of $\boldsymbol{M}$. It is readily seen that $\boldsymbol{M}$ is regressively cut-probabilistic in respect to $\overset{*}{f}$ and that

$$\overset{*}{f}(\boldsymbol{F}) = f(\boldsymbol{F}) \tag{4. 1}$$

whenever $(\exists k)$ $(\boldsymbol{F} \subseteq \boldsymbol{M}_k)$. However, it must be noted that a function $\overset{*}{f}$, which satisfies (4. 1), is not uniquely determinable on the single condition

$$\overset{*}{f}(\boldsymbol{M}) = b.$$

So, for the present, we shall restrain ourselves from supposing that any (non-negative) set function may be found to conform to cut-probabilism.

Mathematical Seminar in the Muroran Inst. Tech., Hokkaido

(Received Apr. 12. 1966)

References

1) Kinokuniya, Y.: Mem. Muroran Inst. Tech. **5** (1), 341–347 (1965).
As a preliminary guide:
Kinokuniya, Y.: Mem. Muroran Inst. Tech. **4** (2), 491–496 (1963).