On the Foundations of Empiricist Logic

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Abstract

To make it exactly applicable to the usual course of practical investigations of mathematics, a system of logic is shown in connection with the empiricist theory of sets. Notion of *implication* is defined by means of ranges and is distinguished from that of conditional implication. Some treatments of undecidable objects are elucidated in reference to the expansion of the universe.

I. Introduction

A predicate may be said to be determinate if and only if objects which are admissible to it and objects which are inadmissible to it, simultaneously, make determinate sets. This view will be essential for making a predicate tightly associated with the universe of its objects of application. Specially, in the theory of confirmation[1], it plays an important role. In this paper, we discuss logical problems in connection with the empiricist theory of sets, and intend to construct a theory of, so to say, *empiricist logic*.

First, we begin with the observer's language $\mathfrak{L}_0$ (here, English), by which we define signs and special terms to make up an object language $\mathfrak{L}$ (in the generalized sense) together with $\mathfrak{L}_0$. In fact, the language $\mathfrak{L}$ may not be a ready fixed language, because there may not be promised any end to the definition of new notions and new signs. Thus $\mathfrak{L}$ may be said to be indeterminate, though there may be no real objection to its practice. It is historical and it is naturally varied depending on specifications made by the author. Since our logical investigations are developed by using $\mathfrak{L}$, we say 'they *stand on* $\mathfrak{L}$'.

If inferences are, under an axiom system $\mathfrak{a}$ standing on $\mathfrak{L}$, made to produce conclusions, the class of these conclusions is called the theory $T(\mathfrak{L}, \mathfrak{a})$. $\mathfrak{L}$ shall at least contain the following signs:

1) *logical connectives*: $\vee$, $\wedge$, $\sim$(*negation*); &; $\equiv$(*definition*)**;

2) *set-theoretical symbols*: $\cup$, $\cap$, $-$(*difference of sets*); $\in$, $\notin$, $\ni$, $\not\ni$, $\subset$, $\not\subset$, $\supset$, $\not\supset$, $\subseteq$, $\supseteq$, $=$, $\neq$;

3) *set-theoretical concepts*: $\emptyset$(*void*), $\boldsymbol{P}(M)$ (*power set of a set* M);

4) *quantifiers*: $\exists$, $\forall$.

These signs or symbols are well-known so that no more of expositions about them may be needed. When we mention $\mathfrak{L}$, $\mathfrak{L}$ is, of course, expected to contain

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** Besides these, signs $\Rightarrow$ and $\Leftrightarrow$, later on given by Definition I. 1 and Definition I. 2 respectively, shall be added as connectives.

signs and concepts occasionally defined besides the above-shown ones, within the limit of need.

If a concatenation (or a chain, or a string) of some symbols and words of $\mathfrak{L}$ is read and is thought to indicate some object, some state or relation of objects, some relation of states (of objects), or some state of relations, it is called an *event*. An event and a concatenation which indicate some abstract meaning to be predicated to an event, are equally called *expressions*.

For any event $\boldsymbol{a}$, the following evaluations are expected:

$$\vdash \boldsymbol{a} \text{ (`}\boldsymbol{a}\textit{ is possible', or `}\boldsymbol{a}\textit{ is ture'});$$

$$\sim\vdash \boldsymbol{a} \text{ (`}\boldsymbol{a}\textit{ is impossible', or `}\boldsymbol{a}\textit{ is false'}).$$

When whether $\vdash \boldsymbol{a}$ or $\sim\vdash \boldsymbol{a}$ is not decided, $\boldsymbol{a}$ is called an *undecidable event*. The following definition may, thereupon, be possible:

Definition. *If cases in which $\vdash \boldsymbol{a}$ cannot be thought to vanish, $\boldsymbol{a}$ is called a possible event.*

If it is stipulated that, for any element $\boldsymbol{p}$ of a set of events $\boldsymbol{U}$,

$$\sim\boldsymbol{p} = \boldsymbol{U} - \{\boldsymbol{p}\},$$

then $\boldsymbol{U}$ is called a *range universe* or simply a *universe*.

If $\boldsymbol{a}$ and $\boldsymbol{b}$ are possible events and if the relation

$$\sim(\vdash \boldsymbol{a} \wedge \vdash \boldsymbol{b})$$

is satisfied, $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be *mutually exclusive*. If any two events from a set of possible events $(\boldsymbol{e}_\lambda)$ $(\lambda \in \Lambda)$ are mutually exclusive, $(\boldsymbol{e}_\lambda)$ is said to be an *exclusive family*.

***Definition.* I. 1.** *For a subset M of a universe $\boldsymbol{U}$, if it is written as*

$$M = R(\boldsymbol{s}),$$

M is called the (deductive) (logical) range[2] *of $\boldsymbol{s}$, which is defined by*

$$\vdash \boldsymbol{s}(x). \equiv . x \in M\,; \quad \sim\vdash \boldsymbol{s}(x). \equiv x \notin M (x \in \boldsymbol{U}).$$

and is called a predicate on $\boldsymbol{U}$.

If an expression $\boldsymbol{s}$ is taken as a predicate on a universe $\boldsymbol{U}$, $\boldsymbol{s}$ may, in a sense, be thought as a *well-formed predicate*. However, we, in this paper, call such an $\boldsymbol{s}$ an *analytic predicate*. A compound of predicates on $\boldsymbol{U}$ is called an *analytic event* on $\boldsymbol{U}$. If $\boldsymbol{s}$ is an analytic predicate, both of $\boldsymbol{s}$ and $\boldsymbol{s}(x)$ are called *events*.

Now, it will be readily be seen that:

$$\text{if } R(\boldsymbol{s}) \neq \emptyset,\ \boldsymbol{s} \text{ is possible on } \boldsymbol{U};$$

$$\text{if } R(\boldsymbol{s}) = \emptyset,\ \boldsymbol{s} \text{ is impossible on } \boldsymbol{U};$$

and if $R(\boldsymbol{s})$ is an indeterminate set, $\boldsymbol{s}$ is undecidable on $\boldsymbol{U}$.

At building an empiricist logic, we will firstly except non-analytic events from our sphere of consideration.

Definition **I. 2.**

$$\boldsymbol{a} \Rightarrow \boldsymbol{b}. \equiv : R(\boldsymbol{a}) \subseteq R(\boldsymbol{b}). \ \& \ .R(\boldsymbol{a}) \neq \emptyset ;$$
$$\boldsymbol{a} \Leftrightarrow \boldsymbol{b}. \equiv : \boldsymbol{a} \Rightarrow \boldsymbol{b}. \ \& \ .\boldsymbol{b} \Rightarrow \boldsymbol{a}.$$

'$\boldsymbol{a} \Rightarrow \boldsymbol{b}$' renders '*$\boldsymbol{a}$ implies $\boldsymbol{b}$*' and '$\boldsymbol{a} \Leftrightarrow \boldsymbol{b}$' renders '*$\boldsymbol{a}$ is equivalent to $\boldsymbol{b}$*'.

Above-defined notions of implication and of equivalence, as they were, are to be called notions of *empiricist implication* and of *empiricist equivalence* respectively, and are to be distinguished from the ones based on the definition

$$\boldsymbol{a} \Rightarrow \boldsymbol{b}. \equiv .(\sim \boldsymbol{a}) \vee \boldsymbol{b} \tag{c}$$

which is used by several symbolic logicians. However, the relation of (c) is essentially what has been called a *conditional relation*[3], and is one of 16 birational operations given between $\boldsymbol{a}$ and $\boldsymbol{b}$, so that it is related to an event which is of different level from the empiricist implication. So, using the sign $\supset$, we apply the definition:

Definition **I. 3.**

$$\boldsymbol{a} \supset \boldsymbol{b}. \equiv .(\sim \boldsymbol{a}) \vee \boldsymbol{b} .$$

Then, as above-mentioned, $\supset$ is a birational operator.

To raise our investigation, the space $\boldsymbol{U}_0$ which consists of primitive objects of level zero, must be given as a determinate set associated with the language $\mathfrak{L}$. Besides, in order to keep connection with the theory of sets, the axiom system $\mathfrak{a}$ must contain set-theoretical axioms to be therein applied. For instance, in order to use the set-theoretical sign $=$, $\mathfrak{a}$ must contain the *axiom of extension* (viz. $(\forall x \in A)\ (x \in B)\ (\forall y \in B)\ (y \in A) \Leftrightarrow .\ A = B$). Moreover, since the set theory on which we are going to found our arguments is the empiricist one, the following stipulations shall naturally be followed:

(i) ordinal numbers are limited within at most the 2nd class;

(ii) for any family of sets (M_ι) $(\iota \in I)$ with a simple-ordered indication I, there exist sequences of indices (λ_k) and (μ_k) $(k=1, 2, \cdots)$ such that

$$\cup M_\iota = \bigcup_{k=1}^{\infty} \Big(\bigcup_{\iota < \lambda_k} M_\iota \Big) \text{ and } \cap M_\iota = \bigcap_{k=1}^{\infty} \Big(\bigcap_{\iota < \mu_k} M_\iota \Big);$$

(iii) if $\lim_\iota F_\iota$ exists, there exists a sequence of indices (ι_k) $(k=1, 2, \cdots)$ such that

$$\lim_\iota F_\iota = \lim_k F_{\iota_k}$$

An expression which is regarded as a declarative statement and what is considered to be the meaning of a declarative statement, are both called *propositions*. An expression which interprets an event, is sometimes called a *description* (in the generalized sense). A proposition is necessarily reckoned to be (i) true, (ii) false, or (iii) undecidable. In case of (i) it is a *theorem*, and in case of (ii) it is a *contradiction*.

The class of theorems which are resulted by logical inferences standing on $\mathfrak{L}$ under the guidance of an axiom system $\mathfrak{a}$, is called the *theory generated on the inference ground* $(\mathfrak{L}, \mathfrak{a})$ and is denoted by $T(\mathfrak{L}, \mathfrak{a})$.

The logic which we here intends to establish, is not a logic which has only tautologies as results of it, but a logic, the content of which makes a logical course of practical inferences in the usual geometry or in the empiricist analysis. Therefore, if $\mathfrak{a}$ is the axiom system of euclidian geometry, $T(\mathfrak{L}, \mathfrak{a})$ coincides with euclidian geometry itself. Since the recent analysis is closely related to the set theory, the analysis may vary depending on whether it bases upon the empiricist ground or not. In this connection, the empiricist logic shall be a logic for the course of empiricist analysis.

II. Hierarchy Branch

When a set of events (or conditions) $\boldsymbol{P}=(\boldsymbol{p}_\lambda)\,(\lambda\in\Lambda)$ is given, let it be that

$$\widetilde{\boldsymbol{P}}=\bigcup_{\lambda\in\Lambda}(\{\boldsymbol{p}_\lambda\}\cup\{\sim\boldsymbol{p}_\lambda\})=\bigcup_{\nu\in N}(\boldsymbol{p}'_\nu)\,.$$

Then, if there is an exclusive universe $\boldsymbol{U}=(\boldsymbol{r}_\mu)\,(\mu\in M)$ and if relations

$$(\forall\mu\in M)\,(\exists N_1\subseteq N)\,(\boldsymbol{r}_\mu=\bigwedge_{\nu\in N_1}\boldsymbol{p}'_\nu)$$

and

$$(\forall\nu\in N)\,(\exists M_1\subseteq M)\,(\boldsymbol{p}'_\nu=\bigvee_{\mu\in M_1}\boldsymbol{r}_\mu)$$

are satisfied, then $\boldsymbol{U}$ is called the ***P**-aspect*. A predicate on $\boldsymbol{U}$ may be thought to be of higher level by 1 than any element of $\boldsymbol{U}$. So, if $\boldsymbol{P}_n$ is a set of predicates on $\boldsymbol{U}_n$ and $\boldsymbol{U}_{n+1}$ is the $\boldsymbol{P}_n$-aspect, and if $\boldsymbol{U}_{n+1}$ and $\boldsymbol{U}_n$ are not essentially equivalent (i.e. they cannot essentially be the same set of objects), then $\boldsymbol{U}_{n+1}$ is said to be of higher level by 1 than $\boldsymbol{U}_n$. In this case, since $\boldsymbol{U}_{n+1}$ may vary depending on the choice of $\boldsymbol{P}_n$, to mean that $\boldsymbol{U}_{n+1}$ is determined by $\boldsymbol{P}_n$ we say '*the type of* $\boldsymbol{U}_{n+1}$ *is determined* (or, *is given*) (*by* $\boldsymbol{P}_n$)' and write it as

$$\boldsymbol{U}_n \overset{(P_n)}{\prec} \boldsymbol{U}_{n+1} \tag{II. 1}$$

or simply as

$$\boldsymbol{U}_n \prec \boldsymbol{U}_{n+1}\,.$$

(II. 1) itself is called a *branching* (*of the type of universe*).

Starting from the primitive universe $\boldsymbol{U}_0$, we may, by succession of branchings, obtain a sequence of universes $(\boldsymbol{U}_n)\,(n=1, 2, \cdots)$ such that

$$\boldsymbol{U}_0\prec\boldsymbol{U}_1\prec\cdots\prec\boldsymbol{U}_n\prec\boldsymbol{U}_{n+1}\prec\cdots.$$

Then $(\boldsymbol{U}_n)$ is called a *branch of type derivation* or simply a *branch*.

If we take an adequate $\boldsymbol{P}'_0$ we may have

$$\boldsymbol{U}_0 \overset{(r_0')}{\prec} \boldsymbol{U}_1' . \ \& \ . \boldsymbol{U}_3 \subseteq \boldsymbol{U}_1' .$$

Therefore, the level of a predicate (or, of a universe) is not to be absolutely fixed, but is relatively determined depending on the branching.

If a theorem, concerned with ranges of predicate, is set-theoretically verified, then it is said to be a theorem (*standing*) *on* the range universe $\boldsymbol{U}$ of which the ranges are subsets. Denoting by $\mathfrak{P}_k$ a set of theorems on $\boldsymbol{U}_k (k=1, 2, \cdots)$, $\mathfrak{P}_{n+1}$ may be considered as being obtained in reference to $\mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_n$, if $\mathfrak{P}_k (k=1, \cdots, n)$ are adequately chosen. In this connection, we denote by

$$\mathfrak{H}_n = (\boldsymbol{U}_0 \prec \cdots \prec \boldsymbol{U}_n \,;\ \mathfrak{P}_1 \cup \cdots \cup \mathfrak{P}_n)$$

the relative construction of $\mathfrak{P}_1, \cdots, \mathfrak{P}_n$, and by

$$\mathfrak{H}_n \xrightarrow{\boldsymbol{U}_{n+1}} \mathfrak{P}_{n+1}$$

or simply by

$$\mathfrak{H}_n \longrightarrow \mathfrak{P}_{n+1}$$

the fact that $\mathfrak{P}_{n+1}$ is obtained in reference to $\mathfrak{H}_n$, $\mathfrak{H}_n$ is called a *hierarchy of order* n.

In most case of usual lectures or text books (of geometry, algebra, differential calculus etc.), the order of hierarchy seems to halt within 4 or so. It may be said that, what makes the content of a theory rich is not the highness of the order of hierarchy, but perhaps is the variousness of possible branchings. As a matter of fact, to prove a theorem (say, T) will necessarily need a special device of hierarchy. So, if $T \in \mathfrak{P}_{n+1}$ and if T is obtained on the way to produce $\mathfrak{H}_{n+1}$ by adding $\mathfrak{P}_{n+1}$ to $\mathfrak{H}$, we denote this addition by

$$(\mathfrak{H}_n , \quad \mathfrak{P}_{n+1})$$

and call it the *process stage* for the proof of T.

Some theorems might need the process of mathematical induction for their proof. In such cases, *inductive ranges* which were introduced in the previous paper[4] (by the present author), should be used instead of deductive ones ; however, in this paper, we will not touch on their details. In the empiricist analysis, a method called '*trans-induction*'[4] is used instead of the method of transfinite induction ; however, we will not touch on it here.

III. Comparison

In order to observe the content of $T(\mathfrak{L}, \mathfrak{a})$ depending on the choice of an axiom system $\mathfrak{a}$, standing on the fixed object language $\mathfrak{L}$, we will simply write $T(\mathfrak{a})$ instead of $T(\mathfrak{L}, \mathfrak{a})$. If it is verified that some contradiction must occur in $T(\mathfrak{a})$, then $T(\mathfrak{a})$ or $\mathfrak{a}$ is said to be *really inconsistent.* If the set of propositions which are put to the proof in reference to $T(\mathfrak{a})$ is an infinite set, the total aspect of $T(\mathfrak{a})$ cannot be obtained; so then, whether it is really inconsistent or consistent may not be ascertained. However, if we assume that $T(\mathfrak{a})$ is not really inconsistent, then it will be the same as to assume that $T(\mathfrak{a})$ is consistent.

$\mathfrak{a}_1$ and $\mathfrak{a}_2$ being two axiom systems standing on $\mathfrak{L}$, if a proposition Q standing on $\mathfrak{L}$ is true in $T(\mathfrak{a}_1)$ and false in $T(\mathfrak{a}_2)$, and if $\mathfrak{a}^*=\mathfrak{a}_1\cap\mathfrak{a}_2\neq\emptyset$, then we say '*$Q$ is undecidable under* $\mathfrak{a}^*$'. If such a Q really exists, then it is evident that $T(\mathfrak{a}_1\cup\mathfrak{a}_2)$ is really inconsistent.

If axiom systems $\mathfrak{a}_1, \cdots, \mathfrak{a}_n$ are exposed for the purpose of performing the comparison of $T(\mathfrak{a}_1), \cdots, T(\mathfrak{a}_n)$, $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}$ is called a *comparison*. If $T(\mathfrak{a}_1\cup\cdots\cup\mathfrak{a}_n)$ is consistent, $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}$ is said to be *compatible*. The set of axioms

$$\mathfrak{a}^*=\mathfrak{a}_1\cap\cdots\cap\mathfrak{a}_n$$

is called the *kernel* of the comparison $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}$. If there are axiom systems $\mathfrak{a}_+$ and $\mathfrak{a}_-$ such that

$$\mathfrak{a}^*\subseteq\mathfrak{a}_+\cap\mathfrak{a}_- \text{ and } \mathfrak{a}_+\cup\mathfrak{a}_-\subseteq\mathfrak{a}_1\cup\cdots\cup\mathfrak{a}_n$$

and if there exists a proposition Q which is evaluated to be true in $T(\mathfrak{a}_+)$ and false in $T(\mathfrak{a}_-)$, then Q is said to be *undecidable on* $\{\mathfrak{a}_1, \cdots, \mathfrak{a}_n\}$. In such a case, $T(\mathfrak{a}_1\cup\cdots\cup\mathfrak{a}_n)$ is, of course, really inconsistent.

***Theorem* III. 1.** *When $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ are axiom systems standing on the same object language $\mathfrak{L}$, though $\{\mathfrak{a}_1, \mathfrak{a}_2\}$ and $\{\mathfrak{a}_2, \mathfrak{a}_3\}$ are both compatible, $\{\mathfrak{a}_1, \mathfrak{a}_3\}$ is not necessarily compatible.*

Demonstration. (0) $1\in\boldsymbol{U}$; (1) $a, b\in\boldsymbol{U} .\Rightarrow. a+b\in\boldsymbol{U}(c=a+b .\equiv. a=c-b)$; (2) $a\in\boldsymbol{U} .\&. ma=na: \Rightarrow. m=n$; (3) $a, b\in\boldsymbol{U} .\Rightarrow. a-b\in\boldsymbol{U}$; (4) $a, b\in\boldsymbol{U} .\Rightarrow. ab\in\boldsymbol{U}$ $(c=ab .\equiv. a=c/b)$; (5) $a, b\in\boldsymbol{U} .\Rightarrow. a/b\in\boldsymbol{U}$; (6) $a-b=c_1, c_2 .\Rightarrow. c_1=c_2$; (7) $a/b=c_1, c_2 .\Rightarrow. c_1=c_2$. Then, if we posit as $\mathfrak{a}_1=\{(0), (1), (2), (3), (6)\}$, $\mathfrak{a}_2=\{(0), (1), (2), (6)\}$ and $\mathfrak{a}_3=\{(0), (4), (5), (7)\}$, $T(\mathfrak{a}_1)$ may stand on the set of rational integral numbers, $T(\mathfrak{a}_2)$ on the set of positive integral numbers, $T(\mathfrak{a}_3)$ on the set of positive rational numbers, $T(\mathfrak{a}_1\cup\mathfrak{a}_2)$ on the set of rational integral numbers, and $T(\mathfrak{a}_2\cup\mathfrak{a}_3)$ on the set of positive rational numbers. However, $T(\mathfrak{a}_1\cup\mathfrak{a}_3)$ is found to be really inconsistent, because it needs the (total) set of (positive and negative) rational numbers, whereas $0/0$ cannot guarantee the consistence of (7).

The above-noticed object $0/0$ is, essentially, very important. In our usual analysis, $0/0$ is not treated as an undecidable object, but is interpreted as an infinitely ramificated object, so that it is excepted as an object of indefinite form and is not thought to be related to an inconsistency of the theory. However, such a treatment is, in the end, only a subsidiary interpretation and is not an essential one directly derivable from the axiom system.

IV. Some Treatments of Undecidable Objects

$\boldsymbol{p}$ being a predicate promised its range $R(\boldsymbol{p})$ in a certain universe $\boldsymbol{U}$, if $R(\boldsymbol{p})=\emptyset$, $\boldsymbol{p}$ must, in fact, an impossible event which has no chance at all to be realized on U. If $R(\boldsymbol{p})$ is meaningless or $R(\boldsymbol{p})=\varnothing$, $\boldsymbol{p}$ is a predicate which cannot be realized in the course of observation. However, in case of $R(\boldsymbol{p})=\varnothing$, if we hypothetically take up a set $R(\boldsymbol{p})$ to be assumed as if $R(\boldsymbol{p})\neq\varnothing$, then, by this assumption, we may only be imposed an extension of the family of sets in accord with the theory for which we are trying. In this connection, the set $R(\boldsymbol{p})$ which shall be added to the family of sets, will then turn out to be treated as an undecidable object for the course of observation. Thus, we may have the following two behaviors to be possible:

(i) we reject $R(\boldsymbol{p})$ as an impossible image;

(ii) we admit the hypothetical set $R(\boldsymbol{p})$ to be added to cause an extension of the family of sets.

In case of (i), $R(\boldsymbol{p})$ is delited out as an exceptible noise for our investigation, whereas, in case of (ii), the addition of $R(\boldsymbol{p})$ must accompany some additional axioms through which the extended space shall be well-reconstructed and hence the extended axiom system shall be found to be compatible. The verification of the compatibility of the extended system, if on an infinite universe, might not be possible without any specific condition. Inc ase of gaussian plane (of complex numbers), it was simply introduced only through some elucidation of the amplitude of a complex number; but, in fact, it had to be passed to Riemann's renovation on the construction. Indeed, the most important thing was the illustrative specification of the notion of the amplitude of a complex number, but that alone could not make a completion, because any prolongative succession of mappings around a singular point of a function necessarily needed Riemann's reconstructive specification of the plane (of complex numbers).

In analizing $R(\boldsymbol{p})$, tracing back to the original universe of primitive objects, if all the intermediate processes are proceeded within a finite number of stages of finitary state, no undecidability can occur, because all the observations must then stay within effective computations. When all the processes, above-mentioned, of tracing back to the orignal universe make only effective computations, $R(\boldsymbol{p})$ is said to be an *effective range*. Then, on the undecidability in case of (ii), the following fact may be stated as a mark of inspection:

***Theorem* IV. 1.** *If $R(\boldsymbol{p})\neq\varnothing$ and $R(\boldsymbol{p})$ is undecidable, $R(\boldsymbol{p})$ cannot be an effective range.*

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