Notched Half-plane under Gravity Force

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Abstract

Stress functions, which W. E. Warren obtained for the stress field created by a point load acting at an arbitrary position in a notched half-plane, are used to cancel the gravity forces on the surface. Mapping function allows consideration of a notch whose radius of curvature varies from 0 to infinite. Numerical and graphical results are presented for the surface stress distribution in the vicinity of the tip.

1. Introduction

The geomertrical configuration of the shallow externally notched half-plane is shown in the complex z plane in Fig. 1. A concentrated force is assumed to act on the surface t, and curvilinear component of stress σ_ξ, σ_η, and $\tau_{\xi\eta}$ are also shown in Fig. 1. The transformation

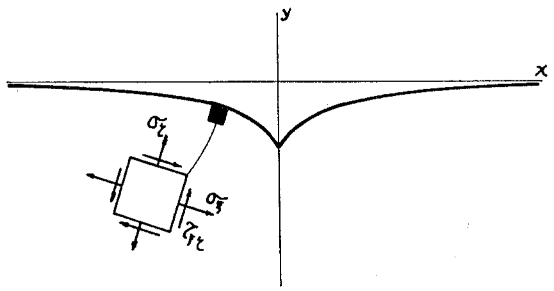

Fig. 1.

$$z = w(\zeta) = c\left(\zeta + \frac{b}{\zeta - i}\right), \qquad b > -1 \tag{1}$$

maps the notched elastic region in the z plane into the lower half plane, the notch surface mapping into the real axis. The depth of the notch is c.b and the radius of curvature at the bottom is expressed as

$$\rho = \frac{c(1+b)^2}{2b} \tag{2}$$

For $b = -1$, the radius of curvatuae is zero and a cusp appears on the notched surface. The case $b=0$ corresponds to the straight half-plane, while $b>0$ represents a protrusion on the half-plane. The notch shape is affected by the parametea b and the effect of b on the stress state in the vicinity of the notch is of physical interest.

2. Stress Functions

The functions $\phi(\zeta)$ $\psi(\zeta)$ are holomorphic in Im $\zeta<0$ and vanish at infinity. The curvlinear components of the complex stress combinations are

$$\sigma_\eta+\sigma_\xi=2[\phi(\zeta)+\bar{\phi}(\bar{\zeta})] \quad (3)$$

$$\sigma_\eta-\sigma_\xi+2i\tau_{\xi\eta}=\frac{2}{\bar{w}'(\zeta)}[\bar{w}(\bar{\zeta})\phi'(\zeta)+w'(\zeta)\psi(\zeta)]$$

addition of the two expressions in Eq. (3) gives

$$\sigma_\eta+i\tau_{\xi\eta}=\phi(\zeta)+\bar{\phi}(\bar{\zeta})+\frac{1}{\bar{w}'(\bar{\zeta})}[\bar{w}(\bar{\zeta})\phi'(\zeta)+w'(\zeta)\psi(\zeta)] \quad (4)$$

which is a useful expression for formulating the boundary condition.

Consider a gravity force $F=X+iY$ acting at z in space the notched elastic region, then in case of the density γ

$$F=y\gamma(\nu+i)$$

and

$$F=(i\nu-1)\frac{cb\gamma}{2}\left(\frac{1}{\bar{\zeta}+i}-\frac{1}{\zeta-i}\right)+c(\nu+i)\eta$$

from which gravity force on the notch surface is given

$$F(t)=(\nu+i)cb\gamma/(t^2+1) \quad (5)$$

To make the problem simple, assume ν in the above be zeao that is for the case gravity force acts in the vertical direction alone. According to W. E. Waaren, the complex functions for the case veatical single load acts at an arbitrary position on the notched surface, are formulated as follows:

$$\phi_0(\zeta)=\frac{ip}{2\pi c[(\zeta-i)^2-b]}\left\{\frac{(\zeta-i)^2}{(\zeta-t)}+\frac{b^2}{2(2+b)(t+i)}+\frac{b(4+b)}{2(2+b)(t-i)}\right\} \quad (6)$$

$$\begin{aligned}\psi_0(\zeta)=\frac{ip(\zeta-i)^2}{2\pi c[(\zeta-i)^2-b]}\Bigg\{&\frac{1}{(\zeta-t)}+\frac{b[2(\zeta-i)(\zeta+i)+b\zeta(\zeta-3i)+b^2]}{(2+b)(t-i)[(\zeta-i)^2-b]^2}\\&+\frac{t(\zeta-i)^2}{(\zeta-t)^2[(\zeta-i)^2-b]}+\frac{2b\zeta(\zeta-i)}{(\zeta-t)[(\zeta-i)^2-b]^2}-\frac{2b^2(\zeta-i)}{(\zeta-t)^2[(\zeta-i)^2+b]^2}\\&+\frac{2b(\zeta-i)^2}{(2+b)(t+i)(\zeta-t)^2[(\zeta-i)^2-b]}+\frac{3ib^2(\zeta-i)}{(2+b)(t+i)[(\zeta-i)^2-b]^2}\\&+\frac{b^2(\zeta-i)[\zeta^3-3i\zeta^2+\zeta+5i+b(\zeta+3i)]}{(2+b)(t+i)(\zeta-t)^2[(\zeta-i)^2-b]^2}\Bigg\}\end{aligned} \quad (7)$$

in which P is intencity of the load and t indicates the position of it. The complex curvilinear stress combinations are obtained by substituting Eqs. (6) and (7) into Eq. (3). One finds

$$\frac{2\pi c}{p}(\sigma_{\xi0}+\sigma_{\eta0})=4Re\left\{\frac{i}{[(\zeta-i)^2-b]}\left[\frac{(\zeta-i)^2}{(\zeta-t)}+\frac{bt(2+b)+2ib}{(2+b)(t^2+1)}\right]\right\} \quad (8)$$

$$\frac{2\pi c}{p}(\sigma_{\eta 0}-\sigma_{\xi 0}+2i\tau_{\xi\eta 0})=\frac{2i(\bar{\zeta}+i)^2(\zeta-i)}{[(\bar{\zeta}+i)^2-b][(\zeta-i)^2-b]^2}\left\{\frac{6b^2}{(2+b)(t^2+1)}\right.$$
$$+\frac{\zeta(\zeta-i)[(\zeta-i)^2-b]-2b^2}{(\zeta-t)^2}+\frac{b[2(\zeta^2+1)+b(\zeta^2+3)+b^2]}{(2+b)(t-i)(\zeta-t)}$$
$$+\frac{b[2(\zeta-i)^3+b(\zeta^3-3i\zeta-\zeta+7i)+b^2(\zeta-3i)]}{(2+b)(t+i)(\zeta-t)^2}$$
$$\left.-\frac{[\bar{\zeta}(\bar{\zeta}+i)+b]}{(\bar{\zeta}+i)}\left[\frac{(\zeta-i)^3+b(\zeta-2t+i)}{(\zeta-t)^2}+\frac{2bt(2+b)+4ib}{(2+b)(t^2+1)}\right]\right\} \qquad (9)$$

3. Solution of the Problem

Straight half-plane, whose surface is assumed to $y=0$, holds the gravity force given by Eq. (5) at the depth where the notch surface would pass. Substitution of $F\cdot dx$ for P in Eqs. (6), (7) leads to the stress functions which will cancel the surface traction along the notch surface, from the gravity as follows

$$\phi(\zeta)=\int_{-\infty}^{\infty}\phi_0(\zeta)\frac{2cb\gamma}{P}\frac{1}{(t^2+1)}Re[w'(t)]\,dt \qquad (10)$$

$$\psi(\zeta)=\int_{-\infty}^{\infty}\psi_0(\zeta)\frac{2cb\gamma}{P}\frac{1}{(t^2+1)}Re[w'(t)]\,dt \qquad (11)$$

where

$$Re[w'(t)]\,dt=c\left(1-\frac{b(t^2-1)}{(t^2+1)^2}\right)dt\,.$$

The complex curvilinear stress combinations are obtained by superposing Eqs. (10) and (11) with Eq. (3):

$$\sigma_\xi+\sigma_\eta=\gamma y+4Re\phi(\zeta)\,,$$
$$\sigma_\eta-\sigma_\xi+i2\tau_{\xi\eta}=\frac{w'(\zeta)}{\bar{w}'(\bar{\zeta})}\gamma y+\frac{2}{\bar{w}'(\bar{\zeta})}[\bar{w}(\bar{\zeta})\phi'(\zeta)+w'(\zeta)\psi(\zeta)]\,,$$

from which one finds

$$\sigma_\xi+\sigma_\eta=\gamma y-4bc\gamma Re\left[\frac{1}{[(\zeta-i)^2-b]}\left\{\frac{\zeta-i}{2i}-\frac{b}{8}\left(3+\frac{\zeta^2 i-i}{\zeta-i}\right)+\frac{b}{4}\right\}\right] \qquad (12)$$

$$\sigma_\eta-\sigma_\xi+2i\tau_{\xi\eta}=\gamma y\frac{(\bar{\zeta}+i)^2[(\zeta-i)^2-b]}{(\zeta-i)^2([(\bar{\zeta}+i)^2-b]}-\frac{2i(\bar{\zeta}+i)^2(\zeta-i)bc\gamma}{[(\bar{\zeta}+i)^2-b][(\zeta-i)^2-b]^2}$$
$$\times\left[\frac{3}{4i}b^2+\left\{\zeta(\zeta-i)[(\zeta-i)^2-b]-2b^2\right\}\left\{\frac{4+b}{8i(\zeta-i)^2}-\frac{b}{4(\zeta-i)^3}-\frac{4i(\zeta-i)^4}{3b}\right\}\right.$$
$$+\frac{b}{2+b}[2(\zeta^2+1)+b(\zeta^2+3)+b^2]\left\{\frac{2+b}{8(\zeta-i)}+\frac{4+b}{8i(\zeta-i)^2}-\frac{b}{8(\zeta-i)}\right.$$
$$\left.-\frac{b}{4i(\zeta-i)^4}\right\}+\frac{b}{2+b}[2(\zeta-i)^3+b(\zeta^3-3i\zeta-\zeta+7i)+b^2(\zeta+3i)]$$
$$\times\left\{-\frac{2+b}{8(\zeta-i)^2}-\frac{b}{4i(\zeta-i)^3}+\frac{3b}{8i(\zeta-i)^4}\right\}+\frac{[\bar{\zeta}(\bar{\zeta}+i)+b]}{(\bar{\zeta}+i)}[(\zeta-i)^2-1]$$

$$\times\left\{\frac{4+b}{8i(\zeta-i)^2}-\frac{b}{4(\zeta-i)^3}-\frac{3b}{4i(\zeta-i)^4}\right\}-\frac{2[\bar{\zeta}(\bar{\zeta}+i)+b]\,b}{(\bar{\zeta}+i)}\left\{\frac{4+b}{8i(\zeta-i)}\right.$$
$$\left.-\frac{b}{8(\zeta-i)^2}-\frac{b}{4i(\zeta-i)^3}\right\}-\left.\frac{b[\bar{\zeta}(\bar{\zeta}+i)+b]}{2(\bar{\zeta}+i)}\right] \tag{13}$$

Equations (12) and (13) are the stress field for the notched half-plane under the gravits, and the stress dlstribution along the surface, which is of practical interest, is expressed by

$$\sigma_{\xi})_{\eta=0}=\frac{cb\gamma}{\xi^2+1}-4cb\gamma\left[\frac{\xi^2-b-1}{(\xi^2-b-1)^2+4\xi^2}\left\{-\frac{1}{2}-\frac{b}{8}\left(3-\frac{\xi^2-1}{\xi^2+1}\right)+\frac{b}{4}\right\}\right.$$
$$\left.+\frac{\xi^2}{(\xi^2-b-1)^2+4\xi^2}\left\{1+\frac{b}{4}\,\frac{\xi^2-1}{\xi^2+1}\right\}\right] \tag{14}$$

and at the base where $\zeta=0$,

$$\sigma_{\xi})_{\zeta=0}=-cb\gamma\frac{1}{1+b} \tag{15}$$

4. Numerical Example

Equation (14) has been evaluated for $b=0.5$, and the surface stress σ_{ξ} is shown as in Table 1 and Fig. 2.

Table 1.

x/c	$\sigma_{\xi}/0.5\,c\gamma$
0.000	2.000
0.104	1.399
0.228	0.768
0.379	0.466
0.556	0.318
0.750	0.235
0.954	0.184
1.164	0.148
1.375	0.123
1.588	0.103
1.800	0.088

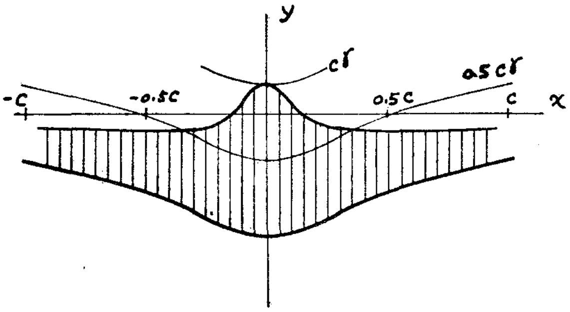

Fig. 2.

5. Remarks

The effect of the notch surface on the stress field created by the gravity, is strongly confined by the mapping function. The result mentioned above is, therefore, for a peculiar case when the notch shape is such that as given by the function (1), but it may be of practical interest on the view point of soil mechanics and geophysics.

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References

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