Set-theoretical Foundations in the Empiricist Pragmatism

Set-theoretical Foundations in the Empiricist Pragmatism

Yoshio Kinokuniya*

Abstract

Annexing the pragmatist dogma to the empiricist theory of sets in connection with the theory of a priori measure, we obtain some important renovations, particularly in the context with ordinal numbers.

1. Introduction

In the previous paper[1)] we posited the following pragmatist dogma: *a completely unfounded mere abstraction can give only a meaningless object.* Applying this dogma as a principle of induction, we have a very powerful device in the analytical logic (i.e., in the set-theoretical logic). The *empiricist pragmatism* is the logical analysis which uses this principle in the empiricist theory of sets (particularly in connection with the theory of a priori measure[2)]). We previously have obtained the following two conclusions in the empiricist pragmatism.

I (**Principle N A**). *If M is a practical*[1)] *set and*

$$(\forall X \subseteq M)(X \text{ is } \widetilde{m}\text{-measurable}^{**}. \Rightarrow . \widetilde{m}X = 0), \tag{1.1}$$

then it must be that M is $\widetilde{m}$-measurable and

$$\widetilde{m}M = 0 .$$

II (**Lemma E**). *If sets*

$$M_1 \subseteq M_2 \subseteq \cdots$$

all are $\widetilde{m}$-measurable and $M = \cup M_k$, then we have

$$\widetilde{m}M = \lim \ \widetilde{m}M_k .$$

If a description Δ_S which defines a collection S of elements in a given universe $\boldsymbol{U}$ implies that

$$(\forall a \in \boldsymbol{U})(a \in S . \vee . a \notin S),$$

then S is called a *descriptive collection* or an *aggregate.* In this paper an aggregate is assumed to be taken in a euclidean space of finite dimension $\boldsymbol{E}$.

To date, some examples of non-measurable sets (with respect to the Lebesgue measure) have been shown through constructions on ordinal numbers. However, in Principle N A, evev if we take 'an aggregate' instead of 'a practical set', it is

---

* 紀国谷 芳 雄

** $\widetilde{m}$-measure is the empiricist generalization of Lebesgue measure.

apparent that any part of M cannot be destined to be of positive $\tilde{m}$-measure. So, in the empiricist pragmatism, we may reasonably adopt the following assertion as an axiom.

Axiom NA. *If an aggregate M satisfies the condition* (1.1), *M is $\tilde{m}$-measurable and*

$$\tilde{m}M = 0.$$

If M is an aggregate and if $\mathfrak{M}$ is the collection of all $\tilde{m}$-measurable subaggregates contained in M, then, as readily seen, $\mathfrak{M}$ is an aggregate (of subaggregates). In this case, if

$$\sup_{X\in\mathfrak{M}} \tilde{m}X = \alpha$$

there exists an increasing sequence of $\tilde{m}$-measurable subaggregates of M

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M$$

such that $\lim \tilde{m}M_k = \alpha$. Then, if N is an $\tilde{m}$-measurable subaggregate contained in $M - \cup M_k$, it must be that

$$\tilde{m}N = 0.$$

So, by Lemma E and Axiom N A, we may reach the following conclusion.

Proposition 1.1. *Every aggregate M is $\tilde{m}$-measurable, under the convention that $\tilde{m}M = \infty$ is also allowed to be a case.*

According to Proposition 1.1 an aggregate is called a (*determinate*) *set* in the meaning that it is descriptive and $\tilde{m}$-measurable. Thus, the empiricist theory of sets may, in the empiricist pragmatism, be renovated in many sides, standing on the foundations above-stated.

2. Framed Increase

Being given a family (or a collection) of collections of points (in $\boldsymbol{E}$) $\mathfrak{A} = (A_\iota)(\iota \in I)$, if I is simply ordered, i.e.,

$$(\forall \iota, \kappa \in I)(\iota \neq \kappa . \Rightarrow : \iota < \kappa . \vee . \iota > \kappa)$$

and if

$$\iota < \kappa . \Rightarrow . A_\iota \subseteq A_\kappa : \ \& \ : A_\iota \subset A_\kappa . \Rightarrow . \iota < \kappa,$$

then $\mathfrak{A}$ is called a *framed increase* of collections (in $\boldsymbol{E}$). If, in addition, I and all A_ι are descriptive, $\mathfrak{A}$ is called a *descriptive increase.*

In case of a descriptive increase, the union

$$A = \cup_{\iota\in I} A_\iota$$

is evidently an aggregate, so that by Proposition 1.1 we have:

Proposition 2.1. *If $\mathfrak{A} = (A_\iota)(\iota \in I)$ is a descriptive increase and*

$$A = \cup_{\iota\in I} A_\iota,$$

then there is found a real number α *such that*

$$\widetilde{m}A=\alpha\,,$$

otherwise

$$\widetilde{m}A=\infty\,.$$

If we have

$$\cup A_{\iota_k}\neq A\,,$$

for every enumerable sequence $(\iota_k)\,(k=1,2,\cdots)$, then $\mathfrak{A}$ is said to be *of unfinishable type.* Proposition 2.1 may be reckoned to hold even when A is of unfinishable type. Now, if $\sup_{\iota\in I}\widetilde{m}A_\iota\equiv\beta<\infty$ (in Proposition 2.1) we have

$$\beta\leqslant\alpha\,.$$

In this case, if $\beta<\alpha$, we evidently have

$$(\forall\iota\in I)\Big(\widetilde{m}(A-A_\iota)\geqslant\alpha-\beta\Big),$$

so that

$$\lim\widetilde{m}\,(A-A_\iota)\geqslant\alpha-\beta\equiv\delta>0\,.$$

Then, there must be promised an atmosphere $(]A-\cup A_\iota[)$[3] for which it is destined that

$$\widetilde{m}(]A-\cup A_\iota[)\geqslant\delta\,.$$

However, if we assert such a peculiar state to be involved in the simple definition '$A-\cup A_\iota$' on our a priori ground, it may not give other than an incompitent assignment for $\widetilde{m}A$. Hence it may be no other than a mere abstract imagination and so it may be taken as meaningless, in the empiricist pragmatism. Thus we may conclude:

Proposition 2.2. *If* $\mathfrak{A}=(A_\iota)(\iota\in I)$ *is a descriptive increase and*

$$A=\cup_{\iota\in I}A_\iota\,,$$

then

$$\widetilde{m}A=\sup_{\iota\in I}\widetilde{m}A_\iota\,.$$

3. On the Ordinal Numbers

If a collection of ordinal type $\bar{A}$ is not a descriptive collection, A is expelled, in our view, from the concept of a determinate collection. So, by grace of Proposition 1.1, any aggregate of ordinal numbers is considered $\widetilde{m}$-measurable, hence as a set. In the empiricist theory the method of transfinite induction is not generally admitted. However, when using the whole arrangement of the ordinal numbers, this method may naturally be recognized to correspond to the ordinal construction of a collection. So, we may take this method as a fictive one restricted within the collections of ordinal numbers. Then, in virtue of Proposition 2. 2, we may readily prove the following lemma.

Lemma 3. 1. *Every well-ordered set must be of $\tilde{\mathfrak{m}}$-measure zero.*
So, we directly have:

Proposition 3. 1. *In empiricist pragmatism, no ordinal number can be admitted to correspond to the real continuum.*

To conclude Proposition 3. 1 the following will also give a demonstration. From the interval [0, 1] let us take a subset L_1 of $\tilde{\mathfrak{m}}$-measure zero, then another subset L_2 of $\tilde{\mathfrak{m}}$-measure zero such as

$$L_1 \subset L_2 .$$

Continuing this process, we may obtain a framed increase of subsets (of [0, 1]) of $\tilde{\mathfrak{m}}$-measure zero $(L_\iota)(\iota \in I)$. Apart from the selection of such an increase, we may observe it only as the existence of such an increase. I may be assumed to correspond to an ordinal (number). Therefore, assuming I as the supremum of such ordinals, we come across a contradiction in that we may then, by virtue of Proposition 2. 2, conclude that

$$\tilde{\mathfrak{m}} \cup L_\iota = 0 .$$

4. Extension

By $\mathfrak{P}(A)$ indicating that a set A has the property $\mathfrak{P}$, if for any set A (in $\boldsymbol{E}$) it is destined that

$$\mathfrak{P}(A). \vee . \sim \mathfrak{P}(A),$$

$\mathfrak{P}$ is said to be *descriptive*. For a descriptive property $\mathfrak{P}$, if it is always destined that

$$A \subset B \subseteq E. \ \& \ . \mathfrak{P}(B) : \Rightarrow . \mathfrak{P}(A), \tag{4. 1}$$

$\mathfrak{P}$ is said to be *regressive* and then, in the relation (4. 1), B is called an *extension* of A in respect to $\mathfrak{P}$. If a descriptive increase of sets $\mathfrak{A} = (A_\iota)(\iota \in I)$ satisfies the condition

$$(\forall \iota \in I)\big(\mathfrak{P}(A_\iota)\big)$$

for a regressive property $\mathfrak{P}$, $\mathfrak{A}$ is called an *extension increase* (in respect to $\mathfrak{P}$).

If a collection A is considered to have a certain descriptive property, A must be descriptive, because a non-descriptive collection is thought, in the empiricist pragmatism, to be meaningless and so to be expelled from our course. Hence, in producing any collection A on condition that $\vdash \mathfrak{P}(A)$*, we may always expect A to be descriptive.

* This renders '$\mathfrak{P}(A)$ *is true*'.

If there exists an extension increase $\mathfrak{A}=(A_\iota)(\iota\in I)$ in respect to $\mathfrak{P}$ and if for any set B (in $\boldsymbol{E}$) it is ascertained that

$$\cup A_\iota \equiv A \subset B . \Rightarrow . \sim \mathfrak{P}(B),$$

$\mathfrak{A}$ is said to be *maximal* and then A is called an *extension limit* in respect to $\mathfrak{P}$. If $A \neq \boldsymbol{E}$, $\mathfrak{P}$ is said to be *maximizable* by $\mathfrak{A}$.

Now, in developing any extension increase $\mathfrak{A}$, we shall necessarily be imposed the following two matters to examine:

(i) *Extensibility*:

$$(\exists \vee \nexists A \subseteq \boldsymbol{E})\big(B \subset A . \ \& \ . \vdash \mathfrak{P}(A)\big),$$

i.e., it is decidable either $\mathfrak{P}$ is extensible beyond a given set B or not;

(ii) *Maximizability*: it is decidable either $\mathfrak{P}$ is maximizable or not.

In view of the above-stated preliminary definitions and investigations, it is readily affirmed that, excepting the problem of real practicality, both of (i) and (ii) can be expected as decidable. Hence, if a concrete property $\mathfrak{P}$ is really proved to be descriptive and to be maximizable, we may certainly be promised a (non-trivial) maximal extension increase. In this case, we say that a *trans-induction* is promised or a *trans-inductive mode* is established for the extension in resrect to $\mathfrak{P}$. In our view of the empiricist pragmatism, it will be specially fair that, throughout the process of trans-induction, all intermediate extensions A_ι and the extension limit A can always be expected as $\widetilde{m}$-measurable.

5. Unexhaustible Null and Metamorphosis

If, for a set $A \subset B$ (in $\boldsymbol{E}$), there exists a set A' such that $A \subset A' \subset B$, $\vdash \mathfrak{P}(A')$ and $\widetilde{m}(A'-A)=0$, then $\mathfrak{P}$ is said to have *unexhaustible null* above A in B. Then, the following theorem is readily proved.

Proposition 5. 1. *If there is a set B of finite $\widetilde{m}$-measure such that*

$$\sim \vdash \mathfrak{P}(B),$$

and if a non-void extension increase $\mathfrak{A}\equiv(A_\iota)(\iota\in I)$ in respect to $\mathfrak{P}$ is such restricted as

$$(\forall \iota)(A_\iota \subset B)$$

and if $\mathfrak{P}$ is unmaximizable by $\mathfrak{A}$, then $\mathfrak{P}$ has unexhaustible null above the limit $A=\cup A_\iota$.

On an extension limit A in respect to a descriptive property $\mathfrak{P}$, following two cases are distinguished:

(i) $\vdash \mathfrak{P}(A)$, then $\mathfrak{P}$ is said to be *closed* in the framed increase $\mathfrak{A}$ of which A is the limit;

(ii) $\sim \vdash \mathfrak{P}(A)$, then it is said that $\mathfrak{P}$ has a *metamorphosis* by $\mathfrak{A}$ or $\mathfrak{A}$ is

$\mathfrak{P}$-metamorphic.

To the question if there always exists a $\mathfrak{P}$-metamorphic extension increase in respect to any property $\mathfrak{P}$ which has unexhaustible null in a certain set of finite $\widetilde{m}$-measure, the answer is negative. For instance, as for the property $\mathfrak{P}$ defined by $\mathfrak{P}(A)\equiv(\widetilde{m}A\leqslant a<\infty)$, if $a<\widetilde{m}B<\infty$, apparently $\mathfrak{P}$ has unexhaustible null in B, but no metamorphosis is found by any extension increase in respect to $\mathfrak{P}$ (in virtue of Proposition 2.2).

Taking $\boldsymbol{E}$ as the set of all real numbers, if $\mathfrak{P}(A)\equiv(\Pr(x\in A)=0)$, it is found that $\mathfrak{P}$ is unmaximizable and has a metamorphosis by the extension increase (A_k) $(k=1, 2, \cdots)$ $(A_k=(-k, k))$, because then $\boldsymbol{E}=\cup A_k$ and $\sim\vdash\mathfrak{P}(\boldsymbol{E})$ while $(\forall k=1, 2, \cdots)(\vdash\mathfrak{P}(A_k))$.

Addendum. If investigations are to be made on a general topological space or on a non-metric space, the problems must accordingly be complicated. In these cases, if sets of real numbers or of points of a euclidean space are made to correspond, by a certain operation, to the aggregates in the original space, the analysis will then be clarified in that all of the figure sets on this correspondence can be expected as $\widetilde{m}$-measurable. When no such means of correspondence is found, it shall be noted that, with no concrete practical instances to be involved, mere abstract processes are possibly disposed to fall into meaninglessness on the empiricist pragmatism.

Mathematical Seminar of the Muroran Inst. Tech., Hokkaido

(Received May 8, 1971)

References

1) Kinokuniya, Y.: Mem. Muroran Inst. Tech. **6** (4) 329–337 (1970).
2) Id. **6** (2) 491–498 (1968).
3) Id. Mem. Muroran Inst. Eng. **2** (3) 783–785 (1957).