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# Some Formulas Derived from Finite Integration

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## Abstract

The finite sine series can sometimes be represented by a function of closed form. And it may conveniently work on the finite difference equation.

Some finite cosine series which can take functions of closed form, are formulated in this paper.

## 1. Mother function

Let take mother function  $F_{\alpha x}$  :

$$\cosh \left\{ x \left( \alpha + \frac{m\pi}{n} i \right) \right\} = F_{\alpha x},$$

where

$$m = 1, 2, 3, \dots, n$$

$i =$  imaginary unit,

then we have

$$\Delta^2 F_{\alpha x-1} = \cosh \left\{ x \left( \alpha + \frac{m\pi}{n} i \right) \right\} \cdot \left\{ 2 \cosh \left( \alpha + \frac{m\pi}{n} i \right) - 2 \right\},$$

in which

$$\Delta^2 F_{x-1} = F_{x+1} - 2F_x + F_{x-1},$$

when  $\tilde{F}_{\alpha x}$  is expressed by

$$\tilde{F}_{\alpha x} = \frac{\cosh \left\{ x \left( \alpha + \frac{m\pi}{n} i \right) \right\}}{2 \cosh \left( \alpha + \frac{m\pi}{n} i \right) - 2},$$

which is written in

$$\Delta^2 \tilde{F}_{\alpha x-1} = F_{\alpha x}.$$

So the finite integration of the above becomes

$$\int F_{\alpha x} \Delta x = \Delta \tilde{F}_{\alpha x-1},$$

or

$$\int \tilde{F}_{\alpha x} \Delta x = \frac{\cosh \left\{ x \left( \alpha + \frac{m\pi}{n} i \right) \right\} - \cosh \left\{ (x-1) \left( \alpha + \frac{m\pi}{n} i \right) \right\}}{2 \cosh \left( \alpha + \frac{m\pi}{n} i \right) - 2}.$$

Considering that

$$\sum_{x=1}^{n-1} \mathbf{F}_{\alpha \cdot x} = \int_1^n \mathbf{F}_{\alpha \cdot x} dx,$$

we have at once

$$\sum_{x=1}^{n-1} \mathbf{F}_{\alpha \cdot x} = -\frac{1}{2} \left\{ \cosh(n\alpha + m\pi i) + 1 \right\} + \frac{\sinh(n\alpha + m\pi i) \sinh\left(\alpha + \frac{m\pi}{n} i\right)}{2 \left\{ \cosh\left(\alpha + \frac{m\pi}{n} i\right) - 1 \right\}}.$$

The imaginary part of the above yields

$$\mathbf{I} \left\{ \sum_{x=1}^{n-1} \mathbf{F}_{\alpha \cdot x} \right\} = -\frac{(-1)^m \sinh n\alpha \sin \frac{m\pi}{n}}{2 \left( \cosh \alpha - \cos \frac{m\pi}{n} \right)},$$

from which we have

$$\sum_{x=1}^{n-1} \sinh x\alpha \cdot \sin \frac{m\pi}{n} x = -\frac{(-1)^m \sinh n\alpha \cdot \sin \frac{m\pi}{n}}{2 \left( \cosh \alpha - \cos \frac{m\pi}{n} \right)}.$$

The inversion theorem leads to the following formula ;

$$\sum_{m=1}^n \frac{(-1)^m \sin \frac{m\pi}{n} \cdot \sin \frac{m\pi}{n} x}{2 \left( \cosh \alpha - \cos \frac{m\pi}{n} \right)} = -\frac{\sinh x \cdot \alpha}{\sinh n\alpha}. \quad (1)$$

In like manner, the real part can be written as follows ;

$$\mathbf{R}_e \left\{ \sum_{x=1}^{n-1} \mathbf{F}_{\alpha \cdot x} \right\} = -\frac{1}{2} \left\{ (-1)^m \cosh n\alpha + 1 \right\} + \frac{(-1)^m \sinh n\alpha \cdot \sinh \alpha}{2 \left( \cosh \alpha - \cos \frac{m\pi}{n} \right)},$$

The inversion formula :

$$f(x) = \frac{2}{n} \sum_{m=0}^n \mathbf{R}_m [f(x)] \cos \frac{m\pi}{n} x,$$

where

$$\mathbf{R}_0 [f(x)] = \frac{1}{2} \left\{ \mathbf{C}_0 [f(x)] + \frac{1}{2} f(n) + \frac{1}{2} f(0) \right\}$$

$$\mathbf{R}_m [f(x)] = \mathbf{C}_m [f(x)] + \frac{1}{2} (-1)^m f(n) + \frac{1}{2} f(0)$$

$$\mathbf{R}_n [f(x)] = \frac{1}{2} \left\{ \mathbf{C}_n [f(x)] + \frac{1}{2} (-1)^n f(n) + \frac{1}{2} f(0) \right\},$$

leads to the following results ;

$$\frac{2}{n} \sum_{m=1}^{n-1} \frac{(-1)^m \cos \frac{m\pi}{n} x}{2 \left( \cosh \alpha - \cos \frac{m\pi}{n} \right)} = \frac{\cosh x\alpha}{\sinh \alpha \cdot \sinh n\alpha} - \frac{1}{2n(\cosh \alpha - 1)} - \frac{(-1)^{n-x}}{2n(\cosh \alpha + 1)}, \tag{2}$$

which is differentiated with respect to  $\alpha$ , to be as follows ;

$$\begin{aligned} \frac{2}{n} \sum_{m=1}^{n-1} \frac{(-1)^m \cos \frac{m\pi}{n} x \cdot \sinh \alpha}{2 \left( \cosh \alpha - \cos \frac{m\pi}{n} \right)^2} &= \frac{x \sinh x\alpha}{\sinh \alpha \cdot \sinh n\alpha} \\ &- \frac{n \cosh x\alpha \cdot \cosh n\alpha}{\sinh \alpha \cdot \sinh^2 n\alpha} - \frac{\cosh x\alpha \cdot \cosh \alpha}{\sinh^2 \alpha \cdot \sinh n\alpha} \\ &- \frac{\sinh \alpha}{2n(\cosh \alpha - 1)^2} - \frac{(-1)^{n-x} \sinh \alpha}{2n(\cosh \alpha + 1)^2}. \end{aligned} \tag{3}$$

### 2. Formulas of Polynomial Function

Letting  $\alpha$  be zero in the Eqs. (2) and (3), we have

$$\begin{aligned} \frac{4}{n} \sum_{m=1}^{n-1} \frac{(-1)^m \cos \frac{m\pi}{n} x}{4 \left( 1 - \cos \frac{m\pi}{n} \right)^2} &= \frac{x^2}{12n} (2n^2 + 2 - x^2) \\ &- \frac{1}{180n} (7n^2 + 17) \cdot (n^2 - 1) - \frac{1}{8n} \left\{ 1 + (-1)^{n-x} \right\}, \end{aligned} \tag{4}$$

and

$$\frac{2}{n} \sum_{m=1}^{n-1} \frac{(-1)^m \cos \frac{m\pi}{n} x}{2 \left( 1 - \cos \frac{m\pi}{n} \right)} = \frac{1}{2n} \left\{ x^2 - \frac{n^2 - 1}{3} - \frac{1 + (-1)^{n-x}}{2} \right\}. \tag{5}$$

Putting  $n-x$  for  $x$  into the Eq. (5), we have

$$\frac{2}{n} \sum_{m=1}^{n-1} \frac{\cos \frac{m\pi}{n} (n-x)}{2 \left( 1 - \cos \frac{m\pi}{n} \right)} = \frac{1}{2n} \left\{ x^2 - \frac{n^2 - 1}{3} - \frac{1 + (-1)^{n-x}}{2} \right\}, \tag{6}$$

which becomes upon the action of  $A_{x-1}^2$

$$- \frac{2}{n} \sum_{m=1}^{n-1} \cos \frac{m\pi}{n} (n-x) = \frac{1}{n} \left\{ 1 + (-1)^{n-x} \right\}. \tag{7}$$

By the well known finite integration formulr, we obtain

$$\sum_{m=1}^{n-1} \cos \frac{m\pi}{n} (n-x) = \frac{\sin \left\{ \frac{\pi}{n} (n-x) \left( m - \frac{1}{2} \right) \right\}}{2 \sin \frac{\pi}{2n} (n-x)} \Bigg|_1^n = -\frac{1}{2} \left\{ 1 + (-1)^{n-x} \right\},$$

which is same with the Eq. (7).

### 3. Advanced Formulas

Denoting that  $D_m = 2 \left( 1 - \cos \frac{m\pi}{n} \right)$  and  $\cosh \alpha = 1 + \xi + \eta i$ , the equation (2) can be written as follows ;

$$\begin{aligned} \frac{2}{n} \sum_{m=1}^{n-1} \frac{(-1)^m (D_m + \xi + \eta i)}{(D_m + \xi)^2 + \eta^2} \cos \frac{m\pi}{n} x &= \frac{\cosh \alpha x \cdot \sinh \bar{\alpha} \cdot \sinh n\bar{\alpha}}{\sinh \alpha \cdot \sinh \bar{\alpha} \cdot \sinh n\alpha \cdot \sinh n\bar{\alpha}} - \frac{1}{n(\xi^2 + \eta^2)} - \frac{(-1)^{n-x} (\xi - \eta i + 4)}{n \{ (\xi + 4)^2 + \eta^2 \}}. \end{aligned}$$

The real part of the above, yields

$$\begin{aligned} \frac{2}{n} \sum_{m=1}^{n-1} \frac{(-1)^m (D_m + \xi)}{(D_m + \xi)^2 + \eta^2} \cos \frac{m\pi}{n} x &= \frac{1}{(\cosh 2\beta - \cos 2\gamma) (\cosh 2n\beta - \cos 2n\gamma)} \\ &\times \left[ \cosh \beta(n+x+1) \cos \gamma(n+1-x) - \cosh \beta(n-x-1) \cos \gamma(n-1+x) \right. \\ &+ \cosh \beta(n+1-x) \cos \gamma(n+1+x) - \cosh \beta(n-1+x) \cdot \cos \gamma(n-1-x) \left. \right] \\ &- \frac{\xi}{n(\xi^2 + \eta^2)} - \frac{(-1)^{n-x} (\xi + 4)}{n \{ (\xi + 4)^2 + \eta^2 \}}, \end{aligned} \tag{8}$$

in which

$$\alpha = \beta + \gamma i,$$

and the imaginary part,

$$\begin{aligned} \frac{2}{n} \sum_{m=1}^{n-1} \frac{(-1)^m \cos \frac{m\pi}{n} x}{(D_m + \xi)^2 + \eta^2} &= \frac{1}{\eta (\cosh 2\beta - \cos 2\gamma) \cdot (\cosh 2n\beta - \cos 2n\gamma)} \\ &\times \left[ \sinh \beta(n+1+x) \sin \gamma(n+1-x) - \sinh \beta(n-1-x) \sin \gamma(n-1+x) \right. \\ &+ \sinh \beta(n+1-x) \sin \gamma(n+1+x) - \sinh \beta(n-1+x) \sin \gamma(n-1-x) \left. \right] \\ &+ \frac{1}{n(\xi^2 + \eta^2)} + \frac{(-1)^{n-x}}{n \{ (\xi + 4)^2 + \eta^2 \}}. \end{aligned} \tag{9}$$

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