Totally Ordered Linear Space Structures and Separation Theorem in Real Linear Topological Spaces

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Abstract

As a sequel to 21)*, this time in a real linear topological space, the author deals with the Hahn-Banach separation theorem** (i. e., so-called Mazur's theorem) and the related problems from the view-point of the totally ordered linear space structures*** of the space.

Introduction. In the preceding note 21)*, in a real linear space (excluding the topological consideration) we have dealt with the geometric form of the Hahn-Banach theorem and the Krein's extension theorem in some detail†. On these subjects, now let the space be equipped with a linear topology (occasionally, locally convex), and let the closed hyperplanes and the continuous linear forms thereof be made mention. Then, still more, by copying 21), there are derived the corresponding versions from a general view via our new (for the author) means. For caution's sake, these resulting versions seem to be somewhat mentionable.

The first part of the matter is concerned with the separation theorem of Mazur type††, and the remainder is so with the extension theorem of Krein-Rutman type†††.

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Preliminaries. Let E be a real linear space with some non-zero vectors. For convenience' sake, notations and terminology employed in 21) are available as they are, except the symbol $\mathscr{T}$ and the Def. 2. $\mathscr{T}$ is merely substituted by $\mathscr{R}$, namely e. g., $(E, \mathscr{R})$ signifies the totally ordered linear space structure of E with respect to $\mathscr{R}$. For the latter, see below.

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* That was written under the direction of the Editors of Hokkaido Math. Jour., and was dedicated to Prof. Y. Katsurada (Hokkaido Univ.) on her 60th birthday.

** By this he means [18), chap. II, § 5, th. 1].

*** For this thought, the author was benefited by D. M. Topping [16), p. 418].

† For the former subject matter, compare 21) with e. g., [9), § 8], [11), p. 460 (Notes and Remarks)] and [12), § 8, Th. 3]. For the latter, compare the same with [13), Th. 3.3], [18), chap. II, § 3, prop. 1] and [19), (V, 5.4), Cor. 1].

†† Cf. 6), 7), 8) and [9), § 8]. Compare the present Theorem 2 with [18), chap. II, § 5, exerc. 3].

††† Cf. e. g., [15), Th. 2.6.3] or [20), Th. XIII. 2.3]. The present Theorem 3 is subsequently compared with [19), (V, 5.4)].

Separation theorems. The said definition is modified as

DEFINITION 1. A system A in E is said to lie (resp., lie semi-positively, lie positively) on one side of a hyperplane $H=\{x\in E: f(x)=\alpha\}$ ($f\in E^*$ being non-zero, α fixed) if $\alpha\leqslant f(a)$ (resp., $\alpha\leqslant f(a)$ and not all be α, $\alpha<f(a)$) for each member a of A.

As a topological version of [21), Th. 1], we have

THEOREM 1. *Let E be a linear topological space and A a positively independent subset of E. A necessary and sufficient condition that A lies (resp., lies semi-positively, lies positively) on one side of a closed maximal subspace $N(f)$ of E is that there exists a t.o.l.s. $(E, \mathscr{R})$, with $A\subset(E, \mathscr{R})^+$, such that (i) holds (resp., (i) plus (ii) holds, (i) plus (iii) holds), where*

(i) *$(E, \mathscr{R})^+$ contains some non-void open subset O of E;*

(ii) *some $a_0\in A$ is an order unit of $(E, \mathscr{R})$;*

(iii) *each $a\in A$ is an order unit of $(E, \mathscr{R})$.*

PROOF. We work with the semi-positive case, and the remains are likewise obtained by [21), Lemmas 1, 2, 3 and 4]. (Necessity) Let $0\leqslant f(a)$ ($a\in A$) and not all be zero. Take $(E, \mathscr{R}_1)$ so that $A\subset(E, \mathscr{R}_1)^+$, then $(E, f(\mathscr{R}_1))$ proves to be a t.o.l.s. as required in view of the "closedness" of $N(f)$. (Sufficiency) Hypothesis implies $A\cup O\subset(E, \mathscr{R})^+$. Besides, not only $a_0\in A$, but also $x\in O$ all are the order units of $(E, \mathscr{R})$ since O is open for linear topology. These lead up to the conclusion.

EXAMPLES. Let the finite sequence space $\boldsymbol{R}^\infty$ be equipped with the local convexity by the usual inner product. Setting as $A=\{(\alpha_1, \alpha_2, \cdots): \alpha_t=0$ for almost all t, $\Sigma\alpha_t>0\}$, an example such that (iii) holds (i.e., the sufficient condition (strict case) of our [21), Th. 1] is met) but (i) fails is furnished. On the other hand, therein taking another $A=\{(0, \alpha_2, \alpha_3, \cdots): \alpha_t=0$ for almost all t, $\Sigma\alpha_t>0\}$, an example such that (i) holds (letting $x_0=0$, the sufficient condition of [18), chap. II, § 5, exerc. 3] is met) or (iii) holds but (i) plus (ii) fails is furnished. These are because of the fact that given positive reals ξ, ε, there are positive integer n and real d satisfying $nd<-\xi$ and $(nd^2)^{1/2}=\varepsilon$.

Now Theorem 1 is, in line with [21), Th. 2], also interpreted in terms of "*absorbing* (syn., *radial*)" by [21), Lemma 4]. Henceforth, we shall proceed from this point of view.

As a general form of the corresponding version of [21), Th. 3], there holds the next theorem. In this theorem, whenever we take into account the topological consideration for quotient space, we let it be equipped with the quotient topology.

THEOREM 2. *Let E be a linear topological space, M a linear subspace of E, and let A be a system in E such that the image $\varphi(A+x_0)$ is positively independent in E/M, where φ is the canonical map of E onto E/M. A ne-*

cessary and sufficient condition that A *lies* (*resp., lies semi-positively, lies positively*) *on one side of a closed hyperplane* H *in* E *with* $H \supset M - x_0$ *is that there exists a t.o.l.s.* $(E/M, \mathscr{R})$, *with* $\varphi(A+x_0) \subset (E/M, \mathscr{R})^+$, *such that* (*i*) *holds* (*resp.,* (*i*) *plus* (*ii*) *holds,* (*i*) *plus* (*iii*) *holds*), *where*

(i) $(E/M, \mathscr{R})^+$ *contains some non-void open subset of* E/M;

(ii) $(E/M, \mathscr{R})^+$ *is absorbing at some point of* $\varphi(A+x_0)$;

(iii) $(E/M, \mathscr{R})^+$ *is absorbing at each point of* $\varphi(A+x_0)$.

PROOF. We work with the case $x_0 \in E$ is equal to zero. The remains are readily verified from this by translation. Now, under the postulate $f(x) = F(x+M)$ $(x \in E,\ x+M \in E/M)$ the following assertions are equivalent:

1) in E, A lies (resp., lies semi-positively, lies positively) on one side of a closed maximal subspace $H = N(f)$ with $H \supset M$;

2) in linear topological quotient space E/M, $\varphi(A)$ lies (resp., lies semi-positively, lies positively) on one side of a closed maximal subspace $N(F)$. Indeed, "$f = F \circ \varphi$" part is clear. Besides, quotient topology for E/M is compatible with the linear structure of E/M, and $(N(f))'$ is open in E if and only if $(N(F))'$ is open in E/M. Therefore the above fact is true and which achieves the desired end by Theorem 1 via [21), Lemma 4].

REMARK 1. In particular, the case where $\varphi(A+x_0)$ is a convex subset of E/M not containing the origin (convex subset A of E not meeting $M-x_0$ is the case) satisfies the initial hypothesis of Theorem 2. Hence, therewith letting $\varphi(A+x_0)$ be open (A is open is the case since φ is open), a fortiori, the Hahn-Banach separation theorem follows.

REMARK 2. For the separation by a (closed) maximal subspace, we are dealing with (cf. [21), Rem. 1]) the positively independent systems in the space instead of the convex subsets not containing the origin. But, moreover, in doing with the convex subsets not radial at the origin for the same purpose, we can proceed by use of Theorem 2 (of course, if possible, alternatively, by taking its non-empty radial kernel).

By the way, we give here a variant of generalized Stiemke theorem.

COROLLARY. *Let* E *be a non-trivial locally convex space and* A *a non-empty finite system in* E. *A necessary and sufficient condition that* A *does not lie positively* (*resp., does not lie semi-positively, does not lie*) *on one side of any closed maximal subspace of* E *is that* $\varphi(A)$ *is positively dependent in* $E/\overline{\{0\}}$ (*resp., positively dependent therein with coefficients all not zero, positively dependent as in just before and further the linear span of* $\varphi(A)$ *is* $E/\overline{\{0\}}$), *where* φ *is the canonical map of* E *onto* $E/\overline{\{0\}}$.

PROOF. To prove the "only if" part of the first assertion, first let E be Hausdorff. Now, let $A = \{a_1, a_2, \cdots, a_n\}$ be positively independent, i.e., the convex hull $\mathrm{co}(A)$ does not contain the origin. While, as it is usually given, $\mathrm{co}(A)$ is compact and hence is closed. With this, take a convex symmetric open 0-neighbourhood U such that $U \cap \mathrm{co}(A) = \emptyset$. Then considering the subset $B = \bigcup \{U + a_i : i = 1, 2, \cdots, n\}$, it follows directly that $\mathrm{co}(B) \not\ni 0$. Hence by Theorem 1 via [21), Lemmas 1 and 4], a fortiori, A lies positively on one side of a closed maximal subspace of E. Now let E be non-Hausdorff. Whereas, Hausdorff space $E/\overline{\{0\}}$ associated with E is at least one dimensional and locally convex. Therefore the present assertion is valid from the fact above and Theorem 2. The converse is clear since $\overline{\{0\}} \subset H$ for any closed maximal subspace H of E. The remains of the proof are attained via these by reductio ad absurdum and by use of (further) quotient topology of $E/\overline{\{0\}}$ (cf. every finite dimensional subspace thereof is closed).

Extension theorem. We next deal with the extension theorem of Krein-Rutman type. To do this, we take the following.

DEFINITION 2. Let $(E, \mathscr{P})$ be a partially ordered linear space. The subset $\{x : 0 < x(\mathscr{P})\}$ of E is called the positive cone of E and is simply denoted by C. But, if necessary, some of them are given by the form $(E, \mathscr{R})^+$ as before.

DEFINITION 3. A partially ordered linear space which is simultaneously a linear topological space is called an ordered linear topological space. By the way, linear topology for E will be denoted by $\mathscr{O}$.

As a corresponding version of [21), Th. 4 (2)]†, there holds the following. This is logically equivalent with [19), (V, 5.4), Theorem (Bauer-Namioka)] excepting the trivial case when $f \in M^*$ is identically-zero, and so too is [21), Th. 4 (2)] with Cor. 1 ibid.

THEOREM 3. *Let E be an ordered linear topological space with positive cone C. Let M be a linear subspace of E and f a non-identically-zero linear form on M. A necessary and sufficient condition that f can be extended to a positive continuous linear form F on E is that there exists a t.o.l.s. $(E, \mathscr{R})$ with the following properties*:

(i) $A_f \cup C \subset (E, \mathscr{R})^+$, *where A_f stands for $\{x \in M : f(x) > 0\}$*;

(ii) $(E, \mathscr{R})^+$ *contains some $O \in \mathscr{O}$ which meets M.*

PROOF. (Necessity) Take $(E, \mathscr{R}_1)$ so that $A_f \cup C \subset (E, \mathscr{R}_1)^+$ by [21), Th. 4 (1) and Lemma 1]. Then $(E, F(\mathscr{R}_1))$ turns out to be a t.o.l.s. as required by [21), Lemma 2] in view of the continuity of F. (Sufficiency) By property (ii), $(E, \mathscr{R})^+$ is absorbing at each point of $M \cap O$ since O is $\mathscr{O}$-open. Hence by [21, Th. 4 (2)], we get a positive linear form F on E extending f. Besides, it follows whereby that $O \subset \{x \in E : F(x) > 0\}$, and F is continuous.

SUPPLEMENT TO THEOREM 3. Our condition (i) plus (ii) above is, as a matter of fact, eqivalent to that $A_f \cup C \cup O$ ($O \in \mathscr{O}$ meets M) holds positive

† Here the author, adds the following. In the case of [21, Th. 4 (2)], on hypothesis, "positiveness of f" was over-imposed. Henceforth, this imposition is rescinded.

linear independence. In view of this, the logical equivalences aforesaid are directly ascertained too. Let us work with the former case, and the latter is similarly done from this. First, our condition is necessary. To see this, letting U be a convex symmetric 0-neighbourhood in hypothesis, take m_1 in M so that $f(m_1-m)>0$ whenever $m\in M\cap(U-(C\cup\{0\}))$. f is positive is immediate, and $A_f\cup C$ is positively independent. Suppose now that $A_f\cup C\cup(U+m_1)$ were no longer so, then there would exist both finite many respective vectors $a_r\in A_f$, $c_s\in C$, $u_t+m_1\in U+m_1$, and corresponding scalars $\alpha_r\geqslant 0$, $\beta_s\geqslant 0$, $\gamma_t>0$ with $\Sigma\gamma_t=1$, such that $p=\Sigma\alpha_r a_r+\Sigma\beta_s c_s+\Sigma\gamma_t(u_t+m_1)=0$. But then, by the above, this yields $f(p)>0$, a contradiction. For the converse, we may assume with ease that $O\in\mathcal{O}$ in hypothesis be convex. Moreover, it follows thereby clearly that $f(x)\geqslant 0$ $(x\in M\cap(O+(C\cup\{0\})))$. Hence by taking $m_2\in M\cap O$, $(-O)+m_2$ serves for a convex 0-neighbourhood as required since $f(m_2-m)\geqslant 0$ whenever $m\in M\cap((-O+m_2)-(C\cup\{0\}))$.

In this connection, the hypothesis in [19), (V, 5.4), Corollary 2 (Krein-Rutman)] explicitly implies that positively independent subset $A_f\cup C$ itself contains an $O\in\mathcal{O}$ which meets M. Hence this cited comes under a special case of Theorem 3 except only when $f\in M^*$ is identically-zero.

We close this note with focusing attention on an extreme case of Theorem 3 (one-sided specializations thereof are also easy). That is, the following is a just consequence of [21), either Th. 2 (semi-strict case) or Th. 4 (2)] (resp., Theorem 1 (semi-positive case) or Theorem 3).

COROLLARY. *Let E be a totally ordered linear topological space with positive cone C. Then there exists a non-zero positive (resp., non-zero positive continuous) linear form on E iff C is absorbing at some point (resp., C contains some non-void $O\in\mathcal{O}$).*

In other words, on putting $C=(E,\mathscr{R})^+$, above positive nature is characterized as $C=(E,f(\mathscr{R}))^+$ for some non-zero $f\in E^*$ (resp., non-zero $f\in E'$), where two non-zero linear forms are the case iff they are positive scalar multiples each of the other.

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References

1) E. STIEMKE: Über positive Lüsungen homogener linearer Gleichungen. Math. Ann. **76** (1915), 340–342.
2) M. B. CARVER: System of linear inequalities. Annals of Math. (2) **23** (1922), 212–220.
3) L. L. DINES: Definite linear dependence. Annals of Math. **27** (1925), 57–64.
4) S. BANACH: Sur les fonctionnelles linéaires II. Studia Math. **1** (1929), 223–239.
5) ——————: Théorie des opérations linéaires. Warszawa (1932), 27–29.
6) S. MAZUR: Über konvexe Mengen in linearen normierten Räumen. Studia Math. **4** (1933), 70–84.
7) M. EIDELHEIT: Zur Theorie der konvexen Mengen in linearen normierten Räumen. Studia Math. **6** (1936), 104–111.

8) S. KAKUTANI: Ein Beweis des Satzes von M. Eidelheit über konvexe Mengen. Proc. Imp. Acad. Tokyo, **13** (1937), 93–94.

9) V. L. KLEE, Jr.: Convex sets in linear spaces. Duke Math. J., **18** (1951), 443–466.

10) KY FAN: On system of linear inequalities. Annals of Math. Study, **38** (1956), 99–156, Part I.

11) N. DUNFORD and J. T. SCHWARTZ: Linear operators, Part I, chaps. II and V. Wiley (Interscience Publishers), Inc, New York, 1958.

12) D. A. RAIKOV: Vector spaces, chap. II. Moscow, 1962. (Russian). (Japanese transl. by Y. YOSHIZAKI: Tokyo Tosho Co., 1966.)

13) J. L. KELLEY, I. NAMIOKA and co-authors: Linear topological spaces, chaps. 1 and 4. D. Van Nostrand Co. Inc., Princeton, 1963.

14) A. WILANSKY: Functional analysis, chaps. 3 and 12. Blaisdell Publishing Co., New York, 1964.

15) R. E. EDWARDS: Functional analysis, chap. 2. Holt, Rinehart and Winston, Inc., New York, 1965.

16) D. M. TOPPING: Some homological pathology in vector lattices. Can. J. Math. **17** (1965), 411–428.

17) C. GOFFMAN and G. PEDRICK: First course in functional analysis, chap. 2. Prentice-Hall, Englewood Cliffs, N. J., 1965.

18) BOURBAKI, N: Espaces vectoriels topologiques, chap. I et II. Éléments de mathématique, livre V. Hermann, Paris, 1966.

19) H. H. SCHAEFER: Topological vector spaces, chaps. II and V. The Macmillan Co., New York, 1966.

20) B. Z. VULIKH: Introduction to the theory of partially ordered spaces, chap. XIII. Wolters-Noordhoff, Ltd., Groningen, The Netherlands, 1967.

21) K. IWATA: Totally ordered linear space structures and separation theorem. Hokkaido Math. Jour. (Sapporo, Japan), Vol. I, No. 2 (1972), 211–217.