Relativities between Sets and Measurements

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Abstract

This is a renovation report on relativities between sets and measurements. The usual outer measure plays an important role in relation to the a priori measure too. Constructions themselves of sets imply many specifications relative to the measurements of sets. The continuum problem, Lebesgue non-measurable sets and the notion of Baire category are specially discussed to gain some lights for the renovation of the foundations of analysis.

0. Introduction

Starting the study under the title of "the theory of a priori measure in connection with the empiricist theory of sets" and afterwards supplementing it by the pragmatist dogma[1], we have more and more been made convinced that there should be found tightly intimate relations between the notions of 'a set' and 'its measurement'. Recently we have arrived at some important synthetic view on the relative construction of the two notions. So we will in this paper state it in several steps of discussion.

Through several previous papers, we have obtained a course of axiomatization which can be sketched as follows.

A collection S of elements in a given universe U is called a *descriptive collection* or an *aggregate* if it is admitted as decidable that

$$(\forall p \in U)(p \in S . \vee . p \notin S).$$

If an aggregate A in a euclidean space is considered as determinate, it should be decidable that

$$(\exists . \vee . \nexists B \subset A)(\widetilde{m} B > 0)$$

$\widetilde{m}$ referring to the apriori measure. If all members of a family of aggregates are contained in a set B and $\widetilde{m} B > 0$, then the family is said to be *uniformly bounded*. A euclidean space is thought to be epistemologically and pragmatisly comprehensive if it is related to the a priori measure such that:

(i) it conforms to the *axiom of size-conformity*, i.e., if an aggregate is regarded as a limit of summation of some uniformly bounded increasing family of aggregates, then its remainder of summation must be measured by $\widetilde{m}$ as tending to zero;

(ii) the *principle of destination* is applicable, i.e., for any aggregate A, if no other value than a can be induced to be equal to $\widetilde{m} A$ on the assump-

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tion that A is $\widetilde{m}$-measurable, then A is $\widetilde{m}$-measurable and $\widetilde{m}A=a$;

(iii) the *a priori construction of $\widetilde{m}$-measurement* is applicable, i.e., for any $\widetilde{m}$-measurable aggregate A the formula

$$\widetilde{m}A=\nu(A)\cdot\mu \tag{0.1}$$

is effectible.

In (0.1) μ referes to the uniform point-measure called the *normal point-dimension*, and $\nu(A)$ is called the *inversion number* of A in respect to μ. $\nu(A)$ is considered as an exactification of the notion of 'power' (of a set), so that, by (0.1), it may be concluded that: for any two aggregates A, B in a euclidean space, if $\nu(A)\leqslant\nu(B)$, it must be that

$$\widetilde{m}A\leqslant\widetilde{m}B,$$

and if $\nu(A)/\nu(B)=\lambda$, then

$$\widetilde{m}A/\widetilde{m}B=\lambda.$$

The aggregates being considered under the above constructions are taken to be called (*determinate*) *sets.* In this view, any euclidean space is taken as an *a priori space*[2] reconstructed by the above constructions.

We have firstly attained the following fundamental theorem.

Theorem 0 (*Theorem of Measurement*). *Any set in a euclidean space is $\widetilde{m}$-measurable, if we admit its $\widetilde{m}$-measure value to be possible to be infinite.*

Subsequently, an important sight of construction has been obtained by the following theorem.

Theorem 1 (*Theorem of Limit*). *If an indexed class of sets (A_ι) $(\iota\in I)$ in a euclidean space is given such that I is simply ordered and*

$$\forall\iota,\ \kappa\in I:\ \iota\leqslant\kappa.\Rightarrow.A_\iota\subseteq A_\kappa,$$

and

$$A=\cup_{\iota\in I}A_\iota, \tag{0.2}$$

and if A is regarded as the limit of (A_ι), then it must be that

$$\widetilde{m}A=\sup\ \widetilde{m}A_\iota.$$

In regard to (0.2), we should thus distinguish two cases: (i) A is the limit of (A_ι); (ii) A is not the limit of (A_ι). However, it is notable that, in case of (ii), A can also be admitted as an aggregate (and hence as a set), because it is demonstrated as follows: Let E be the euclidean space in which A and A_ι $(\iota\in I)$ are containd. Then we have

$$(\forall\iota\in I)(\forall p\in E)(p\in A_\iota.\vee.p\notin A_\iota).$$

Hence

$$(\forall p\in E)(\exists.\vee.\nexists\iota\in I)(p\in A_\iota).$$

So then, defining as

$$\cup A_\iota = \left\{p\in E|(\exists\iota\in I)(p\in A_\iota)\right\},$$

we may have

$$(\forall p\in E)\left(p\in .\vee . \notin (\cup A_\iota)\right).$$

If (i) is the case we call A the *sum* of (A_ι) and (A_ι) *summable*, and if (ii) is the case we call A the *union* of (A_ι).

By grace of Theorem 1 we have previously concluded, in the empiricist pragmatism, that there exists no ordinal number to correspond to the continuum[3]. In this paper, we refer to this subject again in Sect. 2.

Let Q be the set of all rational numbers and

$$Q_x \equiv \{z|z=x+y,\ y\in Q\}$$

and V be a set of real numbers such that

$$\forall x,\ y\in V:\ x\neq y. \Rightarrow .Q_x\cap Q_y=\emptyset$$

and

$$\cup_{x\in V}Q_x=(-\infty,\ \infty).$$

Then V is a *Vitali set*. If a Vitali set V_A is contained in a set A, then V_A is called a *Vitali set in A*. It is well-known, in the classical analysis, that no Vitali set is Lebesgue measurable. However, in our present view, any Vitali set is possibly thought to be a (determinate) set (, therefore $\widetilde{m}$-measurable, by Theorem 0). The reasoning for this assertion is shown in Sect. 2.

Let $U(p,\ \rho)$ be a set (called a *closed ball* (*set*)) in a euclidean space defined as

$$U(p,\ \rho)\equiv\{q|\ \ |q-p|\leqslant\rho\}$$

where $|q-p|$ denotes the distance between the points q and p, and let $d_A(p)$ be defined by

$$d_A(p)=\underline{\lim}\frac{\widetilde{m}A\cap U(p,\ \rho)}{\widetilde{m}U(p,\ \rho)}. \tag{0.3}$$

Then $d_A(p)$ is called the *lower* (*normal*) *density* of a set A at the point p. In this context, one theorem is obtained in comparison with the density theorem*) of Lebesgue, and gives us an interesting example of a set which may be determinate (therefore $\widetilde{m}$-measurable) but not Lebesgue measurable. The proof of the theorem is attained by making a little modification of a proof of the theorem of Lebesgue, that shall be shown in Sect. 3. Incidentally, it will be shown that the usual outer measure (of Lebesgue) plays,

*) Its content is shown in Sect. 3.

in this connection, an important role relative to the a priori measure, too.

In Sect. 5, a counter example of a set is shown to break the distinctiveness of the notion of Baire category.

1. Unfinishing Indication

When a set is taken as a total aggregate of indices, it is called an *indication*. For a simply ordered indication I, denoting as

$$I_{(\kappa)}=\cup_{\iota\leqslant\kappa}\{\iota\} \text{ and } I'_{(\kappa)}=\cup_{\kappa<\iota}\{\iota\},$$

if for evrey intermediate $\kappa\in I$*) it is observed that

$$\nu(I_{(\kappa)})/\nu(I'_{(\kappa)})=0, \tag{1.1}$$

then I is said to be of *unfinishing type* or *unfinishing*.

For an indexed disjoint class of sets $(E_\iota)(\iota\in I)$ (I: simply ordered), if there is a set E such that

$$(\forall p\in E)(\exists\iota\in I)(p\in E_\iota) \text{ and } (\forall\iota\in I)(p\in E_\iota . \Rightarrow . p\in E),$$

(E_ι) is called a *partition* or an *I-partition* of E. For an I-partition (E_ι) denoting as

$$E_{(\kappa)}=\cup_{\iota\leqslant\kappa}E_\iota,$$

if the family $(E_{(\kappa)})(\kappa\in I)$ is summable, we call (E_ι) *summable*.

If $(E_\iota)(\iota\in I)$ is an I-partition of E and if it is destined that

$$\forall\iota,\ \kappa\in I:\ \widetilde{m}E_\iota=\widetilde{m}E_\kappa,$$

(E_ι) is said to be *size-preserving*. In this case, in accordance with (0.1) we may express it as

$$\forall\iota\in I:\ \widetilde{m}E_\iota=\nu\cdot\mu \tag{1.2}$$

μ being the normal point-dimension and $\nu(E_\iota)=\nu$ for all $\iota\in I$. Then, if

$$E_{(\kappa)}=\cup_{\iota\leqslant\kappa}E_\iota \text{ and } E'_{(\kappa)}=\cup_{\kappa<\iota}E_\iota,$$

we may define $\nu(I_{(\kappa)})$ and $\nu(I'_{(\kappa)})$ by the relations

$$\widetilde{m}E_{(\kappa)}=\nu(I_{(\kappa)})\cdot\mu \text{ and } \widetilde{m}E'_{(\kappa)}=\nu(I'_{(\kappa)})\cdot\mu. \tag{1.3}$$

In this case, to emphasize the relation (1.2), we call it a *size-preserving I-partition of E.*

If I is unfinishing, then about $\nu(I_{(\kappa)})$ and $\nu(I'_{(\kappa)})$ defined by (1.3) the relation (1.1) holds. In this case, if

$$0<\widetilde{m}E<\infty$$

we have

*) I.e., $\kappa\neq\inf, \sup\ \iota\ (\iota\in I)$.

$$\frac{\widetilde{m}E_{(\kappa)}}{\widetilde{m}E}=\frac{\nu(I_{(\kappa)})\,\mu}{\nu(I)\,\mu}=\frac{\nu(I_{(\kappa)})}{\nu(I)}\leqslant\frac{\nu(I_{(\kappa)})}{\nu(I'_{(\kappa)})}\,.$$

As the right-most term vanishes by (1.1), it must be that

$$\forall\kappa\in I:\ \widetilde{m}E_{(\kappa)}=0\,. \tag{1.4}$$

From our standpoint, (1.4) is contradictory, because then $\lim\ \widetilde{m}E_{(\kappa)}=\widetilde{m}E>0$ by Theorem 1, whereas $\lim\ \widetilde{m}E_{(\kappa)}=0$ by (1.4). Thus we conclude that:

Theorem 2. *If I is a simply ordered aggregate of unfinishing type, then for any set E such that*

$$0<\widetilde{m}E<\infty\,, \tag{1.5}$$

there can exist no size-preserving I-partition of E to be summable.

The contradictory relation (1.4) may, at the first glance, give us the suggestion that there possibly is an unvanishing atmosphere[4] in the process $\lim\,(E-E_{(\kappa)})$. In effect, if we take, instead of $\widetilde{m}$, some other measure constructed on a special foundation (e.g., the probability measure of homogeneous occurrence of points), the assertion of Theorem 2 may possibly be related to the atmosphere at infinity.

Incidentally, if our work is succeeded by the integral calculus, a non-summable partition of a set may sometimes be reinstated as meaningful. If (E_k) $(k=1, 2, \cdots)$ is a size-preserving partition of a set E which satisfies (1.5) and if a function $f(x)$ is assigned its values by

$$f(x)=(1-\varepsilon_k)\qquad \text{for } x\in E_k\ (k=1, 2, \cdots)$$

and

$$\lim\ \varepsilon_k=0\,,$$

then, for any positive number ε, we may have

$$1-\varepsilon<f(x)<1+\varepsilon \tag{1.6}$$

almost everywhere, because there is a finite integer N such that (1.6) may hold whenever $x\in E_k$ and $k>N$, whereof, if $E_{(N)}=\cup_{k=1}^{N}E_k$, we may, in a similar way to the case of (1.4), have

$$\widetilde{m}E_{(N)}/\widetilde{m}E=0\,.$$

In addition, it is notable that we may then have

$$\int_E f(x)\,dx=\widetilde{m}E\,.$$

2. Vitali Set and the Continuum

Given a set A and a simply ordered indication I, assume that for each $\iota\in I$ there is a mapping φ_ι such that $\varphi_\iota(A)=A_\iota$ and that

$$\iota \neq \kappa . \Rightarrow . A_\iota \cap A_\kappa = \emptyset .$$

Then, defining

$$E = \cup A_\iota ,$$

if (A_ι) is a size-preserving I-partition of E and

$$0 < \widetilde{m} E < \infty ,$$

according to Theorem 2, I cannot be of unfinishing type. However, if we define as

$$E_x = \left\{ x_\iota | \iota \in I, \; x_\iota \equiv \varphi_\iota(x) \right\} ,$$

we may have

$$E = \cup_{x \in A} E_x$$

and this relation may not always be denied even when I is unfinishing.

Now, let $A = [-1, 1]$, V_A be a Vitali set in A and Q_A be the set of all rational numbers contained in A and let

$$A_x = \{ y | y - x \in Q_A \}$$

and

$$E = \cup_{x \in V_A} A_x . \tag{2.1}$$

Then it is obvious that

$$0 < \widetilde{m} E < \infty .$$

In this case, if we define as

$$V_y = \left\{ x \in E | (\exists z \in V_A)(x = z + y) \right\}$$

we may have

$$E = \cup_{y \in Q_A} V_y . \tag{2.2}$$

However, since Q_A is an enumerable infinite set and hence, as easily seen, is a set of unfinishing type, and since $(V_y)(y \in Q_A)$ is apparently sizepreserving Q_A-partition of E, by Theorem 2 (2.2) must be meaningless as a summation formula.

If we denote by Q the set of all rational numbers, by R the set of all real numbers and define Q_x by

$$Q_x = \{ z | z = x + y, \; y \in Q \} ,$$

then we have

$$R = \cup_{x \in R} Q_x$$

to be true. In this context, a Vitali set V_A can be so defined that (Q_x) $(x \in V_A)$ may be a minimal subclass of (Q_x) to satisfy the condition

$$R = \cup_{x \in V_A} Q_x .$$

Then the conception of V_A as a collection may be thought to be consistent in the meaning that V_A is an indication such that $(Q_x)(x \in V_A)$ may fill up R with no overlapping. Such an operative meaning of "*filling up* R" may not be so clearly found in the collection along Q_A, because Q_A is firstly forced its essential property of enumerability which now turns out to be rather independent of the naive meaning of the collection of (2.2). In effect, since the enumerable infiniteness of Q_A implies the unfinishingness of Q_A, the formula (2.2) is, in our view, concluded to give no summation formula.

In the classical analysis, the set V_A has been decided to be Lebesgue non-measurable because of the size-preserving repartition formula (2.2). In our course, though the formula (2.2) is denied by Theorem 2, we may find no reason to reject the set V_A itself as inconsistent. Incidentally, if V^A is admitted to be a (determinate) set, it seems no difficult to demonstrate that if A is an interval of finite length

$$\widetilde{m} V_A = 0 .$$

For all above-stated, if V_A is taken as a well-ordered aggregate to correspond to some regular ordinal, (2.1) too turns to be inconsistent as a summation, because any regular ordinal is apparently of unfinishing type. Moreover, similar relativity is found on the continuum problem too. If the continuum hypothesis of Cantor is true, it must be that, for any interval set E of positive length, we may have

$$\overline{E} = \Omega$$

Ω being the initial ordinal of 3rd class. Then, as Ω is a regular ordinal and hence is unfinishing, by Theorem 2 it is impossible that $0 < \widetilde{m}E < \infty$*), so that it must be that

$$\widetilde{m}E = 0 .$$

This apparently gives a contradiction. Thus we have the following results.

Theorem 3. *If the ordinal of 3rd class is to be admitted, the continuum hypothesis of Cantor cannot hold in the empiricist pragmatism.*

Theorem 4. *If a regular ordinal corresponds to a bounded set A in a euclidean space, then it must be that*

$$\widetilde{m}A = 0 .$$

Subsequently, by Theorem 4, it readily follows that:

Corollary 5. *There can exist no ordinal to correspond to the continuum, in the empiricist pragmatism.*

Corollary 6. *The well-ordering theorem cannot generally be admitted*

*) Because $(\{x\})$ $(x \in E)$ is considered as a size-preserving E-partition of E.

in the empiricist pragmatism.

3. Density Theorems

For a linear set E (of real numbers) if $x \in E$ and

$$\lim_{h \to +0} \frac{m_e E \cap [x-h,\ x+h]}{2h} = 1$$

m_e referring to the outer measure, x is called a *point of density of E.* In relation to this property the following theorem is known.

Theorem 7 (*Lebesgue Density Theorem*) (*1st Density Theorem*). *Almost every point of a Lebesgue measurable set E is a point of density of E.*

It seems very natural if one intends to apply, in any way, the a priori measure in place of the outer measure in a similar construction to that of Lebesgue density. Fortunately we obtained the following proposition to be true by application of the lower normal density defined by (0.3). The proof was attained by making a little modification of the proof of the Lebesgue density theorem cited to a book by J. C. Oxtoby[5]. For any set E in a euclidean space, let the subset E_r of E be defined as

$$E_r \equiv \left\{ p \in E \mid d_E(p) \leqslant r \right\} .$$

Theorem 8 (*2nd Density Theorem*). *For a bounded set E in a euclidean space, if there is a real number* $0 < r < 1$ *for which*

$$m_e E_r > 0 ,$$

then we have

$$\widetilde{m} E_r \leqslant r \cdot m_e E_r .$$

Proof. For any $\varepsilon > 0$, there may be found a bounded open set G such that $E_r \subseteq G$ and

$$m_e E_r > (1-\varepsilon)\, \widetilde{m} G . \tag{3.1}$$

Let $\boldsymbol{S}$ be the class of all closed ball sets of positive radius U such that

$$U \subseteq G$$

and

$$\widetilde{m} E \cap U \leqslant (1+\varepsilon)\, r \cdot \widetilde{m} U . \tag{3.2}$$

Now we first take an arbitrary ball from $\boldsymbol{S}$ as U_1, and choose U_{n+1} in sequence, as follows. $U_1, \cdots, U_n \in \boldsymbol{S}$ are disjoint and $\boldsymbol{S}_n$ denotes the subclass of all members of $\boldsymbol{S}$ that are disjoint to $U_1, \cdots, U_n$. Let δ_n be the supremum value of the diameters of balls of $\boldsymbol{S}_n$. Then we choose U_{n+1} from $\boldsymbol{S}_n$ such that, denoting by $|U|$ the diameter of a ball U, we may have

$$|U_{n+1}| > \frac{1}{2}\delta_n . \tag{3.3}$$

Next, we set the assumption that for the set

$$\overset{*}{E}_r = E_r - \cup_1^\infty U_n \tag{3.4}$$

we have

$$m_e \overset{*}{E}_r > 0 . \tag{3.5}$$

Then, since

$$\Sigma\, \widetilde{m} U_n \leqslant \widetilde{m} G < \infty$$

there exists an integer N such that, denoting by m the dimension of the space*), we may have

$$\Sigma_{n=N+1}^\infty \widetilde{m} U_n < \frac{1}{3^m}\, m_e \overset{*}{E}_r . \tag{3.6}$$

We now take a ball V_{N+k} that is concentric with U_{N+k} and is such that

$$|V_{N+k}| = 3|U_{N+k}| . \tag{3.7}$$

Then we have

$$\widetilde{m} \cup_{k=1}^\infty V_{N+k} \leqslant \sum_{k=1}^\infty \widetilde{m} V_{N+k} = 3^m \Sigma\, \widetilde{m} U_{N+k} ,$$

hence by (3.6)

$$< m_e \overset{*}{E}_r .$$

So then $\cup_{k=1}^\infty V_{N+k}$ cannot cover up the set $\overset{*}{E}_r$, so that

$$\overset{*}{E}_r - \cup_{k=1}^\infty V_{N+k} \neq \varnothing .$$

Hence, there is a point

$$p \in \overset{*}{E}_r - \cup_{k=1}^\infty V_{N+k} . \tag{3.8}$$

Then, in regard to (3.4), we have

$$p \in E_r - \cup_{n=1}^N U_n .$$

As U_n are all closed, $\cup_{n=1}^N U_n$ is closed. So, there must be a ball $U(p) \in \boldsymbol{S}_N$ which has p as its center. Then, if

$$U(p) \cap \cup_{k=1}^\infty U_{N+k} = \varnothing ,$$

by the definition of $\boldsymbol{S}_N$ we have

$$U(p) \in \boldsymbol{S}_{N+k} \text{ for all } k = 1, 2, \cdots,$$

*) I. e., all points in question are contained in the same m-dimensional euclidean space.

hence by (3.3)

$$|U(p)|\leqslant\delta_{N+k-1}<2|U_{N+k}|\,.$$

On the other hand, as $\Sigma\widetilde{m}U_n$ is convergent, we have

$$\lim_{k\to\infty}|U_{N+k}|=0\,,$$

hence

$$|U(p)|=0\,.$$

This is a contradiction. So, there must eventually exist k's such that

$$U(p)\cap U_{N+k}\neq\emptyset\,. \tag{3.9}$$

Now, let k be the smallest of such k's. Then, as

$$U(p)\in\boldsymbol{S}_{N+k-1}\,,$$

by (3.3) we have again

$$|U(p)|\leqslant\delta_{N+k-1}<2|U_{N+k}|\,. \tag{3.10}$$

Besides by grace of (3.9) we have

(the distance between p and the center of U_{N+k})

$$\leqslant\frac{1}{2}|U(p)|+\frac{1}{2}|U_{N+k}|\,,$$

then by (3.10)

$$\leqslant\frac{1}{2}\delta_{N+k-1}+\frac{1}{2}|U_{N+k}|<|U_{N+k}|+\frac{1}{2}|U_{N+k}|=\frac{3}{2}|U_{N+k}|\,,$$

then by (3.7)

$$=\frac{1}{2}|V_{N+k}|\,.$$

Since V_{N+k} and U_{N+k} are concentric, this means that

$$p\in V_{N+k}\,.$$

Therefore

$$p\notin\overset{*}{E}_r-\cup_{k=1}^{\infty}V_{N+k}\,,$$

which is contradictory to (3.8).

This contradiction may firstly be conjectured as caused by the assumption that (U_n) make up an infinite sequence. However, as far as (3.5) holds, we have

$$E_r-\cup_{k=1}^{\infty}U_k\neq\emptyset\,;$$

then, since $\cup_1^n U_k$ is closed, any point of $E_r-\cup_1^n U_k$ and the set $\cup_1^n U_k$ are in a positive distance, so that there may be chosen U_{n+1} from $\boldsymbol{S}_n$ and consequently (U_k) must in fact make up an infinite sequence.

Thus, as the cause of the above-mentioned contradiction is left only the assumption (3.5). So then we have

$$m_e \overset{*}{E_r} = 0$$

i. e.,

$$m_e(E_r - \cup U_n) = 0 . \tag{3.11}$$

Besides, as (U_n) are disjoint closed sets, we have

$$\widetilde{m}E_r \cap (\cup U_n) = \Sigma\, \widetilde{m}E_r \cap U_n ,$$

hence by (3.2)

$$\leqslant (1+\varepsilon)\, r \cdot \Sigma\, \widetilde{m}U_n \leqslant (1+\varepsilon)\, r \cdot \widetilde{m}G ,$$

then by (3.1)

$$< \frac{1+\varepsilon}{1-\varepsilon}\, r \cdot m_e E_r . \tag{3.12}$$

On the other hand

$$\widetilde{m}E_r = \widetilde{m}E_r \cap (\cup U_n) + \widetilde{m}(E_r - \cup U_n)$$
$$\leqslant \widetilde{m}E_r \cap (\cup U_n) + m_e(E_r - \cup U_n) ,$$

so by (3.11) and (3.12)

$$< \frac{1+\varepsilon}{1-\varepsilon}\, r \cdot m_e E_r .$$

Since ε is arbitrary, we ultimately have

$$\widetilde{m}E_r \leqslant r \cdot m_e E_r \qquad \text{Q. E. D.}$$

4. Homogeneous Probability

When observation of points is restricted within a set E in a euclidean space, if the occurrence of points in a special subset A of E is everywhere expected with the same probability π, or, in other words, there is an aleatory variable point P such that

$$\forall p,\ q \in E :\ P_r(P = p) = P_r(P = q)$$

and for every open set $G \subseteq E$

$$P_r(P \in A \cap G) / P_r(P \in E \cap G) = \pi (\leqslant 1) ,$$

then A is said to have *homogeneous probability* π in E. In this case, if E is an open set, it is easily seen that

$$\forall p \in A :\ d_A(p) = \pi .$$

If we use a Vitali set V_I in a bounded interval I, we may really, for any $0<\pi<1$, construct a subset A of I which has homogeneous probability π in I, as follows: Denoting by Q the set of all rational numbers, we may readily divide Q into two sets Q_1 and Q_2 such that $Q_1 \cap Q_2 = \varnothing$ and Q_1 has homogeneous probability π in Q. Then, if we define as

$$A = \left\{x \in I | (\exists y \in V_I)(x-y \in Q_1)\right\},$$

obviously A has homogeneous probability π in I.

Theorem 9. *If a set A has homogeneous probability π in a bounded open set G in a euclidhan space and if $\pi>0$, then*

$$m_e A = m_e G. \tag{4.1}$$

Proof. Since

$$\tilde{m}A = \pi \cdot \tilde{m}G = \pi \cdot m_e G > 0 \tag{4.2}$$

and, by the assumption, apparently

$$A = A_\pi = \left\{p \in A | d_A(p) \leqslant \pi\right\},$$

we have

$$m_e A_\pi > 0.$$

Then, by Theorem 8 and (4.2)

$$0 < \pi \cdot m_e G \leqslant \pi \cdot m_e A$$

i.e.,

$$m_e G \leqslant m_e A.$$

Besides, as $A \subseteq G$

$$m_e A \leqslant m_e G.$$

Consequently it must be that

$$m_e A = m_e G \qquad \text{Q. E. D.}$$

If a set A is Lebesgue measurable, we have

$$m_e A = mA,$$

m referring to the Lebesgue measure. So, if (4.1) holds, by Theorem 8 it must be that $\pi=1$ (because, when A is Lebesgue measurable, $\tilde{m}A = mA$). Thus we see that: if a set A has homogeneous probability π in a bounded open set and $0<\pi<1$, then A cannot be Lebesgue measurable; particularly A cannot be a Borel set (because, as well-known, any Borel set is Lebesgue measurable).

5. Indistinctiveness of the Notion of Baire Category

In analysis, a null set is severally regarded to suggest a degree of negligibility of a property which is taken to be examined for each point of a set whether it is satisfied or not. Similarly, a set of 1st category in the sense of Baire*) has been expected to give a sort of negligibility analogous to that of a null set. But, after all such expectation, it is found notable that the property of 1st category is not so distinctive. We demonstrate it in the following by constructing a counter example.

Let R be the set of all points represented as $p=(x_1, \cdots, x_n)$ ($x_1, \cdots, x_n$ being real numbers) the total of which make up a euclidean space of dimension n, and Q be a subset of R that consists of all points for which all of $x_1, \cdots, x_n$ are rational numbers. Then Q is enumerable, so let it be enumerated as $Q=(q_k)$ $(k=1, 2, \cdots)$.

Now, let it be that

$$U_k^{(\nu)}=\{p\in R \mid \; |p-q_k|<1/2^{\nu k}\} \qquad (\nu, k=1, 2, \cdots).$$

Then sets $R^{(\nu)}$ $(\nu=1, 2, \cdots)$ defined as

$$R^{(\nu)}=(R-\cup_{k=1}^{\infty} U_k^{(\nu)})\cup(\cup_{j=1}^{\nu}\{q_j\})$$

are all, as readily seen, nowhere dense, so that the set

$$R^*=\cup R^{(\nu)}$$

is found to be a set of 1st category. However, it is not difficult to prove that

$$R^*=R,$$

whereas R has generally been thought to be of 2nd category. Thus we find that the notion of (Baire) category is not distinctive.

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References

1) Kinokuniya, Y.: Mem. Muroran Inst. Tch. **7** (1), 329 (1970); **7** (2), 599 (1971).
2) Kinokuniya, Y.: Mem. Muroran Inst. Tch. **7** (3), 636 (1972).
3) Kinokuniya, Y.: Mem. Muroran Inst. Tch. **7** (2), (1972).
4) Kinokuniya, Y.: Mem. Muroran Inst. Eng. **2** (3), 783–785 (1957).
5) Oxtoby, J. C.: *Measure and Category* (Springer-Vertag), (1971).

*) A set is said to be of *1st category* (*in the sense of Baire*) if it can be represented as an enumerable union of nowhere dense sets. If A is not of 1st category, A is said to be of *2nd category*.