Totally Ordered Linear Space Structures and Hahn-Banach Type Extension Theorem

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Abstract

Let E be a real linear (resp. real linear topological) space. By applying 18)* (resp. [19), Th. 3]), from the viewpoint of the *totally ordered linear space structures*** of the product linear space $E\times \boldsymbol{R}$, the author deals with the real Hahn-Banach extension theorem in somewhat general.

Introduction. By means of [18), Th. 4] (resp. [19), Th. 3]), in the real case, we have been concerned with the Krein's (resp. Krein-Rutman) extension theorem in somewhat detail*** from our new (for the author) views. Under the circumstances, but also in the light of the literatures†, we are in a position to formulate the Hahn-Banach extension theorem†† in somewhat general (as one expected). In this article these results are given as Theorems 1 and 2, the former for real linear spaces, the latter for real linear topological spaces. Especially both are also provided with "if" and "only if" parts.

Besides we supplementarily refer to [18), Lemma 3 (2)].

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Preliminaries. Let E be a real linear (later, real linear topological) space ($\neq\{0\}$), and let $\boldsymbol{R}$ be the real field. We first put some definitions.

DEFINITION 1. a) A subset K of E is called a *pointed convex cone* if $K+K\subset K$ and $\alpha K\subset K$ for all $\alpha\geqslant 0$.

b) Let us agree upon the following. By a *gauge function* q (or p) on a pointed convex cone K in E is meant a subadditive positively homogeneous function on K.

DEFINITION 2. The *product linear space* $E\times \boldsymbol{R}$ of linear spaces E and

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** For this thought, the author received suggestions esp. from [10), p. 48 (p. ix)]. Subsequently he was benefited by [7),V. 12] and [14, § 16].

*** For this matter, the author was benefited by [8), § 8.3], [11), Sec. 2.6] and others. As for the implications, *our results are resp. equiv. to the real case of* [15), Cor. 1 of (V, 5.4)] *and the real case of* [15), (V, 5.4) (Bauer-Namioka)] *exc. the trivial case* (with apologies, the author adds "the real case of"). (Cf. [15), p. 227] and [19), Suppl. to Th. 3].)

† By these the author means [6), Th. 12.3], [13), chap. II, § 3, th. 1], [9), Th. 3.4] and others.

†† By this we here quote [14), § 17, 3. (1) (Satz von HAHN-BANACH)].

$\boldsymbol{R}$ is their Cartesian product where vector addition and scalar multiplication are performed coordinatewise. The *topological product* $E_1 \times E_2$ of linear topological spaces $E_i (i=1, 2)$ is their product linear space with the product topology.

In addition, for convenience, notations and terminology employed in 18) and 19) are available unless otherwise specified. Especially, e.g., $(E, \mathscr{R})$ signifies a *totally ordered linear space structure* (above-mentioned) of underlying linear space E with respect to a binary relation $\mathscr{R}$. Structures of this kind have been discussed there in somewhat detail. The following theorems are described in terms of these structures.

Statement of the results. Let us first introduce* our short approach 18) to the argument of the literatures**. Indebted to these literatures for the manner, we now reach the following.

THEOREM 1. *Let M be a linear subspace of E, f a linear form on M. Let K be a pointed convex cone in E, and q a gauge function on K. A necessary and sufficient condition that there exists a linear form F on E extending f and satisfying $F(y) \leqslant q(y)$ for all $y \in K$ is that there exists a t.o.l.s. $(L, \mathscr{R})$ of L with the following properties:*

(i) $B_f \cup C_q \subset (L, \mathscr{R})^+$;

(ii) $(L, \mathscr{R})^+$ *is absorbing at* $(0, 1)$ *for* L;

where L is the product linear space $E \times \boldsymbol{R}$, and $B_f = \{(x, \xi) : f(x) < \xi,\ x \in M\}$, $C_q = \{(y, \eta) : q(y) < \eta,\ y \in K\}$ in L.

PROOF. Under the hypothesis, in L, C_q proves to be a convex cone without vertex zero. And to this end, L can be endowed with a partially ordered linear space structure $(L, \mathscr{P})$ with positive cone C_q. (Necessity) By hypothesis, defining $\Phi(x, \xi) = -F(x) + \xi$, Φ is a positive linear form on $(L, \mathscr{P})$ with $\Phi(0, 1) = 1$. Hence (to be precise), take a t.o.l.s. $(L, \mathscr{R}_1)$ such that $B_f \cup C_q \subset (L, \mathscr{R}_1)^+$ by [18), Th. 4(1) and Lemma 1], then by [18), Lemmas 2, 3(1) and 4], $(L, \Phi(\mathscr{R}_1))$ must become a t.o.l.s. as required. (Sufficiency) Defining $\varphi(x, \xi) = -f(x) + \xi$, φ is a non-identically-zero linear form on $M \times \boldsymbol{R} \ni (0, 1)$. Therefore by hypothesis, now with the aid of [18), Th. 4(2)] (cf. [19), p. 46, footnote]), we get a positive linear form Φ on $(L, \mathscr{P})$ extending φ. Hence there exists a linear form F on E extending f and satisfying $\Phi(x, \xi) = -F(x) + \xi$. And hence $q(y) < \eta$ implies $F(y) \leqslant \eta$ for all $y \in K$, which ensures the assertion.

As for some simple examples

EXAMPLE 1. Let E be $\boldsymbol{R}^2$. Take a pointed convex cone $K = \{(\alpha, \beta) :$

* Such being the case, specifically, our $\mathscr{P}$ below will be of *asymmetry*.

** They are as quoted before; see footnote '†'.

$\alpha>0$, or $\alpha=0$ and $\beta\geqslant 0\}$ in E. Define q on K to mean $q(\alpha, \beta)=\alpha$ if $\alpha>0$ and $q(0, \beta)=\beta$ if $\beta\geqslant 0$, and q is a gauge function on K. With this

(1) let M be the α-axis and define f on M by $f(\alpha, 0)=\alpha$;

(2) let M be $\{(0, 0)\}$ and define $f(0, 0)=0$;

(3) let M be the β-axis and define f on M by $f(0, \beta)=\beta$.

Then in case of (1) (resp. (2)), notwithstanding $B_f\cup C_q$ is not absorbing at $b=((0, 0), 1)$ (resp. at any point of $M\times \boldsymbol{R}$) for L, the sufficient condition of Theorem 1 is met enough. While in case of (3), although f is majorized by q on $M\cap K$, f fails to have desired extension. That is why, choosing the following four vectors b, $c_1=((1, 0), 2)$, $c_2=((1, \rho+1), 2)$ in C_q and $a=(-(0, \rho+1), -\rho)$ in B_f, where ρ being arbitrary, there holds the equality $(\rho b-c_1)+a+c_2=0$. Namely upon appealing to Theorem 1, none of $(L, \mathscr{R})^+$ with $B_f\cup C_q\subset(L, \mathscr{R})^+$ can be absorbing at b for L.

Remark 1. In Theorem 1, let in particular $K=E$ (with gauge p) and $f(x)\leqslant p(x)$ for all $x\in M$. Then it follows (resp.) that C_p is, by itself, absorbing at $(0, 1)$ for L and that $B_f\cup C_p$ is, as above, positively independent in L. Hence by [18), Lemma 1], the sufficient condition thereof is met enough. This corresponds to the usual extension theorem for linear spaces. Moreover, the "if" part of Theorem 1 essentially (and a fortiori) covers [9), Prob. 3 E].

Meanwhile, let $P=(E, \mathscr{R})^+$ be a maximal positive cone in E, which is absorbing at $u_0\in E$. Let us take this opportunity to make mention [18), Lemma 3 (2)] (this plays rather well in conjunction with Lemma 1 ibid.) in connection with the *Minkowski gauge* $p(x)=\inf\{\alpha:\ \alpha>0,\ x\in P-\alpha u_0\}$ of $P-u_0$.

Supplement to [18), Lemma 3 (2)]. At first, needless to say

(1) As usual, using $p(x)$ (resp. in view of the ordered linear space $(E, \mathscr{R})$), one can deduce this lemma also via the Hahn-Banach (resp. Krein's) extension theorem. But as for this lemma, its proof given in 18) is not only self-contained but also simpler than the above.

Secondly this proof, in terms of the *negative part* f^- of $f\in E^*$ i.e., $f^-(x)=\max\{-f(x), 0\}$ $(x\in E)$, now anew verifies

(2) *An $f\in E^*$ required there with $f(u_0)=1$ is given by p in the sense of $f^-=p$, and vice versa. That is, $f(x)$ must be equal to $p(-x)-p(x)$ with $f^-(x)=p(x)$ for all $x\in E$.*

This is known by $p(x)=0$ $(x\in P)$, $p(0)=0$ and $p(x)=\inf\{\alpha:\ -x<\alpha u_0\ (\mathscr{R})\}=\sup\{\beta:\ 0\leqslant\beta u_0<-x(\mathscr{R})\}=-f(x)$ $(x\in -P)$.

Concerning (2), in fact the following will be verified.

(3) *Let K be a convex cone in E which is not identical with E and is absorbing at $b\in E$. Then g^- of $g\in E^*$ is the Minkowski gauge of $K-b$ iff $g(b)=1$ and $\{x\in E:\ g(x)>0\}\subset K\subset\{x\in E:\ g(x)\geqslant 0\}$.*

Returning to the subject, next there holds the following, a topological version of Theorem 1. In this theorem we let $\boldsymbol{R}$ be equipped with the usual topology.

THEOREM 2. *Let E be a linear topological space, and let M, f, K, q be as in the statement of Theorem 1. A necessary and sufficient condition that there exists a continuous linear form F on E extending f and satisfying $F(y) \leqslant q(y)$ for all $y \in K$ is that there exists a t.o.l.s. $(L, \mathscr{R})$ with the following properties*:

(i) $B_f \cup C_q \subset (L, \mathscr{R})^+$;

(ii) $(L, \mathscr{R})^+$ *is a convex neighbourhood at* $(0, 1)$ *for* L;

where L is the topological product $E \times \boldsymbol{R}$ and B_f, C_q are same as in Theorem 1.

PROOF. Proceed as in the proof of Theorem 1, and check that $\Phi(x, \xi)$ is continuous on L if and only if so is $F(x)$ on E. And L now being a linear topological space, to this end, we may consult the proof of [19), Th. 3]. This completes the proof of the theorem.

Notice that, similarly as pointed out in 19), our condition of (i) plus (ii) above is equivalent to that there exists a convex open subset $O \ni (0, 1)$ in L such that $B_f \cup C_q \cup O$ is positively independent. Moreover, this time simple computation gives the following. These simplify our condition of Theorem 2.

REMARK 2. Let U be a convex 0-neighbourhood in E and put (henceforth) $D = U \times \{1\}$, $B = (1/2\, U) \times I$ where $I = \{\rho \in \boldsymbol{R}: |\rho - 1| < 1/2\}$. If $B_f \cup C_q \cup D$ is positively independent the same is true for $B_f \cup C_q \cup B$.

Let us now observe some corollaries about Theorem 2. Corollaries 2 and 3 mentioned below are the usual extension theorems in the context of linear topological spaces.

COROLLARY 1. *Let E, M and f be as in Theorem 2. Let K be a linear subspace of E with $M \subset K$, q a gauge function on K with $f(x) \leqslant q(x)$ for all $x \in M$. If the condition*

(P_1) *there is a convex 0-neighbourhood U in E not meeting $\{y \in K: q(y) = 1\}$*

is enjoyed, the sufficient condition of Theorem 2 is satisfied.

PROOF. Let L, B_f and C_q be as in question. Taking the subset $D = U \times \{1\}$ of L, suppose that $B_f \cup C_q \cup D$ were now positively dependent in L. Then referring to Rem. 1, there would exist both finite many respective vectors, say, $(x_r, \xi_r) \in B_f$, $(y_s, \eta_s) \in C_q$, $(u_t, 1) \in D$ and corresponding scalars $\alpha_r > 0$, $\beta_s \geqslant 0$, $\gamma_t > 0$ (or $\alpha_r = 0$, $\beta_s > 0$, $\gamma_t > 0$) such that

$$(*) \qquad q(\sum \gamma_t u_t) \geqslant q(-\sum \alpha_r x_r) - q(\sum \beta_s y_s)$$
$$\geqslant -f(\sum \alpha_r x_r) - q(\sum \beta_s y_s) > -\sum \alpha_r \xi_r - \sum \beta_s \eta_s = \sum \gamma_t = 1,$$

which contradicts the hypothesis since $\sum \gamma_t u_t \in K \cap U$. Hence by Rem. 2 and by [18), Lemma 1], the proof is completed.

REMARK 3. The converse of this result is not always valid. That is, (P_1) is, under the remaining hypotheses, not always necessary for conclusion. Counterexamples are easily observed (cf. e.g., Ex. 2 below). On the other hand, the condition

(F) *there is a convex* 0-*neighbourhood* U *in* E *not meeting* $\{x \in M: f(x)=1\}$

is rather necessary for this implication (for the proof, cf. Cor. 3 below), but this now fails to be sufficient for it. These facts seem to illustrate the significance of our criterion.

Easily (resp. As a matter of course) Corollary 1 yields the following Corollary 2 (resp. the sufficiency part of Corollary 3).

But of course, to be short, these corollaries are fully done by Theorem 2 itself. For reference, details are given as under.

COROLLARY 2*. *Let* E, M *and* f *be as in Theorem* 2. *Let* p *be a gauge function on* E *with* $f(x) \leqslant p(x)$ *for all* $x \in M$. *If the condition*

(P_2) p *is continuous at the origin*

is enjoyed, the sufficient condition of Theorem 2 *is satisfied.*

PROOF. With the convex 0-neighbourhood $U=\{y \in E: p(y)<1\}$, a priori, $D=U \times \{1\} \subset C_p$ follows. (Alternatively, $C_p \ni (0, 1)$ is readily open in L.) Hence, a fortiori, the assertion follows from Theorem 2.

COROLLARY 3**. *Let* E, M *and* f *be as in Theorem* 2. *A necessary and sufficient condition that* f *can be extended to a continuous linear form* F *on* E *is that* (F) *of Rem.* 3 *is satisfied. If the sufficiency of the condition is met, there exists at least one* F *such that* $F(x) \neq 1$ *for all* $x \in U$.

PROOF. This is viewed as a special case of Theorem 2 ($f(x)=q(x)$ ($x \in M \cap K$) plus $M=K$). (Necessity) By Theorem 2, there are both t.o.l.s. $(L, \mathscr{R})$ and convex 0-neighbourhood U such that B_f, $U \times \{1/2\} \subset (L, \mathscr{R})^+$ hold. If $f(u_0)=1$ for some $u_0 \in M \cap U$, there would follow $(-u_0, -1/2)$, $(u_0, 1/2) \in (L, \mathscr{R})^+$, an obvious contradiction. (Sufficiency) Taking the convex subset $D=U \times \{1\}$ of L, suppose that $B_f \cup D$ were now positively dependent. Then it would follow more simply than (*) that $f(\sum \gamma_t u_t) = -f(\sum \alpha_r x_r) > -\sum \alpha_r \xi_r = \sum \gamma_t = 1$, which is impossible. Sufficiency follows from this by Theorem 2. For the rest, if F is identically-zero, there is nothing to prove. Otherwise, indeed our extension F behaves as $F(x)<1$ for all $x \in U$ since $(1/2\, U) \times \mathrm{I} \subset (L, \mathscr{R})^+$ and since U is open in E. Thus Corollary 3 is proved.

* Cf. [14), § 17, 3. (1)].

** Cf. [17), p. 598].

Incidentally, an examination of this proof directly gives

COROLLARY 4. *Let E, M, f and L, B_f be as in Theorem 2. The condition (F) of Rem. 3 is mutually equivalent to that there exists a convex 0-neighbourhood U in E such that $B_f \cup (U \times \{1\})$ is positively independent in L.*

As a triviality, needless to say

EXAMPLE 2. To extend an identically-zero linear form on M in the sense of Corollary 3, we have at least $U=E$. And to do this in view of Theorem 2, we have at least $D=E\times\{1\}$.

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