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ON A CONFORMAL TRANSFORMATION IN AREAL SPACES

Takanori Igarashi

Abstract

Concept of "conformal transformation" is introduced in areal spaces. Under this transformation, changes of various geometric objects in areal spaces of the general type and of the submetric class are considered.

§1. Conformal transformation in the areal space of the general type.

We assume that there be given two metric m -tensors g_{IJ} and $'g_{IJ}$ in the areal space of the general type.* Let us consider a transformation

$$(A_n^{(m)}, g_{IJ}) \rightarrow (A_n^{(m)}, 'g_{IJ}), \quad (1.1)$$

which satisfies the relation ;

$$'g_{IJ} = \phi^2 g_{IJ}, \quad (1.2)$$

where ϕ is a scalar function such that

$$\phi = \phi(x^j, p_a^j) > 0. \quad (1.3)$$

Differentiating (1.2) by p_γ^k , we have

$$'g_{I,J,k} = 2\phi\phi_{,k} g_{IJ} + \phi^2 g_{I,J,k},$$

and contracting by p^I , we obtain

$$\phi_{,k} = 0, \quad (1.4)$$

by means of Iwamoto's theorem 6)** we can insist that the function ϕ in (1.2) does not depend on arguments p_γ^k but depend on only x^i .

On the other hand, if we assume ϕ be in the form such that

* Latin indices i, j, k, \dots run over $1, 2, \dots, n$; Greek indices $\alpha, \beta, \gamma, \dots$ over $1, 2, \dots, m$; while Latin capital indices I, J, K, \dots denote compound indices $i[m], j[m], k[m], \dots$. In what follows, we use the same notations and terminologies as those in papers 1), 2), ..., 5).

** Numbers in brackets refer to the references of the end of this paper.

*** In this paper, we use the concept "partial differentiation in p^i ", such as

$$\Phi_{ij} = \Phi_{i(m)} = m \Phi_{i_1 i_2 \dots i_m}, \quad \Phi_{,i}^a = \partial \Phi / \partial p_a^i,$$

for any homogeneous function Φ of order ρ in p^i , cf. 4)

$$\phi = \phi(x^i, p^j) > 0 \quad (1.5)$$

instead of (1.3), then differentiating (1.2) partially.*** we have

$$'g_{I,J;K} = 2\phi\phi_{,K}g_{I,J} + \phi^2 g_{I,J;K},$$

and contracting p^j , we get

$$\phi_{,K} = 0.$$

Hence, we can also insist that the function ϕ in (1.2) does not depend on p^j but depend on only x^i .

An angle θ between m-vectors X^I and Y^J is defined as follows;

$$\cos\theta = \frac{g_{I,J} X^I Y^J}{\|X\| \|Y\|} \quad (1.6)$$

where $\|X\|$ is a magnitude of X^I such that $\|X\|^2 = \frac{1}{m!} g_{I,J} X^I X^J$.

Measuring the angle θ between x^i and y^j with use of $'g_{I,J}$, we have

$$\begin{aligned} \cos\theta &= \frac{'g_{I,J} X^I Y^J}{\|X\| \|Y\|} = \frac{'g_{I,J} X^I Y^J}{\sqrt{\frac{1}{m!} 'g_{K,L} X^K X^L} \sqrt{\frac{1}{m!} 'g_{M,N} Y^M Y^N}} \\ &= \frac{\phi^2 g_{I,J} X^I Y^J}{\phi \sqrt{\frac{1}{m!} g_{K,L} X^K X^L} \phi \sqrt{\frac{1}{m!} g_{M,N} Y^M Y^N}} = \frac{g_{I,J} X^I Y^J}{\|X\| \|Y\|} = \cos\theta \end{aligned}$$

Consequently, the angle θ is invariant transformation (1.1) satisfying (1.2). In such a sense, we call this transformation as conformal transformation.

There is a relation such that

$$g_{I,J} p^I p^J = (m!)^2 F \quad (1.7)$$

between the metric m -tensor $g_{I,J}$ and the fundamental function $F(x^i, p^j)$ of $A_n^{(m)}$, it is to say, $F(x^i, p^j)$ represents the magnitude of m -dimensional area element p^j ,

Now, we assume that the fundamental function F is transformed to $'F$ under the conformal transformation. Then;

$$'F = \frac{1}{(m!)^2} 'g_{I,J} p^I p^J = \frac{1}{(m!)^2} \phi^2 g_{I,J} p^I p^J = \phi^2 F,$$

i.e., we obtain

$$'F = \phi F. \quad (1.8)$$

So, we find that (1.8) is hold good when the transformation satisfies (1.2).

Next, let us start from (1.8), conversely. Since there is a function $f(x^i, p_a^i)$ such that $F(x^i, p^j) = f(x^i, p_a^i)$, we may identify f as F unless there is no confusion¹⁾.

Differentiating (1.8), i.e.,

$${}^1F(x^i, p_a^i) = \phi F(x^i, p_a^i) \tag{1.8'}$$

by p_a^i , and making use of (1.4), we have followings;

$${}^1F_{,j}^{\alpha} = \phi F_{,j}^{\alpha} \tag{1.9}$$

$${}^1p_i^{\alpha} = p_i^{\alpha}, \quad (p_i^{\alpha} = F^{-1} F_{,i}^{\alpha}), \tag{1.10}$$

$${}^1L_{ij}^{\alpha\beta} = L_{ij}^{\alpha\beta}, \tag{1.11}$$

where $L_{ij}^{\alpha\beta}$ is a Legendre's form such that $L_{ij}^{\alpha\beta} = p_{i,j}^{\alpha} + p_j^{\beta} p_i^{\alpha}$.

With the help of $p_j^{\alpha}, L_{ij}^{\alpha\beta}$, etc., the metric m-tensor $g_{i,j}$ is expressed ;⁴⁾

$$g_{i,j} = \sum_{\lambda=0}^m \binom{m}{\lambda}^2 \varepsilon_{\alpha_1 \dots \alpha_m} \varepsilon_{\beta_1 \dots \beta_m} F^2 L_{[j_1 [j_1}^{\alpha_1 \beta_1} \dots L_{i_{\lambda} j_{\lambda}}^{\alpha_{\lambda} \beta_{\lambda}} p_{i_{\lambda+1}}^{\alpha_{\lambda+1}} p_{j_{\lambda+1}}^{\beta_{\lambda+1}} \dots p_{i_m}^{\alpha_m} p_{j_m}^{\beta_m}]. \tag{1.12}$$

Taking account of (1.10) and (1.11), we can conclude the following theorem;

THEOREM. 1. *To what $g_{i,j}$ satisfies (1.2), it is equivalent that F satisfies (1.8).*

The contravariant component $g^{l,k}$ of $g_{i,j}$ defined by $g^{l,k} g_{i,j} = \frac{1}{m!} \delta_j^k$ is transformed to ${}^1g^{l,k}$ as

$${}^1g^{l,k} = \phi^{-2} g^{l,k} \tag{1.13}$$

under the conformal transformation.

Next, we have to show how connection coefficients changes. Put

$$g_{ij}{}^{hk} = g_{ii_2 \dots i_m, jj_2 \dots j_m} g^{hh_2 \dots h_m, kk_2 \dots k_m},$$

$$\Lambda_{ij}{}^{hk} = \binom{n-2}{m-1}^{-1} \{ (m-1)! \}^{-2} g_{(ij)}{}^{hk} - \binom{n-2}{m-1} \delta_{(i}^h \delta_{j)}^k \},$$

$$\Lambda_{ij}{}^{\alpha\beta} = \Lambda_{ij}{}^{hk} p_h^{\alpha} p_k^{\beta},$$

$$\Lambda^*{}_{ij}{}^{\alpha\beta} = F^{\frac{2}{m}} \Lambda_{ij}{}^{\alpha\beta}, \quad \Lambda^*{}^{ij}{}_{\alpha\beta} = F^{-\frac{2}{m}} \Lambda^{ij}{}_{\alpha\beta},$$

where $\Lambda^{ij}{}_{\alpha\beta}$ is derived from $\Lambda_{ij}{}^{\alpha\beta} \Lambda^{hj}{}_{\alpha\beta} = \delta_i^h \delta_{\gamma}^{\alpha}$ under the assumption that the mn -rowed det. $|\Lambda_{ij}{}^{\alpha\beta}| \neq 0$.

Then, we give a covariant differential of a vector V^i in the form ;³⁾

$$\delta V^i = dV^i + \Gamma_{jk}^*{}^j V^i dx^k + C_{j,i}^{\delta} V^j \delta p_{\delta}^i, \tag{1.14}$$

where $\delta p_a^i = \gamma_j^i (dp_a^j + B_{jk}^i dx^k)$, $\gamma_j^i = \delta_j^i - p_j^i p_j^{\alpha}$,

$$\Gamma_{jk}^*{}^i = \gamma_{jk}^i - C^{\times ij}{}_{j,h} B_{\gamma k}^h - C^{\times ij}{}_{k,h} B_{\gamma j}^h + C^{\times i\gamma r}{}_{j,k,h} B_{\gamma r}^h, \tag{1.15}$$

$$\gamma_{jk}^i = \frac{1}{2m} \Lambda^*{}^{ir}{}_{\alpha\beta} \{ \Lambda^*{}_{rk}{}^{\alpha\beta} + \Lambda^*{}_{jr}{}^{\alpha\beta} - \Lambda^*{}_{ik}{}^{\alpha\beta} \}, \tag{1.16}$$

$$C^{\times i\gamma r}{}_{k,h} = \frac{1}{2m} \Lambda^*{}^{ir}{}_{\alpha\beta} \Lambda^*{}_{ik}{}^{\alpha\beta}{}_{,\gamma}, \quad C^{\times ij}{}_{k,h} = C^{\times kij}{}_{k,j,h}, \tag{1.17}$$

$$B_{\gamma s}^h = \gamma_{jk}^i p_{\delta}^j \tilde{W}_{js,\gamma}^{hk\delta}, \quad \tilde{W}_{is,\gamma}^{hk\delta}, \quad W_{ik,\delta}^{\gamma\alpha} = \delta_{\alpha}^h \delta_s^r \delta_{\gamma}^{\alpha}, \tag{1.18}$$

$$\begin{aligned}
 W_{lk\delta}^{j\gamma\alpha} &= (C^{\times ia}_{k,l}\delta_j^r + C^{\times ia}_{j,l}\delta_k^r - C^{\times jra}_{j,kl})p_\delta^j + \delta^i\delta_k^r\delta_\gamma^a, \\
 C_{j,k}^{i\gamma} &= C^{\times ir}_{j,k} - p_a^h p_j^a C^{\times ia}_{h,k}.
 \end{aligned}
 \tag{1.19}$$

Under the conformal transformation, quantities g_{ij}^{hk} , $\Lambda_{ij}^{\alpha\beta}$, $\Lambda^{ij}_{\alpha\beta}$ are invariant. Hence, $\Lambda^*_{ij}{}^{\alpha\beta}$ and $\Lambda^{*ij}_{\alpha\beta}$ are transformed as

$${}'\Lambda^*_{ij}{}^{\alpha\beta} = \phi^{\frac{2}{m}}\Lambda^*_{ij}{}^{\alpha\beta}, \quad {}'\Lambda^{*ij}_{\alpha\beta} = \phi^{-\frac{2}{m}}\Lambda^{*ij}_{\alpha\beta},$$

which give us follows ;

$$\begin{aligned}
 {}'\gamma_{jk}^i &= \frac{1}{2m}\phi^{-\frac{2}{m}}\Lambda^{*ir}_{\alpha\beta}\{\phi^{\frac{2}{m}}(\Lambda^*_{rk}{}^{\alpha\beta}{}_{,j} + \Lambda^*_{jr}{}^{\alpha\beta}{}_{,k} - \Lambda^*_{jk}{}^{\alpha\beta}{}_{,r}) \\
 &\quad + \frac{2}{m}\phi^{\frac{2}{m}-1}(\phi_j\Lambda^*_{rk}{}^{\alpha\beta} - \phi_{,r}\Lambda^*_{jk}{}^{\alpha\beta})\}.
 \end{aligned}$$

With use of the fact that $\Lambda^{*ir}_{\alpha\beta}\Lambda^{*rj\alpha\beta} = m^2\delta_j^i$ and of notation such as $\frac{1}{m^2}\Lambda^{*ir}_{\alpha\beta} = \Lambda^{*iv}_{jk}$ and $\phi^{-1}\phi_{,r} = \phi_r$, we obtain the expression

$${}'\gamma_{jk}^i = \gamma_{jk}^i - (\Lambda^{*ir}_{jk}\phi_r - \delta_j^i\phi_r - \delta_k^i\phi_j),
 \tag{1.20}$$

which give us the transformation law of the Christoffel's symbol of the areal space of the general type.

$\widetilde{W}_{is,r}^{hk\delta}$ being invariant under the conformal transformation, the change of B_{as}^h is given by

$${}'B_{as}^h = B_{as}^h - (\Lambda^{*iv}_{jk}\phi_r - \delta_j^i\phi_k - \delta_k^j\phi_j - \delta_k^j\phi_j)p_\delta^i\widetilde{W}_{is,\gamma}^{hk\delta},
 \tag{1.21}$$

by means of (1.18).

On making use of (1.21) and of the invariance of $C_{ij,k}^{\times h\gamma}$, $C_{j,k}^{\times i\gamma}$ and $\widetilde{W}_{is,\gamma}^{hk\delta}$, we can express the transformation law of Γ_{jk}^{*i} in the form ;

$${}'\Gamma_{jk}^{*i} = \Gamma_{jk}^{*i} - U_{jk}^i,
 \tag{1.22}$$

where

$$\begin{aligned}
 U_{jk}^i &= \Lambda_{jk}^{*ir}\phi_r - \delta_j^i\phi_k - (\Lambda_{bc}^{*hr}\phi_d - \delta_b^a\phi_d - \delta_c^a\phi_d) \\
 &\quad \times (C^{\times i\gamma}_{k,r}\widetilde{W}_{a,j,\gamma}^{rc\gamma} + C^{\times i\gamma}_{j,r}\widetilde{W}_{a,k,\gamma}^{rc\gamma} - C^{\times ih\gamma}_{j,k,r}\widetilde{W}_{ah,\gamma}^{rc\alpha})p_a^b.
 \end{aligned}$$

On account of the invariance of $C_{j,k}^{\times i\gamma}$, we can immediately show, from (1.19), that the connection coefficient $C_{j,k}^{i\gamma}$ is a conformal-invariant, i.e.,

$${}'C_{j,k}^{i\gamma} = C_{j,k}^{i\gamma}.
 \tag{1.24}$$

Accordingly, we have the the following theorem ;

THEOREM. 2. *In the areal space of the general type, if there be given a connection (1.14), the connecion coefficient Γ_{jk}^{*i} is transformed such as (1.22), while the another coefficient $C_{j,k}^{i\gamma}$ is invariant under the conformal transformation.*

§ 2. Conformal transformation in the areal space of the submetric class.

In this section, we take up an areal space of the submetric class.

Let us consider a conformal transformation

$$(A_n^{(m)}, g_{ij}) \rightarrow (A_n^{(m)}, 'g_{ij}) \tag{2.1}$$

with

$$'F = \phi F, \phi > 0, \tag{2.2}$$

where g_{ij} is a normalized metric tensor which is given such as

$$g_{ij} = \left(\frac{1}{m} L_{ij}^{\alpha\beta} + p_i^\alpha p_j^\beta \right) g_{\alpha\beta}, |g_{\alpha\beta}| = F^2, \tag{2.3}$$

$g_{\alpha\beta}$ being a metric tensor in an m -dimensional subspace of $A_n^{(m)}$.

$$\text{Since } |'g_{\alpha\beta}| = ('F)^2 = \phi F^2 = \phi^2 |g_{\alpha\beta}| = |\phi_m^2 g_{\alpha\beta}|,$$

(or $|g_{\alpha\beta}|$ is a polynomial homogeneous in each $g_{\alpha\beta}$), we have

$$'g_{\alpha\beta} = \phi_m^2 g_{\alpha\beta}, 'g^{\alpha\gamma} = \phi^{-2} g^{\alpha\gamma}, g_{\alpha\beta} g^{\alpha\gamma} = \delta_\beta^\gamma. \tag{2.4}$$

Taking (2.3), and making use of (2.4), (1.11) and (1.12), it is evident that

$$'g_{ij} = \phi_m^2 g_{ij}.$$

On the other hand, the metric m -tensor $g_{i,j}$ decomposes in the form ;

$$g_{i,j} = m! g_{[i_1 j_1] i_2 j_2 \dots i_m j_m} + J_{i[m],j[m]}, \tag{2.5}$$

in which $J_{i[m],j[m]}$ vanishes in the case that the space is of the metric class.

Taking thought that the expression $'g_{ij} = \psi^2 g_{ij}$ hold good in Riemannian, Finsler and Cartan spaces, it is natural that (2.5) would be transformed to

$$'g_{i,j} = m! (\phi_m^2 g_{[i_1 j_1]}) (\phi_m^2 g_{i_2 j_2}) \dots (\phi_m^2 g_{i_m j_m}) + J_{i[m],j[m]}.$$

Consequently, for the normalized metric tensor g_{ij} and g^{ik} , the transformation laws are given such that

$$'g_{ij} = \phi_m^2 g_{ij}, 'g^{ik} = \phi^{-2} g^{ik}. \tag{2.6}$$

(Otherwise, these expressions hold good, since $C_{ij,k}^{\times h l \gamma} = \frac{1}{2} g^{kl} g_{ij,k}^\gamma$ and (1.4)).

For a Christoffel's symbol

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \frac{1}{2} g^{ih} \{ g_{hk,j} + g_{jh,k} - g_{jk,h} \}, \tag{2.7}$$

substituting (2.6), we have

$$\begin{aligned} \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} &= \frac{1}{2} \phi^{-2} g^{ih} \{ (\phi_m^2)_{,j} g_{hk} + (\phi_m^2)_{,k} g_{ih} - (\phi_m^2)_{,h} g_{jk} \\ &\quad + \phi_m^2 (g_{hk,j} + g_{jh,k} - g_{jk,h}) \}. \end{aligned}$$

Putting $\phi_j = \frac{1}{m} \phi^{-1} \phi_j$, $\phi^i = g^{ij} \phi_j$, we obtain

$$\left\{ \begin{matrix} i \\ jk \end{matrix} \right\} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - (\phi^i g_{jk} - \phi_j \delta_k^i - \phi_k \delta_j^i). \tag{2.8}$$

Under the assumption that the metric tensor g_{ij} is not necessarily real-valued, an affine connection is defined for a contravariant V^i as follows; ⁵⁾

$$DV^i = dV^i + \Gamma_{jk}^{*i} X^j dx^k + C_{j,h}^{i\delta} X^j Dp_\delta^h, \tag{2.9}$$

where $Dp_a^i = \gamma_j^i dp_a^j + B_{ak}^i dx^k$, $\gamma_j^i = \delta_j^i - p_a^i p_j^a$, $\tag{2.10}$

$$\Gamma_{jk}^{*i} = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} - \frac{1}{2} g^{il} (g_{lk,h} B_{aj}^h + g_{jl,h} B_{ak}^h - g_{jk,h} B_{il}^h), \tag{2.11}$$

$$B_{ak}^i = \left\{ \begin{matrix} i \\ jk \end{matrix} \right\} p_j^i \tilde{W}_{pk,a}^{iq\gamma}, \quad \tilde{W}_{pk,a}^{iq\gamma} W_{j\alpha,\gamma}^{pk\beta} = \delta_j^i \delta_k^h \delta_a^\beta, \tag{2.12}$$

$$W_{hk,a}^{i\gamma\tau} = (g^{il} g_{lk,h} \gamma \delta_j^\tau + g^{il} g_{lj,h} \gamma \delta_k^\tau - g^{i\tau} g_{jk,h} \gamma) p_a^j + \delta_a^\gamma \delta_h^i \delta_k^\tau. \tag{2.13}$$

The connection coefficient $C_{j,k}^{j\gamma}$ is defined by

$$C_{j,k}^{j\alpha} = g^{hi} C_{hj,k}^\alpha, \tag{2.14}$$

$$C_{hj,k}^\alpha = \frac{1}{2} (g_{hj,k}^\alpha - 2p_\beta^l g_{li} L_{jk}^{*\beta\alpha}), \tag{2.15}$$

where $L_{ij}^{*\alpha\beta}$ is an "ecmetric tensor" which is defined by by A. KAWAGUCHI such that

$$L_{ij}^{*\alpha\beta} = L_{ij}^{\alpha\beta} g''_{i\beta}, \quad g''_{ij} = \frac{1}{m} L_{ij}^{\alpha\beta} g_{\alpha\beta}, \tag{2.16}$$

and which vanishes in the case that the space is of the metric class.

The ecmetric tensor $L_{ij}^{*\alpha\beta}$ is invariant under the conformal transformation, because $L_{ij}^{\alpha\beta}$ is also invariant.

$C_{hj,k}^\gamma$ in (2.5) is transformed such that

$${}'C_{ij,k}^\gamma = \phi^{\frac{2}{m}} C_{ij,k}^\gamma, \tag{2.7}$$

thus, the connection coefficient $C_{j,k}^i$ given by (2.14) is conformal-invariant.

The transformation law of the another connection coefficient is derived from (2.11), (2.12) and (2.13), in the same way as that in §1.

B_{ak}^j is transformed such as

$${}'B_{ak}^i = B_{ak}^i - (\phi^p g_{hl} - \phi_h \delta_l^p - \phi_l \delta_h^p) p_j^h \tilde{W}_{pk,a}^{j\gamma}, \tag{2.18}$$

hence, Γ_{jk}^{*i} is transformed in the form;

$${}'\Gamma_{jk}^{*i} = \Gamma_{jk}^{*i} - U_{jk}^i, \tag{2.19}$$

with

$$U_{jk}^i = \frac{1}{2} g^{il} \{ (g_{lk,h} B_{aj}^h + g_{jl,h} B_{ak}^h - g_{jk,h} B_{il}^h) - (\phi^p g_{st} - \phi_s \delta_t^p - \phi_t \delta_s^p) (g_{lk,h} \tilde{W}_{pj,\gamma}^{ht\delta} + g_{jl,h} \tilde{W}_{pk,\gamma}^{ht\delta} - g_{jk,h} \tilde{W}_{pl,\gamma}^{ht\delta}) p_\delta^s \}. \tag{2.20}$$

Consequently, we have the following theorem;

THEOREM. 3. *In the areal space of the submetric class, if there be given a connection such as (2.9), then, the connection coefficient Γ_{jk}^{*i} is transformed as in (2.19), while the another connection coefficient $C_{j,k}^{*i}$ is invariant, under the conformal transformation.*

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