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Abstract

R. DeMarr had a conjectur : *if an ordered linear space* X *is order isomorphic to its order dual* X′, *then* X *becomes a real Hilbert space with an inner product which is compatible with the order.* And, in his paper, he proved the conjecture under an assumption. In this note, we prove the conjecture under a simple assumption which is equivalent to DeMarr's assumption.

The conjecture of R. DeMarr is the following.

If a partially ordered linear space X *is order isomorphic to its order dual* X′, *then it is possible to define an inner product* $[\cdot, \cdot]$ *on* X×X *in such a way that* X *becomes a real Hilbert space with this inner product. Furthermore, the inner product can be defined so that it has the following properties :*

(a) *if* $x \in$ X *and* $x \geqq 0$, *then* $[x, \cdot]$ *is a positive linear functional on* X,

(b) *if* $f \in$ X′ *and* $f \geqq 0$, *then there exists* $x \in$ X *with* $x \geqq 0$ *such that* $[x, \cdot] = f$.

Let T be the order isomorphism of X and X′, namely T is a positive linear transformation of X onto X′ and the inverse T^{-1} of T exists. DeMarr showed in his paper that X is Dedekind complete, i. e., for a down-directed set E of positive elements in X, inf E exists. Furthermore, he proved his conjecture under the assumption :

(D) $(x, Ty) = 0$*) *whenever* $x \cap y = 0$ *and* $x, y \in$ X.

In this note, although we can not to be prove that the conjecture is true or not, we shall consider the another assumption to be equivalent to (D), because we are thinking to give something to get a grip of whether the conjecture is true or not.

Let X be a Dedekind complete vector lattice and X′ be its order dual, i. e., the totality of positive linear functionals on X, in this note.

A subset M of X is said to be a normal manifold, if for each $x \in$ X, there exist two elements x_1 and x_2 such that

(#) $x = x_1 + x_2$, $x_1 \in$ M and $x_2 \in$ M$^\perp$

where M$^\perp$ = $\{y$; y is orthogonal to every element in M$\}$ and M$^\perp$ is called the orthognal complement of M.

When M is a normal manifold, the above decomposition (#) for each $x \in$ X is uniquely determined. Therefore, a normal manifold M defines a projection operator [M] of X on M by [M] $x = x_1$. Specially, for an element p in X, if $\{p\}^{\perp\perp}$ is a normal manifold, then the projection operator $[\{p\}^{\perp\perp}]$ is called the projector and denoted by $[p]$. In the Dedekind complete vector lattice X, each element p in X defines always a projector $[p]$ and it is known in [2] that $[p]$ has the property : $[p]\, x = \sup \{n|p| \cap x\}$ for each positive element x in X.

*) Ty means an element in X′ corresponding to an element y in X and (x, Ty) means the value of Ty at x.

The detailed properties on projection operator and the projector are found in Nakano 〔2 ; §4～§6〕

The aim of this note is to prove the following theorem.

Theorem. *Let* X *be a Dedekind complete vector lattice and* X′ *be its order dual. If there exists an order isomorphism* T *from* X *onto* X′ *such that*

(H) $(x, Tx) \neq 0$ *for every non-zero element* x *in* X,

then X *becomes a real Hilbert space with an inner product* 〔・,・〕 *on* X×X *having the above mentioned properties* (a) *and* (b).

If it is proved that (H) implies (D), our theorem comes to be true by DeMarr's result. However, we shall give another proof

Let Λ be an index set. For a set $\{a_\lambda\}(\lambda \in \Lambda)$ in X, we make use of the notation $a_\lambda \uparrow_{\lambda\in\Lambda}$ if for any indices $\lambda, \mu \in \Lambda$ there exists an index $\nu \in \Lambda$ such that $a_\lambda \leqq a_\nu$ and $a_\mu \leqq a_\nu$, and if sup $a_\lambda = a$ exists, then we write $a_\lambda \uparrow_{\lambda\in\Lambda} a$. Similary, we make use of the notation $a_\lambda \downarrow_{\lambda\in\Lambda}$ and $a_\lambda \downarrow_{\lambda\in\Lambda} a$.

Lemma 1. *T is order continuous, i. e., if $a_\lambda \downarrow_{\lambda\in\Lambda} a$ in X, then $Ta_\lambda \downarrow_{\lambda\in\Lambda} Ta$ in* X′.

Proof. This is evident from the fact that T is the order isomorphism of X onto X′.

For $x \in$ X, we define $x^+ = x \cup 0$, $x^- = (-x) \cup 0$ and $|x| = x^+ + x^-$ so that $x = x^+ - x^-$ holds. Then, we can easily see that for each $x \in$ X, we have $|Tx| = T|x|$ in X′.

Now, for each x in X, we denote by N(x) the totality of elements y in X for which $(|x|, T|y|) = 0$ hold We can see that N(x) is a normal manifold as follows. Obviously, N(x) posses the property that $y \in$ N(x) and $|z| \leqq |y|$ implies $z \in$ N(x). Namely, N(x) is a semi-normal manifold in Nakano 〔2〕. Furthermore, if $a_\lambda \in N(x)$, $a_\lambda \uparrow_{\lambda\in\Lambda} a$ and $a \geqq 0$, then by Lemma 1 we have sup $\{(|x|, Ta_\lambda); \lambda \in \Lambda\} = (|x|, Ta)$ and consequently $(|x|, Ta) = 0$ so that we have $a \in$ N(x). Therefore, N(x) is a normal manifold by Nakano's theorem 〔2, Theorem 4.9〕.

Lemma 2. *If* X *satisfies the assumption* (H), *then it follows that* N$(a)^\perp \cdot$ N$(b)^\perp = \{0\}$ *whenever* $0 \leqq a, b \in$ X *and* $a \cap b = 0$.

Proof. Putting $\tilde{\mathrm{N}}(a) = \{f \text{X}'; (a, |f|) = 0\}$ and $\tilde{\mathrm{N}}(b) = \{f \in \text{X}'; (b, |f|) = 0\}$, we get $\tilde{\mathrm{N}}(a) = \{Tx; x \in \mathrm{N}(a)\}$ and $\tilde{\mathrm{N}}(b) = \{Tx; x \in \mathrm{N}(b)\}$ by $|Tx| = T|x|$. For any positive linear funtional h on X, we can make two positive linear functionals $h_1 = h[b]$ and $h_2 = h[a] = h[\{a\}^\perp][\{b\}^\perp]$ **). Then, it follows that $(x, h_1) + (x, h_2) = ([b]x + [a]x + (1 - [b])x, h)$ *for* every x in X, because $a \cap b = 0$ implies $[a][b] = 0$. And hence we have $h = h_1 + h_2$ in X′. Furthermore, since $(a, h_1) = ([b]a, h) = 0$ and $(b, h_2) = ([a]b, h) + [\{a\}^\perp][\{b\}^\perp]b, h) = 0$, we have $h_1 \in$ N(a) and $h_2 \in$ N(b). By this fact, for any positive element z in X, the positive linear functional $h = Tz$ is represented such that $h = h_1 + h_2$ where $0 \leqq h_1 \in$ N(a), $0 \leqq h_2 \in$ N(b) and consequently we have $z = z_1 + z_2$ where $0 \leqq z_1 = T^{-1}h_1 \in N(a)$ and $0 \leqq z_2 = T^{-1}h_2 \in$ N(b). If a positive element z is belong to the intersection N$(a)^\perp \cdot$ N$(b)^\perp$, then, z is orthogonal to both z_1 and z_2 by the definition of the orthogonal complement so that z is orthogonal to z and

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**) For a linear functional f on X and a projection operator 〔M〕, f〔M〕 means a linear functional such that $(x, f[\text{M}]) = ([\text{M}]x, f)$ for every x in X.

hence we get $z=0$. Since N $(a)^{\perp}$ and N $(b)^{\perp}$ are semi-normal manifolds, the intersection N $(a)^{\perp}\cdot$ N $(b)^{\perp}$ is also a semi-normal manifold. Therefore, for any x in N $(a)^{\perp}\cdot$ N $(b)^{\perp}$ the absolute $|x|$ is also belong to N $(a)^{\perp}\cdot$ N $(b)^{\perp}$ so that by the above mentioned fact, we have $x=0$. This is complete the proof of the lemma.

Lemma 3. *If* X *satisfies* (H), *then for every orthogonal elements* x, y *in* X *we have* $(x, Ty)=0$.

Proof. Let x and y be positive elements and be mutually orthogonal. If $(x, Ty)\neq 0$, then putting $y_1=[\mathrm{N}(x)]y$ and $y_2=[\mathrm{N}(x)^{\perp}]y$, y_2 is belong to N $(x)^{\perp}$ and $y_2>0$ by the definition of N (x). On the other hand, we have $\mathrm{N}(x)^{\perp}\subset\mathrm{N}(y)^{\perp\perp}=\mathrm{N}(y)$ from the result $\mathrm{N}(x)^{\perp}\cdot\mathrm{N}(y)^{\perp}=\{0\}$ in Lemma 2. Therefore, we have $y_2\in\mathrm{N}(x)^{\perp}\subset\mathrm{N}(y)$ which contradicts to (H), because $0\leqq(y_2, Ty_2)\leqq(y, Ty_2)=0$ and $0<y_2$. Thus, we get $(x, Ty)=0$ for positive orthogonal elements x, y in X. For any orthogonal elements x, y in X, we obtain $(|x|, T|y|)=0$ so that the values (x^+, Ty^+), (x, Ty^-), (x^-, Ty^+) and (x^-, Ty^-) are all zero by the positivity of T. Consequently, we get the disired result $(x, Ty)=0$ by the linearity of T.

In the following, we shall give a proof of the theorem as aforesaid.

We define a functional $[\cdot, \cdot]$ on X×X as

$$[x,y]=((x, Ty)+(y, Tx))/2 \quad \text{for } x,y\in \mathrm{X}.$$

We can see first that the Schwarz's inequality

$$(1) \qquad [x,y]^2\leqq[x,x][y,y]$$

is satisfies. If X is one dimentional, then the equality holds in (1). Therefore, let X be at least two dimentional. It is enough to prove for linearly independent x, y in X. Since $x+\lambda y\neq 0$ for every real λ, by (H), we have $(x+\lambda y, T(x+\lambda y))\neq 0$. Now, by a simple caluculation, it follows that $(x+\lambda y, T(x+\lambda y))=[x, x]^2+2\lambda[x, y]+[y, y]^2$ and it holds a definite sign for λ. Consequently. we have $[x, y]^2-[x, x][y, y]<0$, because $[x, x]$ and $[y, y]$ are both non-zero by (H). Thus. the Schwarz's inequarity is proved.

Let x be a non-zero element in X. Taking a positive element y in X such that x, y are linearly independent, the above ineqality $[x, y]^2<[x, x][y, y]$ and $0<[y, y]$ yield $0<[x, x]$. Namely, we have

$$(2) \qquad [x, x]>0 \quad \text{for every } 0\neq x\in \mathrm{X}.$$

The following properties (3) and (4) are evident from the definition of $[\cdot, \cdot]$.

$$(3) \qquad [x, y]=[y, x] \quad \text{for every } x, y\in \mathrm{X}.$$

(4) $[\cdot, \cdot]$ is a bilinear form on X×X.

Therefore, $[\cdot, \cdot]$ defines an inner product on X×X and hence X becomes a preHilbert space.

By Lemma 3, it follows that for any $x\in$ X, $(x^+, Tx^-)=(x^-, Tx^+)=0$ so that $(x, Tx)=(|x|, T|x|)$. Therefore, we have

$$(5) \qquad [x, x]=[|x|, |x|] \quad \text{for each } x\in \mathrm{X}.$$

and the norm $\|x\|=[x, x]^{1/2}$ has the property

$$(6) \qquad \|x\|=\||x|\| \quad \text{for each } x \ \mathrm{X}.$$

By this fact, X becomes a normed lattice in the sense of Nakano [2].

Lemma 4. *Each element* f *in the order dual* X′ *is a norm bounded linear functional on* X.

Proof. Let f be a positive linear functional. Putting $u=T^{-1}f$, u is a positive element in X. Then, by (1), we have $|(x, f)| = |(x, Tu)| \leqq (|x|, Tu) \leqq [|x|, u] \leqq \|x\|\|u\|$ for every x in X. This shows that f is norm bounded. For any f in X′, the norm boundedness of f is evident, since f is always represented as difference of two positive elements f^+ and f^-, i. e., $f=f^+-f^-$ in X′. (Cf. see [3, p. 27])

Lemma 5. *The space* X *is complete with respect to the norm induced by the inner product* $[\cdot, \cdot]$

Proof. First, we shall show that the norm is continuous, i. e., $a_n \in X$ $(n=0, 1,\cdots)$ and $a_n \downarrow 0$ implies $\lim\|a_n\|=0$. Under the assumption, we have, by Lemma 1, $\inf\{(a_1, Ta_n)\ ;\ n=1, 2,\ldots\}=0$ and $0 \leqq (a_n, Ta_n) \leqq (a_1, Ta_n)$ for every n. Therefore, we have $\lim\|a_n\|=0$ by the definition of the norm. Next, we shall show that the norm is monotone complete, i. e., if $0 \leqq a_n\uparrow$, $a_n \in X$ and $\sup\{\|a_n\|\ ;\ n=0, 1,\ldots\} < +\infty$, then there exists an element $a \in X$ such that $a_n \uparrow a$. For each positive element x in X, $\{(x, Ta_n)\}$ is a non-decreasing and bounded, because $0 \leqq (x, Ta_n) \leqq [x, a_n] \leqq (\|x\|\,\|a_n\|)^{1/2} < +\infty$ by (1). Therefore, there exists the limit (x, f) such that $\lim\,(x, Ta_n)=(x, f)$. For any x in X, putting $(x, f)=(x^+, f)-(x^-, f)$ we get a positive linear functional f on X by Nakano[2, Theorem 18.2]. If we put $u=T^{-1}f$, we have $a_n \leqq u$ for every n and hence there exists an element a in X such that $a_n \uparrow a$ since X is Dedekind complete. Thus, the norm is complete by Nakano's theorem [2, Theorem 30. 17] on the normed lattice.

We have shown that X is the Hilbert space with the inner product$[\cdot, \cdot]$. Finally, we shall show that the inner product $[\cdot, \cdot]$ satisfies the properties (a) and (b). The property (a) is shown from the definition of $[\cdot, \cdot]$. Let $f \geqq$ o be an element in X′. Since f is norm bounded by Lemma 4, there exists an element a in X such that $(x, f)=[a, x]$for every x in X by the Riesz's representation theorem. From the positivity of f, we have $0 \leqq [a, x]=[a^+, x]-[a^-, x]$ for every positive x in X and hence $0 \leqq [a^-, a^-] \leqq [a^+, a^-]=0$ by Lemma 3. Thus, it follows that $a^-=0$ from (2) and hence $a=a^+ \geqq 0$. This shows that the property (b) is satisfied. The proof of the theorem is completed.

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References

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