Devices for Mathematical Ratiocination

Devices for Mathematical Ratiocination

Yoshio Kinokuniya*

Abstract

The principles of induction and reduction in the philosophical meaning are studied to give some important devices for the objectivist foundations of mathematics. Some redundances of assumption are taken up to promote estimation schemes. Finally, some relative events are discussed with a special view to the methodological dualism.

0. Introduction

If we simply denote by $E(p)$ the assembly of events which fulfil a property p, it may not be other than a mere abstract designation. However, if there factually is found a certain event e fulfilling p, then we may certainly have

$$e \in E(p) \tag{0.1}$$

and may be convinced that

$$E(p) \neq \phi.$$

In this case e is considered as an evidence for the fact (0.1), and $E(p)$ as the extension of $\{e\}$. This procedure of conception will generally underlie mathematical analyses. If $E(p)$ is proved to make a determinate set, it is called the *range* of p.

Now we take up the procedure of '*induction*', which may be defined as follows : By *induction* is meant argument from the particular to the more general concept.[1] According to this definition, the above-mentioned argument may be considered as an induction from $\{e\}$ to $E(p)$. The only problem in here is to examine if $E(p)$ may be considered as a determinate set.

If an assembly $L(p)$ is put to be such that only and all the events of $L(p)$ are capable of being examined on whether p is fulfilled by them or not, and if $L(p)$ is a (determinate) set, then $L(p)$ is called the *level* of p and p is said to be *levelized* by $L(p)$. If $L(p)$ and $E(p)$ are both determinate, and if every element of $L(p)-E(p)$ does not fulfil p, then p is called an *objectivist property.* If no fear of confusion, by 'a property' we mean an objectivist one.

If $E(c)$ is the total aggregate of events which are to be caused by a certain cause c, then $E(c)$ is also called the *range* of c on condition that $E(c)$ is determinate. Induction and levelization too, are analogously argued about c to the case of a property. If an event e is considered to be caused by either one of n causes $c_1, c_2, \cdots, c_n$, and if on examination we find that

---

* 紀國谷 芳雄

$$e \in E(c_1) \cap \cdots\cdots\cdots \cap E(c_m) \tag{0.2}$$

while

$$e \notin E(c_{m+1}) \cup \cdots\cdots\cdots \cup E(c_n),$$

then the cause

$$c_{m+1} \vee \cdots\cdots\cdots\cdots \vee c_n$$

is said to make a *redundant part* of causality for e. The argument which concludes the part

$$c_1 \vee \cdots\cdots\cdots \vee c_m$$

to be sufficient to cause the event e instead of

$$c_1 \vee \cdots\cdots\cdots \vee c_n,$$

is considered as a *reduction,*

In logical calculus, if the clauses '$Pabc$' and '$\sim Pxyz. \wedge. Pyxz$' where p is a predicate and x, y, z are unknown while a, b, c are constants, are given as premises, then it is concluded that $\sim Pbac$. Such a procedure may also be considered as a reduction.

1. Maximization

Given a property p concernig a set, if a set E is found to fulfil p, E is an evidence for p, and thus, if we denote by $\boldsymbol{E}(p)$ the class of sets which fulfil p, we apparently have

$$E \in \boldsymbol{E}(p) \tag{1.1}$$

and then we may be convinced that

$$\boldsymbol{E}(p) \neq \phi.$$

When a set E fulfils the property p, we write

$$E \subset p,$$

which means the same thing with the relation (1.1). If for any two sets A, B (in a certain universe) we always have

$$A \subset B. \&. B \subset p: \Rightarrow. A \subset p,$$

then p is a *regressive property* (of a set). Suppose that the property p is regressive and that there is a family of sets $\boldsymbol{M}$ which satisfies the following conditions :

(i) $\boldsymbol{M} \subset \boldsymbol{E}(p)$;

(ii) $A, B \in \boldsymbol{M}. \&. A \neq B: \Rightarrow: A \subset B. \vee. B \subset A$;

(iii) $U(p) = \cup_{A \in M} A. \&. U(p) \subset K. \&. U(p) \neq K: \Rightarrow. K \not\subset p$.

Such a family $\boldsymbol{M}$ is called an *increase completion* in respect of p, and if $U(p)$ can be considered to be determinate as the sum[2] of $\boldsymbol{M}$, then p is said to be *maximizable* in respect of p.

Now, following facts are provable :

(I) a regressive property p is not always maximizable ;

(II) even if p is maximizable, we do not always have

$$U(p) \subset p.$$

(I) may be verified by the following counterexample. Let us define a property p such that for any subset A of the closed interval $[0,1]$

$$A \subset p. = .\tilde{m}A = 0.^{*)}$$

Then, if $A_0 = \{0, 1\}$ we certainly have $A_0 \subset p$, and we easily see that p is regressive. Now suppose that p is maximizable and $\boldsymbol{M}$ is an increase completion in respect of p. Then it must be that

$$\tilde{m}U(p) \neq 0, \tag{1.2}$$

because : if $\tilde{m}U(p)=0$, then $[0,1]-U(p) \neq \phi$, so that we may take out a point $\beta \in [0,1]-U(p)$ and define U_1 by

$$U_1 = U(p) \cup \{\beta\}$$

for which we apparently have

$$\tilde{m}U_1 = 0.$$

Hence we see here the condition (iii) is not satisfied. On the other hand, for any increasing sequence of sets $(A_k)(k=1, 2, \cdots)$ taken from the family $\boldsymbol{M}$, we have

$$\tilde{m}A_k = 0,$$

hence

$$\tilde{m}(\cup A_k) = 0.$$

Thus, in the light of (1.2), we see that $\boldsymbol{M}$ cannot be a summable family[2], so that $U(p)$ cannot be the sum of $\boldsymbol{M}$. Consequently, p cannot be maximizable.

A counterexample verifying (II) will be shown at the end of the next section.

2. Probabilist Unionization

If there is promised the one and only one ticket to be found as the winning one among n tickets, and if the probability of winning is stipulated as to be uniformly equal for every ticket to the same value p_n, then we have

$$p_n = \frac{1}{n} \tag{2.1}$$

hence

$$\lim p_n = 0. \tag{2.2}$$

However, if we leave the stipulation (2.1) unapplied and directly watch the factual condition, it must utterly be essential that there exists the only one winning ticket. So, let this one be the kth ticket. Then, the other tickets which are not kth, must make up together the redundant part for the probability of winning. Thus the situation must be such that

$$p_n(k) = 1$$

and

$$p_n(j) = 0 \quad (j \neq k)$$

where $p_n(j)$ means the real probability of winning for the jth ticket ($j=1, \cdots, n$).

There has been an argument that, in the light of the evaluation (2.2), we may assert that no ticket would win in the limitless case ; which has been called

---

*) $\tilde{m}$ means the a priori measure which is a generalized extension of Lebesgue measure.

the *lottery paradox*. However, we may say that the stipulation (2.1) should not be so sophistically (or psychologically) treated as such. On such a procedure as (2.1), the probability for a single individual will lose its significance but that it is related to the others in a unionized way. In effect, since by (2.1) we always have

$$np_n=1,$$

the relation

$$\lim np_n=1$$

should also be maintained. So, we may assert that the total possibility of winning is always equal to 1. Hence it must be denied that no ticket would win.

Some similar situation to the above-mentioned is observed in the negation of total additivity of a homogeneous probability measure. If for any two sets A and B of real numbers we have

$$P_r(X\in A):P_r(X\in B)=\tilde{m}A:\tilde{m}B$$

on condition that both $\tilde{m}A$ and $\tilde{m}B$ are finite, then the aleatory variable X is homogeneous and $P_r(X\in A)$ is a homogeneous probability measure. In this case, denoting as $I=(-\infty, \infty)$ and $I_k=\{x \mid k-1\leqslant x<k\}$, we may easily see that

$$P_r(X\in I_k)=0 \quad \text{for} \quad \text{all} \ k=1,2,\cdots,$$

so that

$$\sum_{k=1}^{\infty}P_r(X\in I_k)=0\neq 1=P_r(X\in I).$$

Thus, it has been asserted that the measure $P_r(X\in A)$ cannot be totally additive.

On this problem, parallel to the recognition that

$$P_r(X\in I)=1,$$

we must attach importance to the fact that the total accumulation of the events

$$X\in I_k\ (k=1,2,\cdots)$$

must make an equivalence to the probability of the event

$$X\in I.$$

So then, we sould in any way interpolate the relation

$$I=\bigcup I_k \tag{2.3}$$

in the argument. In this context the summands

$$P_r(X\in I_k)\ (k=1,2,\cdots)$$

should be considered as infinitesimal quantities to make up the total value 1 in connection with (2.3). Hence, let us represent this constitution as

$$P_r(X\in I)=(\mathrm{u})\sum_k P_r(X\in I_k)$$

and call the right side a *unionized summation*. Then, we will naturally have the following relations to be asserted :

$$\text{(i)}\ (\mathrm{u})\sum_{k=1}^{n}P_r(X\in I_k)=0 \quad \text{and} \quad (\mathrm{u})\sum_{k=1}^{\infty}P_r(X\in I_k)=1 \quad (n=1,\ 2,\cdots);$$

$$\text{(ii)} \quad (\mathrm{u})\sum_{k=1}^{\infty} P_r(X \in I_{k\nu}) = \frac{1}{\nu} \quad (\nu = 1, 2, \cdots), \text{etc.}$$

In the above case, if we define a property p such that

$$A \subset p. \equiv . P_r(X \in A) = 0,$$

then p is regressive. Then, it can easily proved that p is maximizable but $U(p) \not\subset p$; which verifies the fact (II) shown in Sect. 1.

3. Redundance in Estimation

In case of the lottery problem, if every real number is a ticket index, and if the ticket indexed by α is the one and only one winning ticket, then we have

$$p_\alpha = 1, p_x = 0 (x \neq \alpha),$$

p_x being the probability of winning of the ticket indexed by x. So, the tickets not indexed by α will factually make up the redundant part of this lottery. Such a redundance is called an *intrinsic redundance*.

Conversely, when we do not know there is the one and only one winning ticket, no other way than the homogeneous estimation is allowed, that is, we can expect no other values than the same one for every p_x. Thus, p_x must necessarily be an infinitesimal quantity. So, if we write it as

$$p_x = \partial p,$$

the constitution of our estimation may be realized by introduction of the *unionized integration*

$$(\mathrm{u}) \int_{x=-\infty}^{\infty} \partial p = 1$$

which may be defined analogously to the concept of the unionized summation. In this context, the intrinsic redundance disappears, but we may say that there instead is an *implicit redundance* observed everywhere homogeneous.

In case of an estimation of some experimental trials, we may draw upon another sort of redundance which grows up practically. For instance, in the case of urn-sampling, if it is unquestionable that the urn contains exactly two balls which are either white or black, then the following three cases are possible : (1) w, w ; (2) w, b ; (3) b, b (w meaning a white ball and b a black one). Suppose that we gain a sample by two times of drawing with replacement. Then the sample will justly be either of the patterns (1), (2), and (3). If the sample is (1), then the urn is possibly estimated to be either (1) or (2), so that the case (3) makes the redundant part ; if the sample is (3), (2) and (3) are possible and (1) is redundant ; and finally, if the sample is (2), the content of the urn is exactly known to be (2) itself. Each redundance above-mentioned may be considered as a sort of *informational redundance* reduced by sampling.

Incidentally, redundance may be considered as an essential source of theoretical noise. When an assumed object is not yet ascertined to be really existent, it may at most give a theoretical noise, and if it is in fact inexistent it must be a redundance to the conception.

When we want to conjecture the cause of an event e, if it is unquestionable that the cause is to be found among m causes $c_1, c_2, \cdots, c_m$, we may refer to the following formula due to T. Bayes :

$$P(c_k,e)=\frac{P(c_k)P(e,c_k)}{\sum_{j=1}^{m}P(c_j)P(e,c_j)}$$

where $P(c_k, e)$ is the probability that e is caused by c_k, $P(e, c_k)$ the probability that c_k causes e, and $P(c_k)$ is the a priori probability by which c_k is to be expected to occur. However, it is usually noted that the assignment of values for $P(c_k)$ ($k=1, 2, \cdots, m$) may scarcely be decided with assurance. Then, the authenticity of the formula is considered to be very faint. Thus being the condition, it may be proposed that the principle of reduction suggested by (0.2) is preferable as a sounder one.

4. Methodological Dualism

In actual mathematical inquiries, there has existed a curious methodological dualism which is distinguished by the opposition between heuristic precept and examinative principles[3]; which may be restated in a practical sketch as follows : for an inquiry, to detect a solution is essentially a different thing from having a demonstrative way to reach a solution. Such a gap between solutions and the demonstrative procedures to obtain them should be eliminated somehow. For this purpose, it seems firstly requisite to study into the total aggregate of solutions of the given inquiry.

Given a special property p, if at least one event e is found fulfilling it, $\{e\}$ may possibly be extended to the locus of p

$$E(p)=\{\alpha|\ \alpha\subset p\}.$$

If $E(p)$ is determinate, then it may certainly be considered as the range of p. However, if $E(p)$ is not allowed to be so, p itself cannot be considered as an objectivist property. Thus being the condition, a simply assumed range $E(p)$ is essentially no more than a mere object abstracted in the annexed set theory[4]. So, in this situation, we may see a dualism between the given property p and its objectivistness which is to be inspected by the examination of $E(p)$; which may eventually be taken as a dualism between p and $E(p)$.

In the theory of operations (or mappings), we have a dualism more complex. Given an operation f, firstly assume that there exists at least one pair of events α and β such that

$$\beta=f(\alpha).$$

Then we may possibly suppose a (determinate) set A such that

$$(\forall\alpha\in A)(f(\alpha)\text{exists}),$$

which is called a *domain* of f if the aggregate

$$f(A)=\{\beta \mid \beta=f(\alpha), \alpha \in A\}$$

is also a set. In this case $f(A)$ is certainly the range of f on the domain A. Such being the condition, we see that the problem whether a given set A may be thought as a domain of f simultaneously draws upon the problem whether a set B may determinately exist as a range of f, that is $B=f(A)$. So then we may see a dualism between A and $f(A)$.

The way starting from the set

$$f(\{\alpha\})=\{\beta \mid \beta=f(\alpha)\}$$

to obtain its extension

$$f(A)=\{y \mid y=f(x), x \in A\},$$

necessitates not only the existence of y's but also the existence of a (determinate) universal set Y such that

$$(\exists x)(y=f(x)) \Rightarrow . y \in Y. \tag{4.1}$$

Then, by taking the dualism, we may moreover have a universal set X to be necessary such that

$$(\exists y)(x=f^{-1}(y)) \Rightarrow . x \in X. \tag{4.2}$$

So, we may say that y is bred by f through the condition (4.1), and inversely x is bred by f^{-1} through the condition (4.2).

If there is an α such that $\beta=f(\alpha)\notin Y$, then β must be but a fictitious object fabricated against α. If we, notwithstanding the criticism, require the existence of such a β, it must be that we create a new element by β. Of course, such a β may not be expected to fit in with the proto-construction[5](or the present system of construction). Thus, on addition of the new element the proto-construction will be revised and extented to a new system which may comprehend β well together with the previous elements. After duch a revision, the universe X too may possibly be changed. If it is to be such that $X=Y$, then the same new elements must be adjoined to both X and Y. A good example of such a case is given by the adjunction of $\sqrt{-1}$ to the real numbers.

If the set $f(A)$ cannot exactly be determined though its existence cannot be denied, then $f(A)$ will possibly be considered as an undecidable object.

Mathematical Seminar of the Muroran Inst. Tch., Hokkaido

(*Received Apr.* 20, 1978)

References

1) Kneebone, G. T., *Mathematical Logic and the Foundations of Mathematics*, 6 (1963).
2) Kinokuniya, Y., Mem. Muroran Inst. Tch. 8(1), 31 (1973).
3) Beth, E. B. & J. Piaget, *Mathematical Epistemology and Psychology,* 93 (1966).
4) Kinokuniya, Y., Mem. Muroran Inst. Tch. 9(1), 328 (1976).
5) Kinokuniya, Y., Mem. Muroran Inst. Tch. 8(3), 730 (1976).