The Application of Galerkin Finite Element Model to Salt Water Dispersion in Porous Media

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The Application of Galerkin Finite Element Model to Salt Water Dispersion in Porous Media

Satoshi Tohma

Abstract

A Galerkin finite element model is applied to investigate the groundwater flow and the salt transport in porous media. The mathematical formulation of this model consists of the coupled form of the flow equation and the convective diffusion equation. The present model uses two dimensional quadrilateral, isoparametric elements with a linear basis function for the salt concentration and the two velocity components. The resultant governing equation is nonlinear owing to the convective terms and is solved at each time step by the iterative method.

A demonstrative sample problem is presented.

1. Introduction

Numerical analysis of the solute transport in groundwater systems has received the attention of many investigators. The two general dimensional models for these problems have been developed and currently being tested. Peaceman and Rachford[1](1962) presented a method based on the finite difference scheme for solving the flow-transport problem. Recently, Lee and Cheng[2](1974), and Huyakorn and Taylor[3](1976) obtained a steady state solution of the solute transport using the variational finite element approximation. For transient processes , Galerkin type finite element technique is derived by Gray and Pinder[4] (1974), and Desai and Contractor[5](1976).

The purpose of this paper is to investigate the steady and transient movement and distribution of a dissolved salt in a saturated unconfined aquifer by means of Galerkin finite element model. To describe the transport of miscible fluids of differing densities, such as salt water and fresh water, the transport equation for the salt and the flow equation for the groundwater must be coupled. As the system is nonlinear, an iterative procedure is adopted to obtain a compatible solution between velocities and concentration. For prescribed initial concentrations, the velocities are first obtained by means of Darcy's law in the flow model. The resulting velocities are used to solve the salt concentration in the transport model. The iterative procedure is repeated until the successive values of the concentration are within a specified tolerance.

2. Modeling Approach

Hydrodynamic dispersion phenomenon is affected by convection, by turblent nature and by the irregularities of the pores in the medium.

The modeling approach developed in this paper is based on the following consideration:
(1) The density of the fluid depends on the concentration.
(2) The contribution of the molecular diffusion to the hydrodynamic dispersion is negligible when compared to the convective diffusion.
(3) The transverse dispersion is less than the longitudinal dispersion.

2-1. Theoretical development

The governing equations for the salt dispersion in porous media can be derived using the basic conservation law and Darcian flow concepts. The flow equation of the saturated aquifer can be written as,

$$V = -\frac{k}{\lambda\mu}(\nabla p + \rho g \nabla y) \qquad \cdots (1)$$

where V is seepage velocity (L/T), k is intrinsic permeabilty (L²),

   p is pressure (M/LT²), g is acceleration of gravity (L/T²),

   λ is porosity, μ is dynamic viscosity (M/LT),

   ρ is fluid density (M/L³) and y is height from the datum line (L).

The conservation equation of the salt water may be written as,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho V) = 0 \qquad \cdots (2)$$

As the movement of the dissolved salt is so slow relative to the rate of propagation of the pressure change, the first term in the Eq.(2) becomes almost negligible compared with the another term.

Therefore, Eq.(2) reduces to

$$\nabla \cdot (\rho V) = \rho \nabla \cdot V + V \cdot \nabla \rho = 0 \qquad \cdots (3)$$

The transport equation for the dissolved salt obtained by substituting the constitutive relationship

$$\rho_s V_s = \rho_s V - D \cdot \nabla \rho_s \qquad \cdots (4)$$

where ρ_s is the mass concentration of the dissolved salt (M/L³),

   ρ is the density of mixture (M/L³),

   V_s is the mass phase average velocity of the dissolved salt (L/T),

   V is the mass phase average velocity of mixture (L/T) and

   D is hydrodynamic dispersion coefficient (L²/T).

The hydrodynamic dispersion coefficient may be expressed as

$$D = D_m + D^* \qquad \cdots (5)$$

where $\mathbf{D}_m$ is molecular diffusion coefficient (L²/T) and

$\mathbf{D}^*$ is convective diffusion (mechanical dispersion) coefficient(L²/T).

Since the contribution of the molecular diffusion to the hydrodynamic dispersion is negligible, the hydrodynamic dispersion coefficient can be estimated as

$$D \simeq D^* \qquad \cdots (6)$$

The conservation equation of the dissolved salt may be expessed as

$$\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \cdot V_s) = 0 \qquad \cdots (7)$$

On substituting (4) into (7), one obtains the transport equation for the salt,

$$\frac{\partial \rho_s}{\partial t} + \rho_s \nabla \cdot V + V \cdot \nabla \rho_s - \nabla \cdot (D \cdot \nabla \rho_s) = 0 \qquad \cdots (8)$$

The additional condition requires to obtain a solution to these equations.
In this analysis, the fluid density is assumed to be a function of concentration as follows:

$$\rho = \rho_0 + (1-E)\rho_s \qquad \cdots (9)$$

where ρ_0 is the density of fresh water and E is an empirical constant and has a value of 0.3 for sea water and ρ_s is the mass concentration of the dissolved salt.

2-2. Galerkin finite element method

In most of the formulation of the fluid flow problems using the finite element method, a variational principle must be found, and the requirement of the variational principle restricts the applicability of the finite element method to self-adjoint partial differential equations.

Unfortunately, for the dispersion problems neither variational principle exists, mor it is not positive definite so that the computational advantage of the variational finite element method are lost.

Galerkin's procedure is a means of obtaining an approximate solution to a differential equation by requiring that the errors between the approximate solution and the true solution are orthogonal to the functions used in the approximation.[6]
Eqs.(1) and (3) can be rewritten as

$$L_1(V, p) = V + \frac{k}{\lambda \mu}(\nabla p + \rho g \nabla y) = 0 \qquad \cdots (10)$$

$$L_2(V, \rho) = \rho \nabla \cdot V + V \cdot \nabla \rho = 0 \qquad \cdots (11)$$

in which L_i is a differential operator.
Similarly, applying the Galerkin's procedure to Eq.(8), one obtains,

$$L_3(V, c) = \frac{\partial c}{\partial t} + c\nabla \cdot V + V \cdot \nabla c - \nabla \cdot (D \cdot \nabla c) = 0 \qquad \cdots (12)$$

where c is mass concentration and is the same quantity that the symbol ρ_s in the Eq.(8).
To solve the above sets of equations using the Galerkin's procedure, approximate solutions of the velocity components, pressure and concentration can be written by linear basis functions

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of the general form:

$$\bar{V} = \sum_{i=1}^{4} N_i(x, \; y) V_i$$

$$\bar{p} = \sum_{i=1}^{4} N_i(x, \; y) p_i \qquad \cdots (13)$$

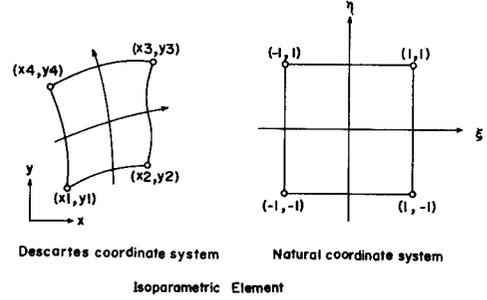$$\bar{c} = \sum_{i=1}^{4} N_i(x, \; y) c_i$$

Fig. 1　Isoparametric Element

where N_i is the shape function and i stands for the corners of the quadrilateral element.
The shape function N_i may be expressed as follows:

$$N_i = \frac{1}{4}(1 + \xi\xi_i)(1 + \eta\eta_i) \qquad \cdots (14)$$

where ξ and η are the natural coordinate variables and the simplified model of this system is shown for **Fig.1**.
Substitution of Eq.(13) into Eqs.(10) through(12) result in the residuals ε_i,

$$\varepsilon_1 = L(\bar{V}, \; \bar{p}) - V - \frac{k}{\lambda\mu}(\nabla p + \rho g \nabla y) \neq 0 \qquad \cdots (15)$$

$$\varepsilon_2 = L(\bar{V}, \; \rho) - \rho\nabla \cdot V - V \cdot \nabla\rho \neq 0 \qquad \cdots (16)$$

$$\varepsilon_3 = L(\bar{V}, \; \bar{c}) - \frac{\partial c}{\partial t} - c\nabla \cdot V - V \cdot \nabla c + \nabla \cdot (D \cdot \nabla c) \neq 0 \qquad \cdots (17)$$

The residuals are forced to be zero by setting weighted integrals of the residual.

$$\int_R \varepsilon_1 W_i dR = 0 \qquad \cdots (18)$$

$$\int_R \varepsilon_2 W_i dR = 0 \qquad \cdots (19)$$

$$\int_R \varepsilon_3 W_i dR = 0 \qquad \cdots (20)$$

where W_i is the weighting function and Galerkin's procedure is formulated by selecting the shape functions N_i as the weighting functions W_i and R is a bounded domain.
The resulting system corresponding to Eqs.(18) and (19), written in a compact matrix form, is

$$[A]\{X\}=\{F\} \qquad \cdots (21)$$

in which,

$$A_{ij}=\iint \begin{bmatrix} \sum_{k=1}^{4}\left(\rho_k N_k\dfrac{\partial N_j}{\partial x}+\rho_k\dfrac{\partial N_k}{\partial x}N_j\right)N_i & \sum_{k=1}^{4}\left(\rho_k N_k\dfrac{\partial N_j}{\partial y}+\rho_k\dfrac{\partial N_k}{\partial y}N_j\right)N_i & 0 \\[2ex] N_iN_j & 0 & N_i\dfrac{k}{\lambda\mu}\dfrac{\partial N_j}{\partial x} \\[2ex] 0 & N_iN_j & N_i\dfrac{k}{\lambda\mu}\dfrac{\partial N_j}{\partial y} \end{bmatrix} dxdy$$

$$X_i=\begin{Bmatrix} u_i \\ v_i \\ p_i \end{Bmatrix} \qquad F_i=\begin{Bmatrix} 0 \\ 0 \\ \iint N_i\dfrac{kg}{\mu}\sum_{k=1}^{4}\rho_k N_k\ dxdy \end{Bmatrix}$$

Similarly, the transport equation can be obtained as follows:

$$[B]\{C\}+[D]\{dc/dt\}=\{G\} \qquad \cdots (22)$$

where

$$B_{ij}=\iint \left[\sum_{k=1}^{4}\left\{\left(u_k\dfrac{\partial N_k}{\partial x}+v_k\dfrac{\partial N_k}{\partial y}\right)N_iN_j+\left(u_k N_k\dfrac{\partial N_j}{\partial x}+v_k N_k\dfrac{\partial N_j}{\partial y}\right)N_i\right\}\right.$$

$$\left.+\left(D_x\dfrac{\partial N_j}{\partial x}\dfrac{\partial N_i}{\partial x}\right)+\left(D_y\dfrac{\partial N_j}{\partial y}\dfrac{\partial N_i}{\partial y}\right)\right]dxdy$$

$$D_{ij}=\iint N_iN_j dxdy \qquad G_i=\int V_m.nN_i ds$$

where V_m is mass flux across s-boundary, n is the outward normal line at the boundary.

The Eq.(22) contains a time-derivative term. While the finite element approximation is very effective in the spatial derivatives, the finite difference scheme is generally used to handle the time derivative.

The time-derivative is replaced by a weighted finite difference scheme, Eq.(22) may be rewritten as

$$[B](W\{C\}_{t+\Delta t}+(1-W)\{C\}_t)+(1/\Delta t)[D](\{C\}_{t+\Delta t}-\{C\}_t)$$
$$=W\{G\}_{t+\Delta t}+(1-W)\{G\}_t \qquad (0\leq W\leq 1) \qquad \cdots (23)$$

where t is time level, Δt is a time step and W is the weight.
In the case of $W=0$, one calls this the explicit method.

Substituting zero for W in Eq.(23), one can get the following equation.

$$([\boldsymbol{B}]-(1/\varDelta t)[\boldsymbol{D}])\{\boldsymbol{C}\}_t+(1/\varDelta t)[\boldsymbol{D}]\{\boldsymbol{C}\}_{t+\varDelta t}=\{\boldsymbol{G}\}_t \qquad \cdots (24)$$

The explicit method generally requires a minimum of computational effort and is usually conditionally stable.

2-3. Computation process

1. For prescribed initial concentrations, the velocities are first computed using Eq.(21). In this step, matrix[**A**] possesses two undesirable properties:
it is unsymmetry and contains zero elements along the diagonal.
Consequently, the matrix must be rearranged so that zeros do not occur on the diagonal.

2. The resulting velocities are used in computing convective terms in the transport equation.   The concentration can be calculated using Eq.(22).   But the velocities are not compatible with the new concentrations.

3. Without advancing in time, the velocities are recomputed using the new concentrations and the cycle is completed by once again solving for the concentrations.   This iterative procedure is repeated until the successive values of concentrations are within a specified tolerance.   In this analysis, a specified tolerance is less than 1 percent of the difference in the dependent variables between successive iterations.

3. Numerical examples

A schematic representation of coastal aquifer is presented in **Fig. 2**.
The dimensions of the solution domain are 31 cm high and 200 cm long.
The dynamic and physical properties used in this analysis are as follows :
Hydraulic conductivities $K_x=K_y=0.29$ cm/sec. Longitudinal hydrodynamic dispersion coefficient $D_x=0.02$ cm²/sec and transverse hydrodynamic dispersion coefficient $D_y=0.002$ cm²/sec.   Porosity $\lambda=0.40$ and density of fresh water $\rho_f=1.00$ g_0/cm³ and salt water $\rho_s=1.025$ g_0/cm³.
In this figure, the mass concentration c is non-dimensionalized by introducing the following dimensionless variable.

$$c=c/c_s \qquad \cdots (25)$$

where c_s is mass concentration of sea water and has a value of 0.035 g/cm³.
The initial concentration in porous media and on the left-hand vertical boundary are c= 0.   For salt water influx on the part of the right-hand boundary, c=1.
Resulting isochlor lines for steady state problem are shown in **Fig. 2** together with the immiscible interface[7] in order to compare the differences to the result of the miscible fluid. At the same time, the velocity distribution of the steady state is shown in **Fig. 3**.   These figures lead to the following :
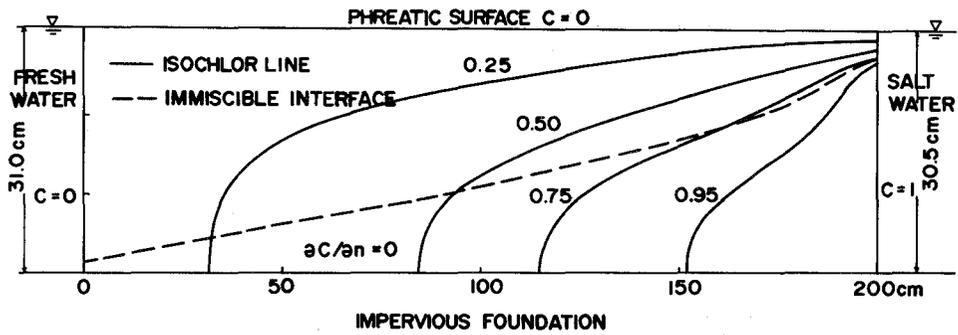The cyclic flow of the salt water in porous media can be found.   This circulation tends to

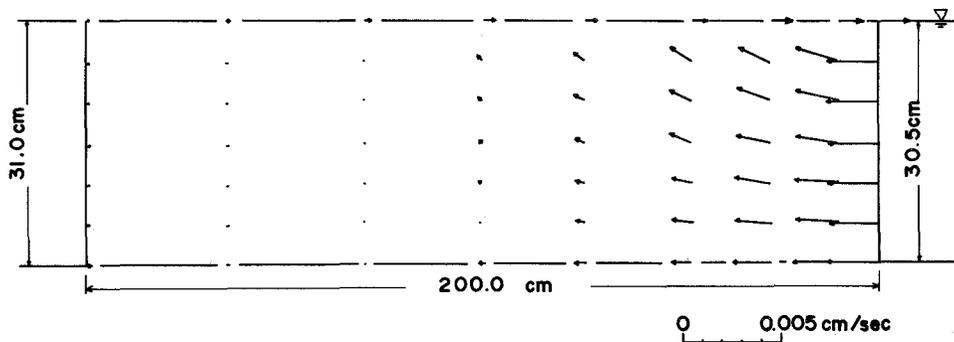**Fig. 2**   Steady state concentration distribution

Fig. 3   Steady state velocity distribution

limit the extent to which salt water penetrates into porous media.   As salt water and fresh water become intimately mixed in porous media, the diluted salt water is less dense than original salt water and rises along a left-ward path.   Consequently, dissolved salt is carried back to the salt water basin by the flow of fresh water.   In regard to the spatail distribution of each isochlor line, the intervals of lines are wider as the velocities become slower.   The front of line becomes vertical in the neighborhood of the base, because the base of porous media is impervious, and the horizontal velocity compoments are predominant.

On the other hand, for the immiscible fluid flow, as may be seen from **Fig. 2**, a abrupt sharp interface separates regions occupied by fluids of different density, and the salt water penetrates porous media to the fresh water basin.

This method is widely used presently to determine the location of the interface for the development of groundwater resouces, but salt water and fresh water actually mix in a region of dispersion.   In case the thickness of the zone of dispersion can not be ignored, the abrupt sharp interface seems to have a weak physical foundation.

Fig. 4 shows the displacement of 0.5-isochlor line up to the steady state.   In case of the transient state, one adopts the hydrodynamic dispersion coefficients in which $D_x=0.2$ cm^2/ sec and $D_y=0.02$ cm^2/sec.
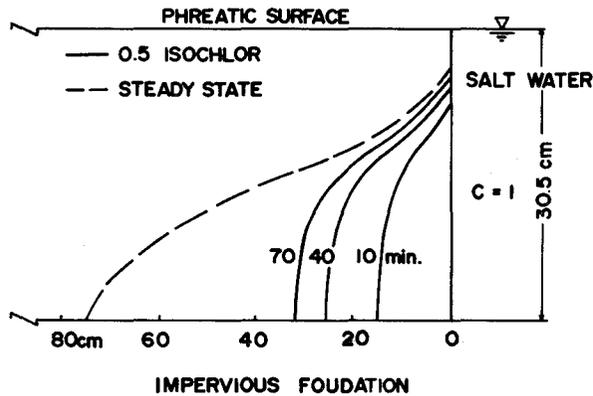
Fig. 4 Transient concentration distribution

As shown in **Fig. 4**, the rate of the intrusion is gradually reduced with time and an increase in dispersion coefficient value tends to lessen the extent to which 0.5-isochlor line occupied porous media when compared to the steady state.

In this paper, the hydrodynamic dispersion coefficients were taken as constant known values, but the coefficient of dispersion depends on the flow patten, on the Peclet number and on the porous medium characteristics.   Peclet number is the ratio of convective diffusion to molecular diffusion, $Pe = Vd/D_m$, where d is the' mean grain size.   As shown in **Fig. 3**, the maximum velocity is about 0.003cm/sec in the steady state.   For the molecular diffusion coefficient, $D_m = 1.5 \times 10^{-5}$ cm²/sec and the mean grain size, d=0.1 cm, Peclet number is Pe $\simeq 20$.

According to the summary of the experimental values obtained by Pfannkuch[8], for 5 < Pe < 300, the main spreading is caused by convective diffusion combined with transverse molecular diffusion.

Experimental results for this range yield :

$$D_x/D_m = \alpha(Pe)^m \; ; \; \alpha \simeq 0.5, \quad 1 < m < 1.2 \qquad \cdots (26)$$

Giving the exponent m=1.2 and the constant α=0.5, one obtains through Eq.(26) the longi tudinal dispersion coefficient $D_x \simeq 20 \times D_m = 0.0003$ cm²/sec.

According to the above results, the dispersion coefficient adopted in this example was overestimated.

4. Conclusions

The Galerkin type finite element model has been presented for the coupled two di mensional problems of the steady and the transient state flow with convective diffusion. Results of numerical analysis clearly illustrate that this model can handle the salt dispersion in porous media.   As the writer did not study the consideration based on the results of ex-

perimentation, the clear conclusion about the applicability of numerical analysis has not been drawn yet.

Though it is almost impossible to solve the dispersion problem analytically, this procedure will provide useful clues for future reseach and development and is one of methods required for predicting and controlling of the quality of groundwater reservoirs.

Acknowledgment

Reference

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