Invariant Extensions for Linear Functionals and Supplement to the Paper "Totally Ordered Linear Space Structures and Extension Theorems"

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Abstract

By modifying the preceding short work 33), some results for invariant extensions for linear functionals are furnished. A supplement is added to my 31).

Introduction. This paper has two aims. First, as a sequel to 33), from the viewpoint of the totally ordered linear space structures of $E \times \boldsymbol{R}$, some problems for invariant extensions of linear functionals are discussed. Subsequently, a correction to a part of my 31), is applied for. As for the former, our hypotheses introduced here seem to be somewhat general. To say more precisely, though e. g. in [33), Cor. 2],

1. *$\mathscr{T}$ is a semigroup of linear transformations of E into E,*
2. *$p(T(y)) \leqslant p(y)$ holds for $y \in E$, $T \in \mathscr{T}$,*
3. *f is invariant under $\mathscr{T}$,*

were assumed, but this time they are weakened to

1. *$\mathscr{T}$ is a set of linear transformations of E into E,*
2. *p is merely a gauge function on E,*
3. *f is merely a linear form on M.*

The main purpose of this part is just to enlarge [33), Cor. 2] in such directions. In consequence, our results relate to Klee [8), $(\alpha) \Leftrightarrow (\delta)$ of (2. 2) Theorem] and to Edwards [14), 3. 3. 2 Remark], and cover them. Some of our results are given in terms of ideals (right or left) of $\mathscr{T}$

For reference, as is apparent, our approaches to extension problems are (were) all based on the following self-contained angle (for this and for its topological version, cf. [21), Th. 2], [26), p. 46, foot-note and Suppl. to Th. 3], [32), Th. 1]).

THEOREM.† *Let E be a real linear space, C a subset of E. Let M be a linear subspace of E, f a non-identically-zero linear form on M. Designated by*

$(>)$ *f can be extended to a linear form F on E so as to $F(c) > 0$ for $c \in C$,*

† The latter statement of this theorem is a Krein type extension theorem. This part, non-topological version of Bauer-Namioka theorem (cf. [17), (V, 5. 4)]), and the some case of Anger-Lembcke [29), Th. 3. 2] are equivalent.

($\geqslant$) *f can be extended to a linear form F on E so as to $F(c)\geqslant 0$ for $c\in C$,*†

(1) *there exists a t.o.l.s. $(E, \mathscr{R})$ such that*

(i) $A\cup C\subset(E,\mathscr{R})^+$, *where* $A=\{x\in M: f(x)>0\}$,
(ii) $(E,\mathscr{R})^+$ *is absorbing at some point of M,*

(2) *there exists a convex absoring (at the origin) set U in E such that $A\cup C\cup(U+a_0)$ is positively independent in E, where $f(a_0)=1$,*

$(>)\Rightarrow(2)\Rightarrow(1)\Rightarrow(\geqslant)$ *holds. If C is a non-pointed convex cone (i.e., $C\ni 0$), $(\geqslant)$, (1), (2) are equivalent.*

It was at this point (the first statement of this theorem) that we were free from the convexity of $C_{pc\mathscr{L}}$ in the proof of [33), Cor. 2]. This like is given in the present work too. With this fact, it appears to me that this angle (including its proof) is somewhat good for our discussion.

To present this note, for the first part, the author was motivated by Agnew-Morse [6), Lemma 2.01], and 8), 14) above cited. For these instructive informations, he is deeply grateful to them.

Preliminaries. For convenience, unless otherwise specified, let the notations and terminology employed in 21), 26), 30), 32), and 33) be available. Especially for E, C, K, M, f, $\mathscr{L}$, L refer to the statement of [33), Theorem 1]. Denote by $\boldsymbol{N}$ the set of all positive integers. We introduce

DEFINITION 1. a) Let q be a *gauge function* on K.

b) By $\mathscr{E}$ is meant a set $(\neq\emptyset)$ of *linear transformations* of E into E such that $T(K)\subset K$ and $q(T(y))\leqslant q(y)$ $(y\in K)$ for all $T\in\mathscr{E}$.

c) By $\mathscr{T}$ is meant a set $(\neq\emptyset)$ of *linear transformations* of E into E such that $T(K)\subset K$ for all $T\in\mathscr{T}$.

d) The *identity map* of E to E is written by I. Since (1. 0) below is synonymous relative to $\mathscr{T}$ and $\mathscr{T}\cup\{I\}$, throughout, $\mathscr{T}$ is taken to be $\mathscr{T}\cup\{I\}$ to our advantage. So is the case for $\mathscr{E}$.

e) Let T_1, T_2, T_3, $\in\mathscr{T}$. $T_2(T_1(y))$ $(y\in E)$ is written by $T_2T_1(y)$. Obviously, *composite mapping* T_2T_1 is a linear transformation of E into E, and $T_3(T_2T_1)=(T_3T_2)T_1$ holds. By T_1^μ $(\mu\in\boldsymbol{N})$ is meant as usual. $\mathscr{T}$ is called *Abelian* if $T_2T_1=T_1T_2$ whenever T_1, $T_2\in\mathscr{T}$. A subset $\mathscr{I}$ of $\mathscr{T}$ is called a *right* (resp. *left*) *ideal* of $\mathscr{T}$ if $\mathscr{I}\mathscr{T}\subset\mathscr{I}$ (resp. if $\mathscr{T}\mathscr{I}\subset\mathscr{I}$), where $\mathscr{I}\mathscr{T}=\{T_2T_1 : T_1\in\mathscr{T}, T_2\in\mathscr{I}\}$ and such.

DEFINITION 2. a) $J_{qC\mathscr{E}}$ is another (*the 2nd*) *quasi-epigraph* of gauge function q on K with respect to C and $\mathscr{E}$: $J_{qC\mathscr{E}}=\Big\{(y,\eta)$: there exist $c\in C$, $T\in\mathscr{E}$, $m\in\boldsymbol{N}$, such that $y+c\in K$ and $\frac{1}{m}q\Big(\sum_{\mu=1}^{m}T^\mu(y+c)\Big)<\eta\Big\}$

† Our symbol "$\geqslant$" is syn. with "$\geqq$".

b) $K_{gC\mathscr{T}\mathscr{S}}$ is the *(3rd) quasi-epigraph* of gauge function g on K with respect to C, $\mathscr{T}$ $\mathscr{S}$: $K_{gC\mathscr{T}\mathscr{S}}=\left\{(y, \eta)\right.$: there exist $c\in C$, $T\in\mathscr{T}$, $m\in \boldsymbol{N}$, such that $y+c\in K$ and $\left.\frac{1}{m}\sup_{S\in\mathscr{S}} g\left(\sum_{\mu=1}^{m} ST^{\mu}(y+c)\right)<\eta\right\}$, where $\mathscr{S}$ is a certain non-void fixed subset of $\mathscr{T}$

c) If $C=\{0\}$, $J_{qC\mathscr{E}}$, $K_{gC\mathscr{T}\mathscr{S}}$ are resp. abbreviated to $J_{q\mathscr{E}}$, $K_{g\mathscr{T}\mathscr{S}}$.

We use $K_{gC\mathscr{T}\mathscr{S}}$ in (1. 1) below. In this case it is well-defined, i.e., $K_{gC\mathscr{T}\mathscr{S}}\subset L$. We note in advance that unlike $C_{gC\mathscr{S}}$, $K_{gC\mathscr{T}\mathscr{S}}$ is not necessarily convex even if $\mathscr{T}$ is Abelian.

Statement of the results. Some modifications of the preceding [33), Theorem 1] establish the following. This theorem simultaneously relates to [8), $(\alpha)\Leftrightarrow(\delta)$ of (2, 2) Theorem] and to [14), 3. 3. 2 Remark].

THEOREM 1. *Under the hypothesis of* [33), *Theorem* 1], *let* $\mathscr{L}$ *be rescinded and let* $\mathscr{T}$ *be considered in place of it. Then the following are equivalent :*

(1. 0) *The statement* (1. 0) *thereof remains valid for* $\mathscr{T}$.

(1. 1) *There exist a gauge function* g *on* K *and a non-void subset* $\mathscr{S}$ *of* $\mathscr{T}$ *such that* $g(ST^{\mu}(y))\leqslant q(y)$ *for* $y\in K$, $T\in\mathscr{T}$, $S\in\mathscr{S}$, $\mu\in \boldsymbol{N}$, *and there exists a t.o.l.s.* $(L, \mathscr{R})$ *whose* $(L, \mathscr{R})^{+}$ *is absorbing at* (0, 1) *for* L *for which*

$$B_{f}\cup K_{gC\mathscr{T}\mathscr{S}}\subset(L, \mathscr{R})^{+}.$$

(1. 2) *There exist a gauge function* g *on* K *and a non-void subset* $\mathscr{S}$ *of* $\mathscr{T}$ *such that* $g(ST^{\mu}(y))\leqslant q(y)$ *for* $y\in K$, $T\in\mathscr{T}$, $S\in\mathscr{S}$, $\mu\in \boldsymbol{N}$, *and there exists a convex absorbing set* U *in* E *for which* $B_{f}\cup K_{gC\mathscr{T}\mathscr{S}}\cup(U\times\{1\})$ *is positively independent in* L.

PROOF is done routinely : (1. 0)$\Rightarrow$(1. 2): Letting g be the restriction of F to K, choose a non-void subset $\mathscr{S}$ of $\mathscr{T}$ Then (1. 0) entails that $F(y)\leqslant\frac{1}{m}\sup_{S\in\mathscr{S}} g\left(\sum_{\mu=1}^{m} ST^{\mu}(y+c)\right)$ for $y+c\in K$, $y\in E$, $c\in C$, $T\in\mathscr{T}$ $m\in \boldsymbol{N}$ and that $g(S T^{\mu}(y))=g(y)\leqslant q(y)$ for $y\in K$, $T\in\mathscr{T}$, $S\in\mathscr{S}$, $\mu\in \boldsymbol{N}$ which show that g, $\mathscr{S}$ and $U=\{x\in E : F(x)<1\}$ are as required. For (1. 2)$\Rightarrow$(1. 1), under g, $\mathscr{S}$ of (1. 2), appeal to [30), Rem. 2] (an analogue of) and to [21), Lemma 1]. (1. 1)$\Rightarrow$(1. 0) : Likewise as in the case of [30), Th. 1 ("if" part)], indeed we obtain an $F_0\in E^{*}$ such that extending f and satisfying $F_0(y)\leqslant\frac{1}{m}\sup_{S\in\mathscr{S}} g\left(\sum_{\mu=1}^{m} ST^{\mu}(y+c)\right)$ $(y+c\in K, y\in E, c\in C, T\in\mathscr{T} m\in \boldsymbol{N})$. F_0 meets the (b) of (1. 0) is clear. F_0 does the (c) of (1. 0) is obtained thanks to but *mutatis mutandis* from the final part of the proof of Agnew-Morse [6), Lemma 2. 01]. (For the details, $F_0(y-T(y))\leqslant\frac{1}{m}\sup_{S\in\mathscr{S}} g\left(\sum_{\mu=1}^{m} ST^{\mu}(y-T(y))\right)=\frac{1}{m}\sup_{S\in\mathscr{S}} g(ST(y)-ST^{m+1}(y))\leqslant\frac{1}{m}\{q(y)+q(-y)\}$ follows for $y\in K$, $T\in\mathscr{T}$ $m\in \boldsymbol{N}$.)

REMARK 1. Compare the above estimate with the original.

Needless to say,

REMARK 2. [33), Theorem 1] is essentially found (is proved by) in this theorem. The Hahn-Banach extension theorem itself is as well found in this theorem.

Theorem 1 enables us to generalize [33), Corollary 2 to Theorem 1] in three directions :

COROLLARY. *Let in particular* $K=E^{\dagger}$, $C=\{0\}$ *in Theorem 1. Then*[††] (1. 0) *and the following statements are equivalent :*

(P 1) *There exist a gauge function* g *on* E, *a non-void* $\mathscr{S}\subset\mathscr{T}$ *and an* $F_1\in E^*$ *extending* f *such that*

(i) $\sup_{S\in\mathscr{S}} g(ST^{\mu}(y))\leqslant \sup_{S\in\mathscr{S}} g(S(y))\leqslant p(y)$ *for* $y\in E$, $T\in\mathscr{T}$, $\mu\in\boldsymbol{N}$,

(ii) $F_1(T_1T_2\cdots T_{j-1}T_j\cdots T_k(y))=F_1(T_1T_2\cdots T_jT_{j-1}\cdots T_k(y))$ *for* $y\in E$, $T_S\in\mathscr{T}$; $j, k\in\boldsymbol{N}$ $(2\leqslant j\leqslant k)$, *and they are equal to* $f(y)$ *if* $y\in M$,

(iii) $F_1(y)\leqslant \sup_{S\in\mathscr{S}} g(S(y))$ *for* $y\in E$.

(1. 2)′ *There exist a gauge function* g *on* E *and a non-void* $\mathscr{S}\subset\mathscr{T}$ *such that* $\sup_{S\in\mathscr{S}} g(ST^{\mu}(y))\leqslant p(y)$ *for* $y\in E$, $T\in\mathscr{T}$, $\mu\in\boldsymbol{N}$; *and* $B_f\cup K_{g\mathscr{T}\mathscr{S}}$ *is positively independent in* L.

PROOF. Since $K_{g\mathscr{T}\mathscr{S}}\supset C_p$, by Theorem 1, it suffices to prove (1. 0)′⇒(P 1)⇒(1. 2)′. Assuming $g=F_1=F$, choose a non-void subset $\mathscr{S}$ of $\mathscr{T}$, and the first implication is self-evident. (P 1)⇒(1. 2)′ is done under g and $\mathscr{S}$ of (P 1) : To begin with, let $x+\alpha_1y_1+\alpha_2y_2=0$ for $(x, \xi)\in B_f\cup\{(0,0)\}$; (y_1, η_1), $(y_2, \eta_2)\in K_{g\mathscr{T}\mathscr{S}}$; $\alpha_1, \alpha_2>0$; where say $\frac{1}{m}\sup_{S\in\mathscr{S}} g\Big(\sum_{\mu=1}^{m} ST_1^{\mu}(y_1)\Big)<\eta_1$, $\frac{1}{n}\sup_{S\in\mathscr{S}} g\Big(\sum_{\nu=1}^{n} ST_2^{\nu}(y_2)\Big)<\eta_2$ for some $m, n\in\boldsymbol{N}$; $T_1, T_2\in\mathscr{T}$. Then since part (i) of the hypotheses guarantees[†††]

$$(*)\qquad \frac{1}{m}\sup_{S\in\mathscr{S}} g\Big(\sum_{\mu=1}^{m} ST_1^{\mu}(y_1)\Big)+\frac{1}{n}\sup_{S\in\mathscr{S}} g\Big(\sum_{\nu=1}^{n} ST_2^{\nu}(y_2)\Big)$$
$$\geqslant\frac{1}{mn}\sup_{S\in\mathscr{S}} g\Big(\sum_{\mu,\nu=1}^{m,n}(ST_2^{\nu}T_1^{\mu}(y_1)+ST_1^{\mu}T_2^{\nu}(y_2))\Big),$$

it follows from the hypotheses that

$$\xi+\alpha_1\eta_1+\alpha_2\eta_2>f(x)+\frac{1}{mn}\sup_{S\in\mathscr{S}} g\Big(\sum_{\mu,\nu=1}^{m,n}(ST_2^{\nu}T_1^{\mu}(\alpha_1y_1)+ST_1^{\mu}T_2^{\nu}(\alpha_2y_2))\Big)$$
$$\geqslant f(x)+\frac{1}{mn}F_1\Big(\sum_{\mu,\nu=1}^{m,n}(T_2^{\nu}T_1^{\mu}(\alpha_1y_1)+T_1^{\mu}T_2^{\nu}(\alpha_2y_2))\Big)$$
$$=f(x)+\frac{1}{mn}\sum_{\mu,\nu=1}^{m,n}F_1\Big(T_2^{\nu}T_1^{\mu}(\alpha_1y_1+\alpha_2y_2)\Big)$$
$$=f(x)+f(-x)=0.$$

† In this case, q is written by p.

†† In this case, (1. 0) is written by (1. 0)′.

††† The original form of this estimate is due to Agnew-Morse [6), pp. 21-22].

Now that thus the above is all correct, let us generalize (*) by the induction, and the proof is carried out in line with the above.

The following remarks are immediate consequences of this corollary.

REMARK 3. If we are concerned with the case when $\mathscr{T}$ has a right ideal $\mathscr{I}$, (P 1) may take the shape of

(P 2) *There exist a gauge function g on E and an $F_1 \in E^*$ extending f such that*

(i) $F_1(y) \leqslant \sup_{S \in \mathscr{I}} g(S(y)) \leqslant p(y)$ *for* $y \in E$,

(ii) *(ii) of (P 1) holds, where* $Ts \in \mathscr{I}$, $k=2$.

And if, moreover, $\mathscr{I}$ is $\mathscr{E}$, (i) is weakened to

$$F_1(y) \leqslant p(y) \text{ for } y \in E.$$

REMARK 4. Sometimes F_1 is substituted† for g in (P 1). In this case it should be noted that (iii) thereof is dispensable (reconsidering the proof above (cf. $\mathscr{S} \subset \mathscr{T}$)). Thus, if $\mathscr{T}$ has a left ideal $\mathscr{I}$, (P 1) may take the shape of

(P 3) *(Klee type condition) There is an $F_1 \in E^*$ extending f such that*

(i) $F_1(S(y)) \leqslant p(y)$ *for* $y \in E$, $S \in \mathscr{I}$.

(ii) *(ii) of (P 1) holds.*

This means that letting in particular $\mathscr{T}$ be a semigroup (for composite, and so on) and f be invariant thereunder, the present corollary *a fortiori* supplies an alternative proof to Klee [8), $(\alpha) \Leftrightarrow (\delta)$ of (2. 2) Theorem]. We note in passing that under some modifications, the method of [6), Lemma 2. 01] acquires (P 3).

On the other hand

REMARK 5. In this corollary, let in particular $\mathscr{T}$ be replaced by $\mathscr{E}$. Then in a sense that $g=p$ and $\mathscr{S}=\{I\}$, (P 1) takes the shape of

(P 4) *There exists an $F_1 \in E^*$ extending f such that*

(i) *(ii)* of *(P 1)* holds,

(ii) $F_1(y) \leqslant p(y)$ *for* $y \in E$.

As a result, letting in particular f be invariant, the present corollary proves the substance of Edwards [14), 3.3.2 Remark] (qua left s) without difficulty. Reasons are known by appealing to Γ_i by 2^{j-2} $(2 \leqslant j \leqslant k)$ times. By the way, for short [33), Cor. 2] proves the same as well. The reason is that this problem is synonymous relative to $\mathscr{E}$ and $\langle \mathscr{E} \rangle$, where $\langle \mathscr{E} \rangle$ is the generated semigroup of $\mathscr{E}$. For reference, if we are concerned with (1. 2)′, viewing p as g, noticing that $K_{p\mathscr{E}\mathscr{E}} = J_{p\mathscr{E}}$, it is simplified by

(1. 2)″ $B_f \cup J_{p\mathscr{E}}$ *is positively independent.*

We note again that unlike $C_{pC\mathscr{S}}$, $J_{pC\mathscr{E}}$ is not always convex even if $\mathscr{E}$ is Abelian.

REMARK 6. Let in particular $\mathscr{T}$ be $\langle \mathscr{E} \rangle$. Then (P 1) is reduced to :

† To avoid meaninglessness, let $\mathscr{S} \neq \{I\}$.

(P 5) *There exists an* $F_1 \in E^*$ *extending* f *such that*

(i) $F_1(T_1 T_2(y)) = F_1(T_2 T_1(y))$ *for* $y \in E$; $T_1, T_2 \in \langle \mathscr{E} \rangle$,

(ii) $F_1(T(x)) = f(x)$ *for* $x \in M$, $T \in \langle \mathscr{E} \rangle$,

(iii) $F_1(y) \leqslant p(y)$ *for* $y \in E$,

which corresponds to [33), Cor. 1 (with $C = \{0\}$)]. And if f is invariant, (P 5) (whence it is equiv. to [33), (1. 0)′ of Cor. 2]) is equivalent to [8), (ε) following (2. 2) Theorem] (as was given ibid) and to :

(1. 2)‴ $B_f \cup C_{pC}$ *is positively independent, where* $C = \left\{ \sum_{i=1}^{k} (S_i T_i y_i - T_i S_i y_i) : S_i, T_i \in \mathscr{T} ;\ y_i \in E,\ k \in N \right\}$.

If $\langle \mathscr{E} \rangle$ is Abelian, (P 5) is reduced to :

(P 6) *There exists an* $F_1 \in E^*$ *extending* f *such that*

(i) $F_1(T(x)) = f(x)$ *for* $x \in M$, $T \in \langle \mathscr{E} \rangle$,

(ii) $F_1(y) \leqslant p(y)$ *for* $y \in E$,

which corresponds to [33), Cor. 4 (with $C = \{0\}$)]. If f is invariant, (P 6) is reduced to :

(P 7) *There exists an* $F_1 \in E^*$ *extending* f *such that* $F_1(y) \leqslant p(y)$ *for* $y \in E$.

This is equivalent to "$f(x) \leqslant p(x)$ *for* $x \in M$".

We close the first part of this paper with the following.

REMARK 7. Replacing E, $F \in E^*$, U, etc. by preordered linear topological space E, $F \in E'$, 0-neighbourhood U etc. respectively, we can state and prove the topological version of Theorem 1 with ease. We call this Theorem 2.

And, I here, with an apology, submit a part of 31) for correction :

SUPPLEMENT. Each proof of

1° the 1st and the 2nd assertions of Cor. 1 of Th. 1,

2° the 1st and the 2nd assertions of Cor. 1 of Th. 2,

3° Cor. 2 of Th. 1,

in my paper 31) is defective (the assertions themselves are available). I wish to correct them as follows.

PROOFS OF 1°. For the 1st assertion, it suffices to examine the logical equivalence of the following conditions, where $f \not\equiv 0$.

(α) *There exists a convex absorbing set* U *in* E *such that* $A \cup C \cup (U + a_0)$ *is positively independent, where* $f(a_0) = 1$.

(β) *There exists a convex absorbing set* V *in* E *such that* $B_f \cup C_C \cup (V \times \{1\})$ *is pos. ind.*

(α)$\Rightarrow$(β) : Pos. independence of $A \cup C$ is inherited to that of $B_f \cup C_C$ (and *vice versa*). Clearly $C_C \cup ((-U) \times \{1\})$ is pos. ind. Let now $x - c + v = 0$ for $x \in X$, $c \in C \cup \{0\}$, $v \in -U$. Then since $A \cup C \cup \{-v + a_0\}$ is pos. ind., so is $A \cup C \cup \{x - c + a_0\}$ which induces $f(x + a_0) \geqslant 0$, i.e., $f(x) \geqslant -f(a_0) = -1$. These mean that $-U$

is suited for V of (β). For the converse, since $f(-a-a_0)\leqslant -1$ $(a\in A\cup\{0\},\ a_0\in A,\ f(a_0)=1)$, by hypothesis, it never happens that $(-a-a_0)-c-u=0$ $(c\in C\cup\{0\},\ u\in -\frac{1}{2}V)$. This means that $-\frac{1}{2}V$ is suited for U of (α). The 2nd assertion is now clear via (α).

The proofs of 2° are analogized.

PROOF OF 3°. To answer this via (β), we employ an absolutely convex absorbing set V of E such that $|f(x)|<1$ for $x\in X\cap V$.

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