On a characterization of some function space as Banach lattices under the topological equivalence

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Abstract

Characterizations of Orlicz spaces as Banach lattices are interesting problem. In previous paper [5], we considered this problem from the point of view of some relations between Banach lattices and their conjugate spaces.

In this paper, we give another conditions under which Banach lattices are toplogically isomorphic to some Orlicz space.

1. Introduction. Let $\Phi(\xi)$ be a continuous Young function, i. e., in the interval $[0, \infty)$, $\Phi(\xi)$ is a real valued, non-decreasing, continuous and convex function with $\Phi(0)=0$. Let μ be a non-atomic, completely additive measure on a set Ω with $\mu(\Omega)=1$.

The Orlicz space $L_{\Phi}(\Omega, \mu)$ consists of all real valued functions $f(t)$, μ-measurable on Ω, for which

$$\rho(\alpha f)=\int_{\Omega} \Phi(\alpha|f(t)|)\,d\mu<\infty \quad \text{for some resl number } \alpha>0.$$

This space is a conditionally complete*) vector lattice and becoms a Banach space with the Luxemburg norm

$$\|f\|_{\Phi}=\inf\left\{\frac{1}{|\xi|}:\rho(\xi f)\leqq 1\right\}.$$

If Φ satisfies the (Δ_2)-condition,**) we have $\rho(f)<\infty$ for all f in L_{Φ} and more the norm has following properties:

1) the norm is continuous, i. e., $f_n \downarrow_{n=1}^{\infty} 0$***) implies $\|f_n\| \downarrow_{n=1}^{\infty} 0$,

2) the norm is monotone complete, i. e., if $f_n \uparrow_{n=1}^{\infty}$ and $\sup_{n\geqq 1}\|f_n\|_{\Phi}<\infty$ there exists a function $f \in L_{\Phi}$ such that $f_n \uparrow_{n=1}^{\infty} f$,

3) $\rho(f)=1$ is equivalent to $\|f\|_{\Phi}=1$****)

In the preceding papers [4] and [5], we considered two characterizations of the Orlicz space, namely the former is given by making use of the N-function and the latter is considered, under the topological equivalence, by making use of some transformation from a Banach lattice to its conjugate space.

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*) A vector lattice R is said to be conditionally complete, if $R \ni a_\lambda \geqq 0 (\lambda \ni \Lambda)$ there exists $a \in R$ such that $a=\inf_{\lambda\in\Lambda} a_\lambda$. Cf. [9]

**) Cf. [7]

***) The notation $f_n \downarrow_{n=1}^{\infty} (f_n \uparrow_{n=1}^{\infty})$ means that the sequence $\{f_n\}$ is non-increasing (non-decreasing). $f_n \downarrow_{n=1}^{\infty} f (f_n \uparrow_{n=1}^{\infty} f)$ means that f is the limit of f_n in the order.

****) This property is equivalent to 1)

In this paper, we also deal with the same problem. In a vector lattice R, for any $x \in$ R, we define $x^+ = x \cup 0$ and $x^- = (-x) \cup 0$. Then, we have $x = x^+ - x^-$, $x^+ \cap x^- = 0$ and the absolut $|x|$ of x is defined by $|x| = x^+ + x^-$. When $|x| \cap |y| = 0$, we say "x and y are mutually orthogonal.

R is said to be a condisionally complete Banach lattice when R is a conditional complete vector lattice and has a complete norme $\|\cdot\|$ on R such that $|x| \leqq |y|$ implies $\|x\| \leqq \|y\|$ (x, $y \in$ R).

For any $0 \neq p \in$ R, the projector $[p]$ is defined by

$$[p]x^+ = \sup_{n \leqq 1}\{n|p| \cap x^+\},\ [p]x^- = \sup_{n \geqq 1}\{n|p| \cap x^-\} \text{ and } [p]x = [p]x^+ - [p]x^-$$

The projector $[p]$ is a linear projection operator in R.

The norm on R is said to be smooth, if for every element $a \in$ R with $\|a\| = 1$ there exists only one linear functional (we write it by a^*) on R such that $(a, a^*) = 1$ and $\|a^*\| = \sup\{(x, a^*) : \|x\| = 1, x \in \text{R}\} = 1$, where (x, a^*) means the value of a^* at x. Then, it is seen that

$$(x, a^*) = \lim_{\varepsilon \to 0} \frac{\|a + \varepsilon x\| - 1}{\varepsilon} \qquad \text{for every } x \text{ in R.}$$

The purpose of this paper is to prove the following theorem.

THEOREM. *Let* R *be a conditionally complete Banach lattice and the norm* $\|\cdot\|$ *on* R *be continuous, monotone complete and smooth. If* R *has a positive complete element* s (i. e., $s \cap |x| = 0$ *for* $x \in$ R *implies* $x = 0$.), *with* $\|s\| = 1$, *and satisfies the following conditions*:

1) $\|[p]s\| = \|[q]s\|$ *implies* $([p]s, s^*) = ([q]s, s^*)$,

2) *there exists a constant number* A>0 *such that*

$$\left\|\sum_{i=1}^{n} \frac{[p_i]s}{\|[q_i]s\|}\right\| \leqq 1 \quad \textit{implies} \quad \sum_{i=1}^{n} \frac{([p_i]s, s^*)}{([q_i]s, s^*)} \leqq \left\|\sum_{i=1}^{n} \frac{[p_i]s}{\|[q_i]s\|}\right\|$$

and

$$\sum_{i=1}^{n} \frac{([p_i]s, s^*)}{([q_i]s, s^*)} \leqq 1 \quad \textit{implies} \quad \left\|\sum_{i=1}^{n} \frac{[p_i]s}{\|[q_i]s\|}\right\| \leqq \text{A}$$

where $\{[p_i]\}$ *are any mutually othogonal projectors and* $\{[q_i]\}$ *are any nonzero projectors, then* R *is topologically isomorphic to some Orlicz space.*

2. Preliminaries. Before to prove the theorem, we restate some results in *Nakano's spectral theory.* (Cf. [8], [9] and [10])

The set $\mathfrak{P}$ of projectors in R is called an ideal, if (i) $0 \notin \mathfrak{P}$, (ii) $[x] \in \mathfrak{P}$ and $[x] \leqq [y]$ (i. e., $[x]z \leqq [y]z$ for all $0 \leqq z \in$ R) implies $[y] \in \mathfrak{P}$, (iii) $[x], [y] \in \mathfrak{P}$ implies $[x][y] \in \mathfrak{P}$ ($[x][y]$ means $[|x| \cap |y|]$).

Let $\mathcal{E}$ be the space consisting of all maximal ideals $\mathfrak{P}$ of projectors in R. Then, $\mathcal{E}$ is a compact Hausdorff space and $\mathcal{J} = \{U_{[x]} : x \in R\}$ is a neighbourhood system in $\mathcal{E}$, where $U_{[x]} = \{\mathfrak{P} \in \mathcal{E} : [x] \in \mathfrak{P}\}$. Furthermore, each $U_{[x]}$ is both open and closed in $\mathcal{E}$ and it is valid that

$[p][q] = 0$ implies $U_{[p+q]} = U_{[p]} + U_{[q]}$ ($+$ means the union of disjoint sets).

For $x \in$ R, the function $(x/s, \mathfrak{P})$ on $\mathcal{E}$ is defined by

$$\left(\frac{x}{s},\mathfrak{P}\right)=\begin{cases}\lambda & \text{if } \mathfrak{P} \in \prod_{\varepsilon>0} (U_{[x_{\lambda+\varepsilon}]}-U_{[x_{\lambda-\varepsilon}]}) \\ +\infty & \text{if } \mathfrak{P} \in \prod_{-\infty<\lambda<+\infty} (\mathscr{E}-U_{[x_\lambda]}) \\ -\infty & \text{if } \mathfrak{P} \in \prod_{-\infty<\lambda<+\infty} U_{[x_\lambda]}\end{cases}$$

where $[x_\lambda]=[(\lambda s-x)^+]$.

This function is called the relative spectrum, and the following properties are shown.

Lemma 1. [10;Th. 19.2 and 19.3] (i) $(x/s,\mathfrak{P})$ *is almost finite* (*i. e., finite in an open dense set in* $\mathscr{E}$, *and continuous in* $\mathscr{E}$).

(ii) $(x/s,\mathfrak{P})=([p]x/s,\mathfrak{P})$ *on* $U_{[p][x]}$ *for any projectoe* $[p]$ (Cf. [10;Th. 18.4]),

(iii) *the set* $\{(x/s,\mathfrak{P}):x\in R\}$ *is linear and lattice isomorphic to* R ([10;Th. 18.5-Th. 18.10]).

For a bounded continuous functions $f(\mathfrak{P})$ on $U_{[p]}$, the integral of $f(\mathfrak{P})$ by $x\in R$, denoted by $\int_{[p]} f(\mathfrak{P})d\,\mathfrak{P}\,x$, is defined as the order limit of partial sums

$$\sum_{j=1}^{n_i} f(\mathfrak{P}_{ij})[p_{ij}]x$$

for every sequence of orthogonal partitions $\{[p_{ij}]\}$ of $[p]$ such that for $\varepsilon_i>0$,

$$\operatorname*{Osc}_{\mathfrak{P}\in U_{[p_{ij}]}} f(\mathfrak{P}) \leqq \varepsilon_i \quad (i=1,2,\cdots;j=1,2,\cdots,n)$$

and for any $\mathfrak{P}_{ij}\in U_{[p_{ij}]}$, where $\lim_{i\to\infty}\varepsilon_i=0$.

For an unbounded continuous function $f(\mathfrak{P})$ on $U_{[p]}$, if there exists an increasing sequence of bounded continuous functions $f_n(\mathfrak{P})$ on $U_{[p]}$ such that

$$\lim_{n\to\infty} f_n(\mathfrak{P})=f(\mathfrak{P}) \text{ and } \lim_{n\to\infty}\int_{[p]} f_n(\mathfrak{P})d\,\mathfrak{P}\,x$$

exists, then we shall say that $f(\mathfrak{P})$ is integrable by x on $U_{[p]}$ and denote this order limit by $\int_{[p]} f(\mathfrak{P})d\,\mathfrak{P}\,x$. (Cf. [10; § 20]).

We have, as an integral representation, the following fact.

Lemma 2. [10;Th. 21 .1 and 21.2] *For any* $a\in R$, $(a/s,\mathfrak{P})$ *is integrable by s and we have*

$$a=\int_{[s]}\left(\frac{a}{s},\mathfrak{P}\right)d\,\mathfrak{P}\,s.$$

Conversely, if a continuous function $f(\mathfrak{P})$ *is integrable by s and*

$$b=\int_{[s]} f(\mathfrak{P})d\,\mathfrak{P}\,s,$$

then $f(\mathfrak{P})=(b/s,\mathfrak{P})$ *for all* $\mathfrak{P}\in\mathscr{E}$.

Lemma 3. *For any* $0\neq a\in R$, *there exists a non-decreasing sequence* $\{x_n\}$ *of step elements in* R *such that* $\sup_{n\geqq 1} x_n=|a|$.

Proof. By virtue of [9;Th. 11.6], putting

$$U_{[p_{n,i}]}=\{\mathfrak{P}:\frac{i-1}{2^n}<(\frac{|a|}{s},\mathfrak{P})<\frac{i}{2^n}\}^-(i=1,2,\cdots,2^n n)$$

and

$$U_{[p_n]}=\{\mathfrak{P}:n<(\frac{|a|}{s},\mathfrak{P})\}^-, (X^- \text{ means the closure of } X)$$

we get an orthogonal system $\{[p_{n,1}], \cdots, [p_n, 2^n n], [p_n]$. Accordingly, for the increasing sequence of step elements

$$x_n = \sum_{i=1}^{2^n n} \frac{i-1}{2^n}[p_{n,i}]s + n[p_n]s$$

we have

$$\lim_{n\to\infty}\left(\frac{x_n}{s}, \mathfrak{P}\right) = \left(\frac{|a|}{s}, \mathfrak{P}\right) \text{ for each } \mathfrak{P} \in \mathscr{E}.$$

Therefore, the desired result is obtained by Lemma 2.

3. The proof of the theorem. We define a function $\Phi(\xi)$ by the following:

$$\Phi(\xi) = \begin{cases} \xi & , \text{ if } 0 \leqq \xi < 1 \\ \dfrac{1}{([p]s, s^*)} & , \text{ if } \xi \geqq 1, \text{ where } \xi = \dfrac{1}{\|[p]s\|}. \end{cases}$$

First, we shall see that $\Phi(\xi)$ are well defined. For each $\xi \geqq 1$, we can find a projector $[p]$ with $\xi = 1/\|[p]s\|$ by the continuity of the norm and the non-atomicity of R. Furthermore, by the condition 1), $([p], s^*)$ is uniquely determined for every projectiors with $\xi = 1/\|[p]s\|$. Thus, $\Phi(\xi)$ are well defined.

Next, we shall investigated some properties of Φ.

It is obvious that for any real number $\varepsilon > 0$, $(\|s+\varepsilon[p]s\|-1)/\varepsilon \leqq (\|s+\varepsilon[q]s\|-1)/\varepsilon$ if $[p] \leqq [q]$. Therefore, by the smoothness of the norm on R, we have $([p]s, s^*) \leqq ([q]s, s^*)$ for any projectors $[p], [q]$ with $[p] \leqq [q]$.

For any projectors $[r], [q]$with $\|[r]s\| \leqq \|[q]s\|$, we can find a projector $[p]$ such that $\|[r]s\| = \|[p]s\|$ and $[p] \leqq [q]$, since R is non-atomic and has continuous norm. Then, we have $([p]s, s^*) \leqq ([q]s, s^*)$ and $([r]s, s^*) = ([p]s, s^*)$ by the condition 1) and consequently $([r]s, s^*) \leqq ([q]s, s^*)$.

Therefore, by the definition of Φ, we have properties of Φ:

(i) $\Phi(\xi) \leqq \Phi(\eta)$ if $1 \leqq \xi \leqq \eta$,

(ii) $\Phi\left(\dfrac{1}{\|[p]s\|}\right) \geqq \dfrac{1}{\|[p]s\|}$ for any projector $[p] \neq 0$.

Namely, $\Phi(\xi)$ is a non-negative, non-decreasing function in $\xi \geqq 0$ with $\Phi(0)=0$. Since the norm continuous and s^* is the norm bounded linear functional on R, $\Phi(\xi)$ is also a continuous function in $\xi \geqq 0$.

We shall prove the following property:

(iii) $\Phi(\lambda\xi) \leqq \lambda\Phi(\xi)$ for $0 < \lambda < 1$ and for any $\xi \geqq 0$.

If $0 < \lambda\xi < 1$, obviously $\Phi(\lambda\xi) = \lambda\Phi(\xi)$. For $\xi \geqq 1$, we take a projector $[p]$ satisfying $\xi = 1/\|[p]s\|$. Then, if $0 < \lambda\xi < 1$, we have

$$\Phi\left(\frac{1}{\|[p]s\|}\right) = \frac{\lambda}{\|[p]s\|} \leqq \lambda\Phi\left(\frac{1}{\|[p]s\|}\right) = \lambda\Phi(\xi) \text{ by (ii),}$$

if $\lambda\xi \geqq 1$, taking a projector $[p]$ with $\lambda\xi = 1/\|[q]s\|$, we have, by condition 2)

$$\frac{\|[p]s\|}{\|[q]s\|} = \lambda < 1 \quad \text{implies} \quad \frac{([p]s, s^*)}{([q]s, s^*)} \leqq \lambda$$

and hence

$$\Phi\left(\frac{1}{\|[q]s\|}\right)=\frac{1}{([q]s, s^*)}\leqq\frac{\lambda}{([p]s, s^*)}=\lambda\ \Phi\left(\frac{1}{\|[p]s\|}\right),$$

namely $\Phi(\lambda\ \xi)\ \leqq\ \lambda\,\Phi(\xi)$.

Thus, we obtain a non-negative, non-decreasing, continuous and convex function $\Phi(\xi)$ with $\Phi(0)=0$ and $\Phi(\xi)>0$ for $\xi>0$.Such $\Phi(\xi)$ is a so-called *continuous Young function.* Cf. [12])

Now, we define a functional ν on σ-ring $\mathscr{F}$ in the preliminaries as

$$\nu(U_{[x]})=([x]s, s^*)\quad (x\in R).$$

Obviously, ν is a completely additive and $\nu(\mathscr{E})=1$ by $\mathscr{E}=U_{[s]}$. As in [4] we have a regular Borel measure μ on $\mathscr{E}$ as extension of ν.

Let $L_\Phi(\mathscr{E}, \mu)$ be the totality of all μ-measurable functions $f(\mathfrak{P})$ such that

$$\rho(\alpha f)=\int_{\mathscr{E}}\Phi(\alpha|f(\mathfrak{P})|)d\mu<\infty\quad \text{for some positive number } \alpha.$$

Then, the L_Φ is a Banach lattice by the Luxemburg norm:

$$\|f\|_\Phi=\inf\ \{\frac{1}{|\xi|}:\rho(\xi\ f)\leqq 1\}.$$

Since every $(x/s, \mathfrak{P})(x\in R)$ is continuous by Lemma 1 and hence μ-measurable in $\mathscr{E}$. Consequently,

$$\rho(x)=\int_{\mathscr{E}}\Phi(|(\frac{x}{s}, \mathfrak{P})|)d\mu$$

has a sence, and we have $0\leqq\rho(x)\ \leqq\ \rho(y)$ if $|x|\ \leqq\ |y|$ for $x, y\in R$.

We shall call x a step element in R such that its from is $x=\sum_{i=1}^{n}\xi_i[p_i]s$ for orthogonal family $\{[p_i]; \mathrm{i}=1, 2, \cdots, n\}$.

For a step element $x=\sum_{i=1}^{n}\xi_i[p_i]s$, by rhe Nakano's spectral theory

$$\begin{aligned}\rho(x)&=\int_{\mathscr{E}}\Phi(|(\frac{x}{s},\ \mathfrak{P})|)d\mu\\&=\sum_{i=1}^{n}\Phi(|\xi_i|)([p_i]s, s^*),\end{aligned}$$

because $\mu(U_{[p]})=([p]s, s^*)$ and $(\xi[p]s/s, \mathfrak{P})\ \equiv\xi$ for $\mathfrak{P}\in U_{[p]}$.

Therefore, if $\|x\|\leqq 1$ and $|\xi_1|\leqq\cdots\leqq|\xi_R|<1\leqq|\xi_{R+1}|\leqq\cdots\leqq|\xi_n|$, selecting projectors $[q_i]$ with $|\xi_i|=1/\|[q_i]s\|(i=k+1, \cdots, n)$, we have then, on account of the definition of Φ and the assumptions 1) and 2) in the theorem

$$\begin{aligned}\rho(x)&=\sum_{i=1}^{k}|\xi_i|([p_i]s, s^*)+\sum_{i=k+1}^{n}\frac{([p_i]s, s^*)}{([q_i]s, s^*)}\\&\leqq(\sum_{i=1}^{k}[p_i]s, s^*)+\|\sum_{i=k+1}^{n}|\xi_i|[p_i]s\|\leqq 1+\|x\|\leqq 2.\end{aligned}$$

Conversely, if $\rho(x)\leqq 1$ for the step element $x=\sum_{i=1}^{n}\xi_i[p_i]s$, then we have

$$\|x\|\leqq\|\sum_{i=1}^{k}\xi_i[p_i]s\|+\|\sum_{i=k+1}^{n}\xi_i[p_i]s\|$$

$$\leqq \left\|\sum_{i=1}^{k} [p_i]s\right\| + \left\|\sum_{i=k+1}^{n} \frac{[p_i]s}{\|[q_i]s\|}\right\|$$

$$\leqq 1+A,$$

because we have $|\xi_i|=1/\|[q_i]s\|$for $i\geqq k+1$ and $\sum_{i=k+1}^{n}$ $([p_i]s, s^*)/([q_i]s, s^*)\leqq\rho(x)\leqq 1$ and hence$\|\sum_{i=k+1}^{n}[p_i]s/\|[q_i]s\|\|\leqq$A by the assumption 2) in the theorem.

Thus, it is seen that for any step element x, $\|x\|\leqq 1$ implies $\rho(x)\leqq 1$ and conversely $\rho(x) \leqq 1$ implies $\|x\| \leqq 1+$A.

We shall prove that above results are also valid for any element x of R. By Lemma 3, there exists a non-decreasing sequence $\{x_n\}$ of step elements in R such that $\sup_{n\geqq 1} x_n=|x|$ and hence by the Lebesgue bounded sequence theorem we have $\lim_{n\to\infty} \rho(x_n)=\rho(x)$. Using this fact and the continuity of the norm we have for aky element x in R,

$$\|x\| \leqq 1 \quad \text{implies} \quad \rho(x) \leqq 1$$

and

$$\rho(x) \leqq 1 \quad \text{implies} \quad \|x\| \leqq 1+A.$$

From the above considerations, R is toplogically isomorphic to the subspace $\Lambda\equiv\{(x/s, \mathfrak{P}):x\in R\}$of $L_\Phi(\mathcal{E}, \mu)$. Espetially, we have $\|x\|_\Phi\leqq\|x\|\leqq(1+A)\|x\|_\Phi$ for $x\in R$, where we identify any $x\in R$ and $(x/s,\mathfrak{P})\in\Lambda$ under the linear and lattice isomorphism.

Now, let us take an $f(\mathfrak{P})\in L_\Phi(\mathcal{E}, \mu)$. Since $f(\mathfrak{P})$ is almost finite in $\mathcal{E}$, by the Lussin's theorem (for example, see [3;p. 243] or [12;Chap. 5, exercise]) there exists a sequsnce of compact sets C_n such that

$$\mu(\mathcal{E}-C_n)\leqq 1/n,\ C_1\subset C_2\subset\cdots\subset C_n\subset\cdots \text{and } f(\mathfrak{P}) \text{ is bounded and continuous on each } C_n.$$

On the otherhand, it is known that for each $p\in R$, $U_{[p]}$ is both open and closed and more compact. Accordingly, the proper space $\mathcal{E}$ is a regular topological space, i. e., for any open set $G \subset\mathcal{E}$, if $\mathfrak{P}_0 \in G$ then there exists an open set E such that $\mathfrak{P}_0 \in E \subset E^- \subset G$, where E^- is the closure of E.

And also, for compact sets C_n, there exist projectors $[p_n]$ such that $[p_n]\uparrow_{n=1}^{\infty}$ and $C_n\subset U_{[p_n]}(n=1, 2, \cdots)$. (Cf. [10;Th. 16.3])

Therefore, for each continuous function $f(\mathfrak{P})\chi_{C_n}(\mathfrak{P})$ (χ_E means the characteristic function on the set E), we have the bounded continuous extension $g_n(\mathfrak{P})$ of $f(\mathfrak{P})$ $\chi_{C_n}(\mathfrak{P})$ over $\mathcal{E}$ such that $g_n(\mathfrak{P})=0$ for $\mathfrak{P} \notin U_{[p_n]}$. (Cf. [10;p. 16])

Consequently, by Lemma 2, for each $n=1, 2, \cdots$

$$x_n=\int_{\mathcal{E}} g_n(\mathfrak{P})d\mathfrak{P}\, s \qquad \text{exists in } R$$

and

$$\left(\frac{x_n}{s}, \mathfrak{P}\right)=g_n(\mathfrak{P})=f(\mathfrak{P})\chi_{C_n}(\mathfrak{P}) \qquad \text{on } C_n \subset U_{[p_n]}.$$

Obviously, the sequence $\{x_n\}$in R is non-decreasing and we have

$$g_n(\mathfrak{P})\uparrow_{n=1}^{\infty} f(\mathfrak{P}) \qquad \text{a. e.}$$

so that

$$\rho(f)=\lim_{n\to\infty}\rho(g_n)=\lim_{n\to\infty}\ \rho(x_n).$$

From this relation, we have

$$\|g_n\|_\Phi\uparrow_{n=1}^{\infty}\|f\|_\Phi \text{ and hence } \|x_n\|_\Phi\uparrow_{n=1}^{\infty}\|f\|_\Phi.$$

By $\|x_n\|\leqq(1+A)\|x_n\|_\Phi(n=1,2,\cdots)$ and the monotone completeness of the norm $\|\cdot\|$ on R. there exists an element $x\in R$ such that $x_n\uparrow_{n=1} x$ and $(x/s,\mathfrak{P})=f(\mathfrak{P})$a. e. Thus, R is topologically isomorphic to the Orlicz space $L_\Phi(\mathcal{E},\ \mu)$. The theorem is proved.

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