ON AREAL SPACES BASED ON THE FUNDAMENTAL FUNCTION F = α^2/β (II)

ON AREAL SPACES BASED ON THE FUNDAMENTAL FUNCTION $F = \alpha^2/\beta$ (II)

by

Takanori IGARASHI

Abstract

We consider the necessary and sufficient condition for a sufficient for a special areal space $A_n^{(m)}$ to belong to the semi-metric class.

§ 0. INTRODUCTION. In the Finsler geometry, a Finsler space with (α, β)-metric is, as well known, a space of which fundamental function is given in the form

(0.1) $F(x, p)=f(\alpha, \beta), \quad \alpha = [\det(a_{ij}(x)y^i y^j)]^{1/2}, \quad \beta = b_i(x)y^i$

where $a_{ij}(x)$ is a Riemannian metric and $b_i(x)$ is non-zero covariant vector.

We know, as typical (α, β)-metrics, so-called Randers' metric $F = \alpha + \beta$ [1]*), and Kropina's metric $F = \alpha^2/\beta$ [2].

On areal spaces $A_n^{(2)}$, G. T. Bollis [3] gave metric $F = \alpha + \beta$, $\alpha = (\det[\tilde{g}_{ij}(x)p^i_\lambda p^j_\mu])^{1/2}$, $\beta = b_{ij}(x)\, p^i_1 p^j_2$. where $\tilde{g}_{ij}(x)$ is a Riemannian metric and $b_{ij}(x)$ is a skew-symmetric tensor.

Recently, the author [4] treatedan areal space $A_n^{(m)}$ equipped a fundamental function in the form

(0.2) $F = \alpha^2/\beta, \quad \alpha = [\det(a_{\lambda\mu})]^{1/2}, \quad a_{\lambda\mu} = a_{ij}(x)p^i_\lambda p^j_\mu, \quad a_{ij} = a_{ji},$

$\beta = \varepsilon^{\lambda\mu} b_{\lambda\mu}/2, \quad b_{\lambda\mu} = b_{ij}(x)p^i_\lambda p^j_\mu, \quad b_{ij} = -b_{ji}.$

In that paper, the main result which we obtained is such that

THEOREM. *When a fundamental function of an area space $A_n^{(m)}$ is given by (0.2), then the following two conditions are equivalent:*

(i). $A_n^{(m)}$ is of semi-metr ic class.

(ii). The relation $(\rho^{(\alpha}_i - \sigma^{(\alpha}_i)(\rho^{\beta)}_{i\ j} - \sigma^{\beta)}_j) = 0$ holds good.

However, it was found that the above theorem holds good, even if we rewtite β as $\beta = [\det(b_{\lambda\mu})]^{1/2}$, what we give from now on.

§ 1. PRELIMINARY. We coonsider an n-dimensional areal space $A_n^{(m)}$ based on the notion of the m-dimensional surface-element p.

Let (x^i) be local coordinates and (p^i_α) be local representations of p. In this paper, Latin indices

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*) *Number in brackts refer to the referrences at the end of the paper.*

run over 1, 2, ..., n; Greek indices over 1, 2, ..., m; where $1<m<n$, and we adopt the Einstein's summation convention. Other notations and terminologies are employed as same as those of the work of A. Kawaguchi [5].

We put a fundamental function of $A_n^{(m)}$ as

(1.1) $F(x,p)=\alpha^2/\beta$

(1.2) $\begin{cases} \alpha=[\det(a_{\lambda\mu})]^{1/2}, & a_{\lambda\mu}(x,p)=a_{ij}(x)p^i_\lambda p^j_\mu, \quad a_{ij}=a_{ji} \\ \beta=[\det(b_{\lambda\mu})]^{1/2}, & b_{\lambda\mu}(x,p)=b_{ij}(x)p^i_\lambda p^j_\mu, \quad b_{ij}=-b_{ji}. \end{cases}$

Next, we define a Legendre's form of a function $\varphi(x,p)$ as follows;

(1.3) $L^{\alpha\beta}_{ij}[\varphi]=(\ln\varphi);^{\alpha}_{i};^{\beta}_{j}+(\ln\varphi);^{\beta}_{i}(\ln\varphi);^{\alpha}_{j}$

where the notation $;^{\alpha}_{i}$ means the partial differentiation with respect to p^i_α.

Differentiating (1.2) by p^i_α, we have

(1.4) $\alpha;^{\alpha}_{i}=(1/2)\alpha a^{\lambda\mu}a_{\lambda\mu};^{\alpha}_{i}$, where $a^{\lambda\mu}a_{\lambda\nu}=a^{\lambda\mu}a_{\nu\lambda}=\delta^{\mu}_{\nu}$

(1.5) $\beta;^{\alpha}_{i}=(1/2)b^{\lambda\mu}a_{\lambda\mu};^{\alpha}_{i}$, where $b^{\lambda\mu}b_{\lambda\nu}=b^{\mu\lambda}b_{\nu\lambda}=\delta^{\mu}_{\nu}$

If we introduce quantities ρ^{α}_{i}, σ^{α}_{i} such that

(1.6) $\rho^{\alpha}_{i}=(\ln\alpha);^{\alpha}_{i}=\alpha^{-1}\alpha;^{\alpha}_{i}, \quad \sigma^{\alpha}_{i}=(\ln\beta);^{\alpha}_{i}=\beta^{-1}\beta;^{\alpha}_{i},$

then we obtain:

PROPOSITION 1. $\rho^{\alpha}_{i}=a^{\alpha\lambda}a_{ik}p^k_\lambda, \quad \sigma^{\alpha}_{i}=b^{\alpha\lambda}b_{ik}p^k_\lambda.$

Proof). From (1.4), it follows

$$\rho^{\alpha}_{i}=(1/2)a^{\lambda\mu}a_{\lambda\mu};^{\alpha}_{i}=(1/2)a^{\lambda\mu}a_{\lambda\mu}(a_{hk}p^h_\lambda p^k_\mu);^{\alpha}_{i}$$
$$=(1/2)a^{\lambda\mu}a_{hk}(\delta^k_i\delta^\alpha_\lambda p^k_\mu+a^{\lambda\mu}a_{hi}p^h_\lambda)$$
$$=a^{\alpha\lambda}a_{ik}p^k_\lambda,$$

and analongously on σ^{α}_{i}.

PROPOSITION 2. $\rho^{\alpha}_{i};^{\beta}_{j}=-a^{\alpha\beta}a_{\gamma\delta}\delta^{\alpha}_{i}\delta^{\beta}_{j}-\delta^{\beta}_{i}\delta^{\alpha}_{j}+a^{\alpha\beta}a_{ij}$

$\sigma^{\alpha}_{i};^{\beta}_{j}=-b^{\alpha\beta}b_{\gamma\delta}\sigma^{\alpha}_{i}\sigma^{\beta}_{j}-\sigma^{\beta}_{i}\sigma^{\alpha}_{j}+b^{\alpha\beta}b_{ij}.$

proof). It is sufficient that we do with $\rho^{\alpha}_{i};^{\beta}_{j}$. Differentiating ρ^{α}_{i} by p^j_β partially, we have

$$\rho^{\alpha}_{i};^{\beta}_{j}=(a^{\alpha\varepsilon}a_{ik}p^k_\varepsilon);^{\beta}_{j}=a^{\alpha\varepsilon};^{\beta}_{j}a_{ik}p^k_\varepsilon+a^{\alpha\varepsilon}a^{ik}\delta^k_j\delta^\beta_\varepsilon=a^{\alpha\varepsilon};^{\beta}_{j}a_{\varepsilon\gamma}\delta^{\gamma}_{j}+a^{\alpha\beta}a_{ij}.$$

substituting the relation

$$a^{\alpha\varepsilon};^{\beta}_{j}a_{\varepsilon\gamma}=(a^{\alpha\varepsilon}a_{\varepsilon\gamma});^{\beta}_{j}-a^{\alpha\varepsilon}a_{\varepsilon\gamma};^{\beta}_{j}=-a^{\alpha\varepsilon}a_{\varepsilon\gamma};^{\beta}_{j}$$

into the above representation, we can rewrite as follows;

$$\rho^{\alpha}_{i};^{\beta}_{j}=-a_{\varepsilon\gamma};^{\beta}_{j}a^{\alpha\varepsilon}\delta^{\gamma}_{i}+a^{\alpha\beta}a_{ij}=-(a_{hk}p^h_\varepsilon p^k_\gamma);^{\beta}_{j}a^{\alpha\varepsilon}\delta^{\gamma}_{i}+a^{\alpha\beta}a_{ij}$$
$$=-a_{jk}p^k_\gamma a^{\alpha\beta}\rho^{\gamma}_{i}-a_{hj}a^{\alpha\varepsilon}p^h_\varepsilon\rho^{\beta}_{i}+a^{\alpha\beta}a_{ij}=-a_{jk}a_{\gamma\delta}\rho^{\gamma}_{i}\rho^{\delta}_{j}-\rho^{\beta}_{i}\rho^{\alpha}_{j}+a^{\alpha\beta}a_{ij}.$$

About $\sigma^{\alpha}_{i};^{\beta}_{j}$, we can obtain the right hand analogously. Q.E.D.

Then, with use of Proposition 1 and 2, we can represent the Legender's forms of α and β such that

$$(1.7)\quad L^{\alpha\beta}_{ij}[\alpha]=(\ln\alpha);^{\alpha\beta}_{ij}+(\ln\alpha);^{\beta}_{i}(\ln\alpha);^{\alpha}_{j}=\rho^{\alpha\beta}_{i;j}+\rho^{\beta}_{i}\rho^{\alpha}_{j}$$
$$=-a^{\alpha\beta}a_{\gamma\delta}\rho^{\gamma}_{i}\rho^{\delta}_{j}-\rho^{\beta}_{i}\rho^{\alpha}_{j}+a^{\alpha\beta}a_{ij},$$

$$(1.8)\quad L^{\alpha\beta}_{ij}[\beta]=-b^{\alpha\beta}b_{\gamma\delta}\sigma^{\gamma}_{i}\sigma^{\delta}_{j}-\sigma^{\beta}_{i}\sigma^{\alpha}_{j}+b^{\alpha\beta}b_{ij}.$$

If we define tensors $a''_{ij}(x,p)$ and $b''_{ij}(x,p)$ as

$$(1.9)\quad \begin{cases} a''_{ij}=a_{ij}-a_{\gamma\delta}\rho^{\gamma}_{i}\rho^{\delta}_{j}, & \mathrm{rank}(a''^{ij})=n-m,\\ b''_{ij}=b_{ij}-b_{\gamma\delta}\sigma^{\gamma}_{i}\sigma^{\delta}_{j}, & \mathrm{rank}(b''^{ij})=n-m,\end{cases}$$

then we have:

PROPOSITION 3. *Legendere's form of α and β are given in the form svch that*

$$L^{\alpha\beta}_{ij}[\alpha]=a^{\alpha\beta}a''_{ij},\quad L^{\alpha\beta}_{ij}[\beta]=b^{\alpha\beta}b''_{ij}.$$

§2. RESULTS.

First of all, we show;

PROPOSITION 4. *The Legendere's form of the fundamental fundamental function given by (1.1) together with (1.2) is*

$$L^{\alpha\beta}_{ij}[F]=2L^{\alpha\beta}_{ij}[\alpha]-L^{\alpha\beta}_{ij}[\beta]+2(\rho^{\beta}_{i}-\sigma^{\beta}_{i})(\rho^{\alpha}_{j}-\rho^{\alpha}_{j}).$$

Proof). Starting from $F;^{\alpha}_{i}=(\alpha^2/\beta);^{\alpha}_{i}=2\alpha\beta^{-1}\alpha;^{\alpha}_{i}-\alpha^2\beta^{-2}\beta;^{\alpha}_{i},$

we rewrite the quantity p^{α}_{i} defined by $p^{\alpha}_{i}=(\ln F);^{\alpha}_{i}$ as

$$(2.1)\quad p^{\alpha}_{i}=F^{-1}F;^{\alpha}_{i}=2\alpha^{-1}\alpha;^{\alpha}_{i}-\beta^{-1}\beta;^{\alpha}_{i}=2\rho^{\alpha}_{i}-\sigma^{\alpha}_{i}$$

by means of (1.3). Applying (1.6) to the fundamenetal fundamental function F, we have the Legendre's form of F such that $L^{\alpha\beta}_{ij}[F]=p^{\alpha\beta}_{i;j}+p^{\beta}_{i}p^{\alpha}_{j}$, to which we substitute (2.1), then it follows;

$$(2.2)\quad L^{\alpha\beta}_{ij}[F]=2\rho^{\alpha\beta}_{ij}-\sigma^{\alpha\beta}_{ij}+(2\rho^{\beta}_{i}-\sigma^{\beta}_{i})(2\rho^{\alpha}_{j}-\rho^{\alpha}_{j}).$$

With use of (2.2) and Proposition 3, we can conclude this proposition. Q.E.D.

By means of the symmetry of $a^{\alpha\beta}$ and (1.7) (respectively by means of antisymmetry of $b^{\alpha\beta}$ and (1.8)), we obtain:

PROPOSITION 5. *The symmetric part of α (resp. β) statisfies the relation*

$$L^{\alpha\beta}_{ij}[\alpha]=a^{\alpha\beta}a''_{ij},\quad (\textit{resp. } L^{\alpha\beta}_{ij}[\beta]=0).$$

From this proposition, it yields:

PROPOSITION 6. *The symmetetric part of the Legender's form of F satisfies the relation*

$$L^{\alpha\beta}_{ij}[F]=2a^{\alpha\beta}a''_{ij}+2(\rho^{(\alpha}_{i}-\sigma^{(\alpha}_{i})(\rho^{\beta)}_{j}-\sigma^{\beta)}_{j}).$$

An areal space in which the relation $L^{(\alpha\beta)}_{i\ j}[F]=g^{\alpha\beta}g''_{ij}$ holds good is said to be of "semi-metric class", where $g''_{ij}=a_{ij}-a_{\gamma\delta}p^{\gamma}_{i}p^{\delta}_{j}$, $\mathrm{rank}(g''_{ij})=n-m$, and $g^{\alpha\beta}$ is symmetric.

Now, in conclusion, we obtain the following theorem wich is the same in appearence as the theorem in [4].

THEOREM. *When the fundamental function of an areal space $A^{(m)}_{n}$ is given by (1.1) together with*

(1.2), then following two conditions are equivalent.

(i). $A_n^{(m)}$ *belongs to the semi-metric class.*

(ii). *The relation* $(\rho^{(\alpha}_{i} - \sigma^{(\alpha}_{i})(\rho^{\beta)}_{j} - \sigma^{\beta)}) = 0$ *holds good.*

Especially we have

COROLLARY. *When the fundamental function of an areal space* $A_n^{(m)}$ *is given by (1.1) together with (1.2), in addition, when the relation, when the relation* $\rho^{\alpha}_{i} = \sigma^{\alpha}_{i}$ *holds good, then the space* $A^{(m)}_n$ *belongs to the metric class and class and it is conformal to the Riemannian space whose metvic is* $a_{ij}(x)$.

Proof). Substiting the relation $\rho^{\alpha}_{i} = \sigma^{\alpha}_{i}$ into (2.2), we have $L^{\alpha\beta}_{ij}[F] = 2a^{\alpha\beta}\, a''_{ij}$ what explains that $A_n^{(m)}$ belongs metric class. Moreover, from $\rho^{\alpha}_{i} - \sigma^{\alpha}_{i} = (\ln \alpha/\beta);^{\alpha}_{i} = 0$, it yields $\ln(\alpha/\beta) = c(x)$. Putting $c_0(x) = \exp(c(x))$, we have $F = \alpha^2/\beta = c_0(x)\,\alpha = c_0(x)\,[\det(a_{ij}(x)\,p^i_\gamma p^j_\mu)]^{1/2} = [\det(\tilde{a}_{ij}(x)\,p^i_\lambda p^j_\mu)]^{1/2}$, where $\tilde{a}_{ij}(x) = \exp((2/m)c(x))a_{ij}(x)$, it shows the conformality.

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