CHARACTERISTICS IN AN RF SUPERCONDUCTING QUANTUM INTERFERENCE DEVICE AS A FUNCTION OF APPLIED MAGNETIC FLUX : SYSTEMATIC CALCULATIONS I

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Abstract

There has been a lot of discussion of characteristics in superconducting quantum interference device (SQUID). However, much less information is available on systematic calculations of these behavior. In this report, we describe various features in a superconducting ring having one Josephson junction. Systematic computer calculations of static behavior of the rf − SQUID have been carried out. The characteristic features depend strongly on a parameter $\beta = (2\pi L I_0) / \Phi_0$, where I_0 is a critical current of the junction, L is a self − inductance of the ring and Φ_0 is the flux quantum. In the regime $\beta > 1$, the quantum states are discrete and the transitions between the quantum states are irreversible. The present work is focused on the correspondence between energy of the system and the characteristics in the rf − SQUID over the range of $\beta = 0.20$ to 2π. The results of the calculations are shown in the following No. 2 paper.

1. Introduction

The superconducting quantum interference device (SQUID) is originated from the epoch − making theoretical prediction of the Cooper pairs tunneling between two superconductors.[1] We will describe here the details of the physical bases and the numerical calculations for the SQUID.

The SQUID has been investigated from various viewpoints and by various kinds of experimental techniques since the first observation by Jaklevic et al. in 1964.[2-4] These subjects represent some of the purest and most fundamental aspects of the superconductivity.[5-9]

However, much less information is available on systematic calculations of the characteristics of the SQUID. The present work is concerned with systematic computer calculations of the static behavior of the rf − SQUID, which contains one ideal Josephson junction in the superconducting ring.

The SQUIDs are based on the two physical pillars. The first is fluxoid quantization and the second is Josephson effect. Figure 1 shows a superconducting ring with a single Josephson weak link. We shall make the simplification that the ideal Josephson junction area is small enough for the current density to be uniform, and that it never contains a significant fraction of a flux quantum. The internal magnetic flux Φ passing through the ring includes the magnetic flux $L I_s$ gener-

ated by the current I_s circulating in the ring, where L is the self $-$ inductance of the ring. As shown in Fig. 1, the internal flux Φ threading the ring is then related to the applied flux Φ_x by

$$\Phi = \Phi_x - LI_s ,$$

where Φ_x is the applied flux intercepted by the ring, and LI_s is the screening flux generated by the induced supercurrent.

In the present paper, many physical quantities have been calcuated as a function of applied magnetic flux Φ_x. Their behavior depends on the dimensionless parameter $\beta = (2\pi LI_0)/\Phi_0$, where I_0 is the critical current of the junction and Φ_0 is the flux quantum. For $\beta < 1$, Φ is a single $-$ valued function of Φ_x, whereas in the regime $\beta > 1$, it is three $-$ valued around half integer values of Φ_x. Then hysteresis appears, for transitions in increasing and decreasing field occur at different Φ_x values. Namely the quantum states are discrete and the transitions between
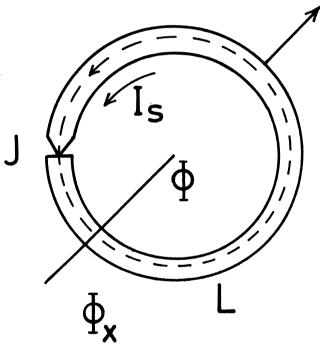

Fig. 1 Superconducting ring with a ideal Josephson junction denoted by J. The contour used for integration is shown by the broken line. Internal magnetic flux Φ, circulating current I_s, self $-$ inductance L and applied magnetic flux Φ_x are related by $\Phi = \Phi_x - LI_s$. Typical values are $L = 5\text{nH}$, and $I_0 = 1\ \mu\text{A}$. The junction resistance in the normal state is $R = 10\ \Omega$, and the diameter of the ring is about 2mm.

the quantum states are irreversible.

Most practical rf − SQUIDs have typical values of $L \sim 5 \times 10^{-10}$ H, $I_0 \sim 1 \times 10^{-6}$ A ; hence the magnitude of β is about 1.50. Our numerical calculations have been carried out for values of β from 0.20 to 2π.

We emphasize the correspondence between the energy of the ring and the various characteristics in the rf − SQUID. For $\beta > 1$, when the applied magnetic flux Φ_x is increased, the potential energy barrier preventing transition from the initial fluxoid quantum state to an adjacent one decreases. Actually, the transition occurs as the energy barrier vanishes at a critical value of Φ_{xc}.

In section 2 the fluxoid quantization, the Josephson tunneling and the energy of the system are briefly reviewed in order to recall their physical meaning and to define physical quantities for the following discussion.

2. Basic Equations

2.1 Fluxoid quantization

2.1.1 Bohr − Sommerfeld quantum condition

A closed line integral of the canonical momentum along a path in a superconducting ring can be derived in the presence of a magnetic field. Then the Bohr − Sommerfeld quantum condition gives the fluxoid quantization, as follows :

$$\oint p \cdot \mathrm{d}l = nh,$$

$$p = m^* v + e^* A \qquad (e^* < 0),$$

where p is the canonical momentum of a Cooper pair ($m^* = 2m$, $e^* = 2e$), A is the vector potential and n is an integer. If the superconducting ring is sufficiently thick in comparison to the penetration depth then the contribution $\int m^* v \cdot \mathrm{d}l$ for the supercurrent vanishes except in the Josephson junction. The dashed line in Fig. 1 represents the integration path. Hence the integral can be written as

$$(1 / e^*) \oint p \cdot \mathrm{d}l = [m^* / (\rho^* e^{*2})] \int_{\text{junction}} j \cdot \mathrm{d}l + \int_{\text{surface}} B \cdot \mathrm{d}S,$$

which is equal to $n(h / e^*) = n(h / 2e) = n\Phi_0$. Here j is the current density given by $(\rho^* e^* v)$, ρ^* being the number of the Cooper pair per unit volume. The enclosed flux will be called internal magnetic flux and denoted by Φ and the flux quantum is defined to be $\Phi_0 (= h / 2e)$.

For simplicity we assume that the Josephson junction is sufficiently small in area. Hence we in-

troduce a following definition of phase difference across the junction θ :

$$\theta \equiv - [m^* / (e^* \rho^* \hbar)] \int_{\text{junction}} j \cdot \mathrm{d}l,$$

$$= 2\pi [\Phi / \Phi_0 + n].$$

The minus sign indicates that the direction of current j is opposite to that of the increase in Φ, as can be seen in Fig. 1.

2.1.2 Ginzburg — Landau viewpoint

The order parameter is a complex quantity as

$$\Psi = | \Psi (x, y, z) | \exp [\mathrm{i} \, \phi (x, y, z)],$$

where the amplitude is $| \Psi (x, y, z) | = \sqrt{\rho^*}$ and $\phi (x, y, z)$ is the phase. A relationship between the current density flowing in the superconductor and the order parameter in the presence of a magnetic field is given by[5]

$$j = [e^* \hbar / (2m^* \mathrm{i})] \, \{\Psi^* \bigtriangledown \Psi - \Psi \bigtriangledown \Psi^*\} - (e^{*2} / m^*) A | \Psi |^2.$$

Substituting Ψ into this equation, we get

$$\bigtriangledown \phi = (e^* / \hbar) A + [m^* / (e^* \rho^* \hbar)] j.$$

From the viewpoint of GL theory, the fluxoid quantization is based on the existence of a single — valued complex superconducting order parameter Ψ. This requires that the phase $\phi (x, y, z)$ must change by an integral multiple of 2π when a complete close circuit has been covered.

$$2\pi n = (e^* / \hbar) \int_{\text{surface}} B \cdot \mathrm{d}S + [m^* / (e^* \rho^* \hbar)] \int_{\text{junction}} j \cdot \mathrm{d}l,$$

$$= (e^* / \hbar) \int_{\text{surface}} B \cdot \mathrm{d}S + \theta .$$

Then, the value of the phase difference θ and defind by

$$\theta \equiv - [m^* / (e^* \rho^* \hbar)] \int_{\text{junction}} j \cdot \mathrm{d}l,$$

$$= 2\pi [\Phi / \Phi_0 + n].$$

2.2 Josephson current

The superconducting current I across the junction shown in Fig. 1 depends on the phase differ-

ence θ as[10]

$$I = I_0 \sin \left\{ \theta \ (0) + e^* \ Vt \ / \ \hbar \right\}$$

where I_0 is the maximum zero − voltage current that can be passed by the junction and V is a constant dc bias voltage. With no bias voltage (the dc Josephson effect) a dc current will flow across the junction with a value between I_0 and $-I_0$ according to the value of the phase difference θ. This phase difference θ adjusts to the current I, according to the above Josephson relation. If a current greater than I_0 is passed through the junction, a voltage appears across it.

2.3 Energy of the system

If the current through the junction is varying with time, the phase difference θ must also be changing with time, and it can be shown that a voltage V is developed across it. This voltage is related to the time rate of θ by[10]

$$\frac{d \theta}{dt} = \frac{2e}{\hbar} V.$$

Consider a junction through which a constant current I_s is flowing, the current having been raised from zero to this value over a time t. During the time the current is increasing, the rate of change of current $dI \ / \ dt$ corresponds to a voltage V across the junction. Therefore, power IV is being delivered to it and work, $dE_J = IVdt$, is performed in setting up the current and the consequent phase difference. The value of dE_J is the increase in energy of the junction due to the passage of a current through it and dE_J is given by[10]

$$dE_J = IVdt,$$

$$= (I_0\hbar) \ / \ (2e) \sin \theta \ \frac{d \theta}{dt} dt.$$

Then, we obtain the junction coupling energy E_J which depends on the phase difference, as

$$E_J \fallingdotseq (I_0\hbar) \ / \ (2e) \left[1 - \cos \theta \right],$$

$$\equiv - E_0 \cos \theta + \text{constant}.$$

Provided E_J is large compared with the thermal fluctuation energy $k_B T$, phase coherence extends across the barrier and a supercurrent can be passed through up to the critical current I_0.

On the other hand if the current I goes through the superconducting ring, the magnetic energy of the current flowing in the inductance L is $(1 \ / \ 2) LI^2$. The energy of the system of the supercon-

ducting ring is then expressed by two terms,[11)]

$$U = \frac{1}{2}LI_s{}^2 - E_0\cos\theta \, ,$$

$$= \left(\frac{1}{2L}\right)(\Phi - \Phi_x)^2 - E_0\cos\left(\frac{2\pi\Phi}{\Phi_0}\right) \, ,$$

where the first term is the magnetic energy E_m associated with a current and the second is the junction coupling energy E_J. Here the electro−static Coulomb energy associated with any differ-ence in charge density between the two sides of the barrier is neglected. The extra amount of constant energy is also assumed to be zero.

2.4 Basic equations

The basic equations are summarized and are described below. The main characteristics of the *rf* − SQUID are the behaviors of the internal flux Φ and of the screening circulating current I_s as a function of the external flux Φ_x. They are derived from the next equations,

$$\Phi = \Phi_x - LI_s , \tag{1}$$

$$\theta = 2\pi[\Phi/\Phi_0 + n], \tag{2}$$

$$I_s = I_0\sin\theta . \tag{3}$$

Equations (1), (2) and (3) are linked equations for the three unknown quantities Φ, I_s and θ in terms of the applied flux Φ_x. Here we introduce dimensionless parameter β, defined as

$$\beta = (2\pi LI_0)/\Phi_0, \tag{4}$$

where β depends on the value of LI_0. The limiting forms of the equations are $\Phi = \Phi_x$ for $LI_0 = 0$, which corresponds to an open ring, and complete flux quantization $\Phi = n\Phi_0$ for $LI_0 \gg \Phi_0$, which corresponds to a closed ring with no weak link. Making the substitution of eqs. (2) and (3) into eq. (1), we get a next relation,

$$\Phi = \Phi_x - LI_0\sin(2\pi\Phi/\Phi_0). \tag{5}$$

Substituting eqs. (1) and (2) into eq. (3) gives

$$I_s = I_0\sin(2\pi\Phi/\Phi_0). \tag{6}$$

For the ring with a junction the energy of the system is given by

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$$U = \left(\frac{1}{2L}\right)(\Phi - \Phi_x)^2 - E_0\cos\left(\frac{2\pi\Phi}{\Phi_0}\right). \tag{7}$$

3. Numerical Computer Calculations for the Characteristics in *rf* — SQUID

3.1 The system energy $U(\Phi, \Phi_x)$

The system energy U, normalized by $(\Phi_0^2 / 2L)$, is written as

$$U / (\Phi_0^2 / 2L) = (\Phi / \Phi_0 - \Phi_x / \Phi_0)^2 - \beta / (2\pi^2)\cos(2\pi\Phi / \Phi_0). \tag{8}$$

The energy $U(\Phi, \Phi_x)$ of the system given by eq. (8) is calculated. Figures 2 to 11 show the results of $U(\Phi, \Phi_x)$ for various values of parameter β. Using partial derivative of $U(\Phi, \Phi_x)$ with respect to Φ, the condition at the local minimums of $U(\Phi, \Phi_x)$ is

$$\frac{\partial U(\Phi, \Phi_x)}{\partial \Phi} = 0, \tag{9}$$

which leads to the next equation :

$$\Phi = \Phi_x - LI_0\sin\left(\frac{2\pi\Phi}{\Phi_0}\right). \tag{10}$$

This is exactly the same equation to eq. (5). We can get the following relation :

$$\frac{d\Phi}{d\Phi_x} = \frac{1}{1 + \beta\cos(2\pi\Phi / \Phi_0)}. \tag{11}$$

For $\beta < 1$, the denominator of eq. (11) has always positive value and then there are no portions of the curve with negative slope in Φ. Therefore, Φ is a single — valued function. As a result, the ring has only one stable state whose Φ value is obtained by eq. (10) for any value of Φ_x. On the other hand, for $\beta > 1$, Φ in eq. (10) is three — valued for some parts of the range Φ_x. Then meta-stable states can exist. Consequently, hysteresis can occur since the transitions in increasing and decreasing flux occur at different Φ_{xc}.

For an example, let explain the behavior of the hysteresis in Fig. 9 in the case of $\beta = 1.50$. Figure 9 shows the behavior in the low energy portion of U for $\beta = 1.50$ in greater detail. There are two specific inflection points at $\Phi / \Phi_0 = 0.366$ and 0.634 in Fig. 9. When increasing the external flux Φ_x, the transition occurs at $\Phi_{xc} / \Phi_0 = 0.544$, whereas it takes place at $\Phi_{xc} / \Phi_0 = 0.456$ when decreasing the flux. Up to flux Φ_{xc} the system is maintained in the lower flux side of the potential valley by the central barrier, even if it is in the unstable equilibrium state (see $\Phi_x = 0.52$ in Fig. 9). Finally the barrier vanishes at Φ_{xc} and the transition takes place. During the back process the system change occurs in the inverse order.

The behavior of the system can be understood by considering its trajectory. The essential feature is that the external flux jumps from an initial fluxoid quantum state to the next fluxoid state.

3.2 The junction coupling energy E_J vs. external flux Φ_x

The junction coupling energy is

$$E_J = -E_0 \cos(2\pi\Phi / \Phi_0). \tag{12}$$

Using the expressions of E_0 and β, we get the normalized form of E_J

$$E_J / (\Phi_0^2 / 2L) = -[\beta / (2\pi^2)] \cos(2\pi\Phi / \Phi_0). \tag{13}$$

Figures 12 to 17 display the E_J vs. Φ_x for the following values of parameter β : 0.20, 0.50, 1.00, 1.50, 3.00 and 2π. The derivative of E_J with respect to Φ_x becomes

$$\frac{dE_J}{d\Phi_x} = \frac{I_0 \sin(2\pi\Phi / \Phi_0)}{1 + \beta \cos(2\pi\Phi / \Phi_0)}. \tag{14}$$

For $\beta > 1$, if $\cos\theta = -1/\beta$, the denominator of eq. (14) goes to zero, leading to an infinite slope of E_J at Φ_{xc}. Φ_{xc} is given by the next two equations :

$$\Phi_{xc} / \Phi_0 = \Phi / \Phi_0 \pm (\beta / 2\pi)(1 - \beta^{-2})^{1/2}, \tag{15}$$

$$\Phi / \Phi_0 = (1 / 2\pi) \cos^{-1}(-1/\beta). \tag{16}$$

As an example, in the case $\beta = 1.50$, we get two values of Φ_{xc} / Φ_0, namely 0.544 and 0.456, in the range $0 < \Phi / \Phi_0 < 1$. These critical values are exactly the same as those obtained from the analysis of the system energy U, as explained previousy. That is, hysteresis occurs since the transitions in increasing and decreasing flux takes place at different Φ_x values. These critical values Φ_{xc} correspond to the position at which the slope of E_J takes infinite value.

3.3 The magnetic energy E_m vs. external flux Φ_x

The magnetic energy of the ring is

$$E_m = \left(\frac{1}{2L}\right)(\Phi - \Phi_x)^2. \tag{17}$$

The normalized form is

$$E_m / (\Phi_0^2 / 2L) = [\beta / (2\pi)]^2 \sin^2(2\pi\Phi / \Phi_0). \tag{18}$$

Figures 12 to 17 show the E_m vs. Φ_x behavior with parameter β from 0.20 to 2π.

The derivative of E_m with respect to Φ_x becomes

$$\frac{dE_m}{d\Phi_x} = \frac{(\beta / L)(\Phi_x - \Phi)\cos(2\pi\Phi / \Phi_0)}{1 + \beta\cos(2\pi\Phi / \Phi_0)}. \tag{19}$$

For $\beta > 1$, if $\cos\theta = -1/\beta$, the denominator of eq. (19) goes to zero, leading to an infinite slope of E_m at Φ_{xc}. The value of Φ_{xc} is the same as obtained by eqs. (15) and (16).

Hysteresis occurs when the transitions in increasing and decreasing flux take place at different Φ_x values. These critical value Φ_{xc} corresponds to the position at which the slope of E_m gives infinite value.

The total energy U, which is normalized, is also shown in Figs. 12 to 17. The value of U in these figures corresponds to the maximum or minimum value in Figs 6 to 11

3.4 E_J, E_m vs. phase difference θ

The junction coupling energy E_J, the magnetic energy E_m and the system energy U are expressed as a function of the phase difference θ as follows :

$$E_J / (\Phi_0{}^2 / 2L) = -[\beta / (2\pi^2)]\cos\theta, \tag{20}$$

$$E_m / (\Phi_0{}^2 / 2L) = [\beta / (2\pi)]^2\sin^2\theta, \tag{21}$$

$$U / (\Phi_0{}^2 / 2L) = -[\beta / (2\pi^2)]\cos\theta + [\beta / (2\pi)]^2\sin^2\theta, \tag{22}$$

$$= -[\beta / (2\pi^2)]\cos(2\pi\Phi / \Phi_0) + [\beta / (2\pi)]^2\sin^2(2\pi\Phi / \Phi_0). \tag{23}$$

Equation (23) can be also obtained by making a substitution of eq. (10) into eq. (8). Consequently the physical meaning of eqs. (22) and (23) is that the minimum, the maxiimum, local minimum or local maximum energies of the superconducting ring are given by eqs. (22) or (23). Figures 18 to 23 show the behavior of these functions for several values of β.

The derivative U with respect to θ gives

$$d[U / (\Phi_0{}^2 / 2L)] / d\theta = [\beta / (2\pi^2)]\sin\theta\,[1 + \beta\cos\theta]. \tag{24}$$

When $\beta < 1$, the superconducting ring has a minimum energy at $\theta = 0$ and a maximum energy at $\theta = \pi$. On the other hand, when $\beta > 1$, the system has a minimum energy at $\theta = 0$ and a maximum energy at $\theta = \cos^{-1}(-1/\beta)$.

3.5 Internal flux Φ vs. external flux Φ_x

In Fig. 24 to 29, results of calculations for Φ versus Φ_x relation are shown. These behavior depends on the dimensionless parameter $\beta = (2\pi LI_0) / \Phi_0$. Taking the derivatives of Φ with re-

spect to Φ_x, we can get the following relations using eqs. (5) or (10) :

$$\frac{d\Phi}{d\Phi_x} = \frac{1}{1 + \beta \cos (2\pi\Phi / \Phi_0)}. \tag{25}$$

For $\beta < 1$, the denominator of eq. (25) is always positive and then, there are no portions of the curve with negative Φ vs. Φ_x slope. Namely, the value of Φ increases monotonically as a function of Φ_x. Therefore, Φ is a single–valued functions of Φ_x. The slope of Φ is maximum at Φ_x / Φ_0 = 1 / 2. On the contrary, for $\beta > 1$, the Φ vs. Φ_x curves have regions of negative slope. The portions with negative slope in these curves are unstable. This point has been already discussed from the viewpoint of the system energy U. Namely the quantum states are discrete and the transitions between the quantum states are irreversible. Consequently, hysteresis exists when the transitions in increasing and decreasing field occur at different Φ_x values.

For $\beta > 1$, if $\cos\theta = - 1 / \beta$, the denominator of eq. (25) goes to zero leading to an infinite slope of Φ vs. Φ_x at Φ_{xc}. These values Φ_{xc} are given by the two equations (15) and (16). As an example, in the case $\beta = 1.50$, values 0.544 and 0.456 for Φ_{xc} / Φ_0 correspond to the critical positions of the transitions when Φ_x increases or decreases in a range of $0 < \Phi_x / \Phi_0 < 1$, respectively. These critical values are exactly the same as those obtained from the analysis for the system energy U.

When increasing β, the flux ranges of successive hysteresis path overlaps partly as can be seen in Fig. 29. The critical value of β leading to overlap of the hysteresis path is given by the criterion that the critical external flux Φ_{xc} of the one fluxoid quantum state reaches zero in the decreasing flux process. Then the critical value of β_0 is deduced by the two eqs. (15) and (16), which give the next equation.

$$(1 / 2\pi) \cos^{-1} (- 1 / \beta_0) = (\beta_0 / 2\pi)(1 - \beta_0^{-2})^{1/2}. \tag{26}$$

We can obtain the critical value of β_0 by numerical calculation,

$$\beta_0 = 4.6033. \tag{27}$$

3.6 Induced flux LI_s vs. external flux Φ_x

In Fig. 24 to 29, results of calculations of LI_s versus Φ_x are shown. The numerical calculations have been made by using eq. (1). The calculated curves depend on the dimensionless parameter β. In these figures the positive or negative sign of LI_s correspond to the current direction in the ring. By considering eq. (1), the value of LI_s is obtained by subtracting the value of Φ from the diagonal line in the Φ vs. Φ_x relation. Hence the criterion for hysteresis is derived in

the same way as for the Φ vs. Φ_x curves. For $\beta < 1$, I_s is a single − valued functions of Φ_x, whereas for $\beta > 1$, LI_s is three − valued in some parts of the Φ_x range. Hysteresis appears since the transitions in increasing and decreasing flux occur at different Φ_x values. The critical value Φ_{xc} corresponds to the position at which the slope of I_s becomes infinite. These values of Φ_{xc} are exactly the same as those obtained by the analysis of the system energy U as well as the case of Φ vs. Φ_x.

It should be noticed that the transition does not take place at the maximum value I_0 but at the value I_{sc} given by

$$I_{sc} = I_0 (1 - \beta^{-2})^{1/2} \qquad (\beta > 1). \tag{28}$$

For $\beta = 1.50$, the critical value of the current I_{sc} is $0.745 I_0$. In addition, the screening current has the form $I_0 \sin(2\pi\Phi / \Phi_0)$. Then the maximum current appears always at $\Phi / \Phi_0 = 1 / 4$.

3.7 Phase difference θ vs. external flux Φ_x

In Figs. 30 to 35, the values of the phase difference θ across the junction as a function of external flux Φ_x are shown. Taking the derivatives of θ with respect to Φ_x, we obtain

$$\frac{d\theta}{d\Phi_x} = \frac{(2\pi / \Phi_0)}{1 + \beta \cos(2\pi\Phi / \Phi_0)} . \tag{29}$$

For $\beta < 1$, the denominator of eq. (29) is always positive and there are no portions of the θ vs. Φ_x curves with negative slope. Therefore, θ is a single − valued functions of Φ_x. Instead, for $\beta > 1$, θ is multi − valued for some parts of the Φ_x range. Hysteresis appears since the transitions in increasing and decreasing flux occur at different Φ_x values. These critical values Φ_{xc} correspond to the position at which the slope of θ takes an infinite value.

3.8 Fluxoid vs. external flux Φ_x

In Figs. 30 to 35, results of the absolute value of $\left| \text{fluxoid} / \Phi_0 \right|$ are shown as a function of Φ_x. The applied flux Φ_x is able to drive the transition from one fluxoid quantum state to another successive quantum state. From eq. (2), $\left| \text{fluxoid} / \Phi_0 \right|$ is integer n which is expressed as

$$\left| \text{fluxiod} / \Phi_0 \right| = n = (\theta / 2\pi) - (\Phi / \Phi_0). \tag{30}$$

The derivative of n is

$$\frac{dn}{d\Phi_x} = (1 / 2\pi) \frac{d\theta}{d\Phi_x} - (1 / \Phi_0) \frac{d\Phi}{d\Phi_x},$$

$$= \frac{1}{1 + \beta \cos(2\pi\Phi / \Phi_0)} [(1 / \Phi_0) - (1 / \Phi_0)]. \tag{31}$$

The derivative can have a non zero value only if $\cos \theta = -1 / \beta$, which corresponds to the transitions between two fluxiod quantum states. These critical values have also the same maginitudes as those given by the analysis of the system energy U. The derivative of eq. (31) is zero if $\cos \theta \neq -1 / \beta$, which means for n to be constant.

For $\beta < 1$, the fluxoid can change at $\Phi_x / \Phi_0 = 1 / 2$: the system energy reaches its maximum value corresponding to $\theta = \pi$ and one flux quantum can enter into the ring.

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Figure Captions

Figures of the numerical calculations of the characteristics in the rf – SQUID and the figure captions are given in the following paper.