グラフにおける中心的サイクル

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Central Cycles in Graphs

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We consider the problems of identifying subgraphs which are centres, medians, or centroids of a given simple, connected graph. The case where the subgraph comprises a discrete set of vertices is well known. However concepts which measure centrality in a graph, such as: eccentricity, distance, and subset cardinality, can be extended to connected subgraphs such as: paths, trees, and cycles. Methods have been reported which deal with the requirement that the subgraph is a path or a constrained tree. In this paper we extend this work to the case where the subgraph is required to be a cycle. We first examine the case where the underlying graph is a grid-graph. We then report on a tabu search-based heuristic and on randomized exchange methods for the identification of cycle centres, medians and centroids in general graphs.

Keywords : Graph, Network, Cycle, Loction Problem

1 Introduction

This paper introduces some cycle location problems on graphs. Concepts which measure the centrality of a vertex in a graph (such as : eccentricity, distance and component cardinality) are extended to a cycle in a graph. Locating cycles with minimum eccentricity, distance and component cardinality may be viewed as multicentre, multimedian, and multicentroid problems respectively, where the facilities are located on vertices that must constitute a cycle. These problems are related to the traveling salesman and Hamiltonian cycle problems on graphs. We now introduce some notation and terminology.

Let $G = (V, E)$ denote a simple, connected, undirected graph with vertex set V (with $|V| = n$), and edge set E. For the graph theoretic notation and terminology used in this paper, see Foulds(1998). An edge between u and v is denoted by uv. The *distance* between vertices x and y, denoted by $d(x, y)$, is defined to be the length of a shortest path in G between vertices x and y, expressed as the number of edges.

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The *eccentricity* of a vertex x, denoted by $e(x)$, is defined to be $e(x) = \max\{d(x,y) : y \in V\}$. The *diameter* of G, denoted by $diam(G)$, is defined to be $diam(G) = \max\{e(x) : x \in V\}$. The *radius* of G, denoted by $rad(G)$, is defined to be $rad(G) = \min\{e(x) : x \in V\}$. A vertex x of G is called a *centre* if $e(x) = rad(G)$. The distance of a vertex x, denoted by $d(x)$, is defined by $d(x) = \sum\{d(x,y) : y \in V\}$. The distance of G, denoted by $d(G)$, is defined by $d(G) = \sum\{d(x,y) : x, y \in V\}$.

We now extend these concepts from a single vertex to subsets of vertices, and also to cycles. For any subset $U \subseteq V$, and any $x \in V$, let $d(x, U) = \min\{d(x, u) : u \in U\}$. The *eccentricity* $e(U)$ of U is defined by $e(U) = \max\{d(x, U) : x \in V\}$, and the *distance* $d(U)$ of U is defined by $d(U) = \sum\{d(x, U) : x \in V\}$. The *weight* $w(U)$ is the largest number of vertices in a component of $G - U$.

We denote the vertex set of any cycle C in G by $V(C)$. Then $e(C) = e(V(C)), d(C) = d(V(C))$ and $w(C) = w(V(C))$ are called the *eccentricity*, the *distance* and the *weight* of the cycle C, respectively.

The length of a cycle C is the number of edges of C. A cycle with the length p is called by p-cycle.

Definition 1

A cycle C of G is:
(1) A *p-cycle center* of G if $e(C) \leq e(C')$ for any p−cycle

C'.
(2) A *p-cycle median* of G if $d(C) \leq d(C')$ for any p−cycle C'.
(3) A $p-$*cycle centroid* of G if $w(C) \geq w(C')$ for any p−cycle C'.

Definition 2

A cycle C of G is:
(i) A *cycle centre* if C has minimum eccentricity among all cycles of G and has minimum length among cycles of G having minimum eccentricity.
(ii) A *cycle median* if C has minimum distance among all cycles of G and has minimum length among all cycles of G having minimum distance.
(iii) A *cycle centroid* if C has minimum weight among all cycles of G and has minimum length among all cycles of G having minimum weight.

We now discuss applications of these ideas, beginning with some which are well-known for original concepts of : centre, median, and centroid. In general, the theory of facilities location, well known in operations research and industrial engineering, is concerned with the location of one or more facilities which are sited so as to service a number of given clients. We confine our attention here to the location of facilities on graphs or networks, rather than in the plane.

The first application is concerned with the location of emergency facilities (e.g. emergency clinics, police stations, or fire stations) on a network. The objective is to minimize the greatest distance between any of the facilities and any of the clients. In terms of what we have discussed previously, this corresponds to the identification of the set of centres of the underlying graph, which will represent the locations of the facilities.

The second application is concerned with the location of non-emergency facilities (e.g. libraries, post offices, or government bureaus). The objective is to minimize the total distance between each client and the facility closest to it. In terms of what we have discussed previously, this corresponds to the identification of the set of medians of the underlying graph.

The third application is concerned with the location of special facilities (e.g. distribution, or communication centres). The objective is to minimize the size of any identifiable neighborhood of connected clients which is not part of the facility. In terms of what we have discussed previously, this corresponds to the identification of the set of centroids of the underlying graph.

Most facility location models assume that the facilities to be located can be represented by a disconnected set of vertices, possibly a singleton. However it is sometimes productive for the set of vertices to have a given structure within the underlying graph. For instance, the circumstance when the set constitutes a path or tree has been studied by Buckley and Harary(1990), Richey(1990), Hakimi(1993) and Slater(1980,1981,1982 and 1983). There are applications when the vertex set is a tree, such as: the design of systems for gas reticulation. irrigation pipelines, or freeway systems. There are also applications when the vertex set is a path (a special type of tree), such as: the design of a new superhighway or a new subway line. We are concerned in this paper with the new special case in which the vertex set is a cycle. Examples of applications of this scenario include: the design of circular human communication networks in organizational structures, circular bus routes, and the creation of express ring roads in an urban environment.

Observation :
The following are equivalent:

(1) G has a Hamiltonian cycle C.
(2) G has a p-cycle centre C with $e(C) = 0$.
(3) G has a p-cycle median C with $d(C) = 0$.
(4) G has a p-cycle centroid C with $w(C) = 0$.

The problem of determining whether a given graph contains a Hamiltonian cycle is NP-complete(Garey and Johnson(1979)). Therefore, the problem of determining whether (2), (3), or (4) are true for a given graph are also NP-complete.

In the next two sections we discuss p-cycle centres and p-cycle medians in grid-graphs. In Section 4 some properties of the eccentricity, the distance and the weight of a cycle in general graphs will be established. In Section 5 we report on a tabu search technique which can be used for the same purpose. We summarize our findings, present some conclusions on them, and suggest directions for further research in Section 7.

2 Cycle centres in grid-graphs

For given positive integers $n_1 \geq 1, n_2 \geq 1$, a grid-graph $G_{n_1 \times n_2}(V, E)$ is defined as follows:

$$V = \{v_{ij} = (i,j) : 0 \leq i \leq n_1, 0 \leq j \leq n_2, i, j \text{ integers}\},$$
$$E = \{\{v_{ij}, v_{kl}\} : \{|i-k|, |j-l|\} = \{0, 1\}\}.$$

Theorem 1
(1) If at least one of n_1, n_2 are odd, there exists a Hamilton cycle C with $e(C) = 0$.
(2) If the numbers n_1, n_2 are both even, there exists a cycle C with $e(C) = 1$.
Now we restrict the problem to smaller cycles:

Problem 1
Among cycles with the rectangular form:
$C : P(p_1, p_2), Q(p_1 + m_1, p_2), R(p_1 + m_1, p_2 + m_2)$ and $S(p_1, p_2 + m_2)$
of $G_{n_1 \times n_2}(V, E)$, how can one characterize all m_1, m_2, p_1, and p_2 with $\min e(C)$ such that $L(C)$ is least among all such cycles?
Here, n_1, n_2, m_1, m_2 are positive integers and $1 \leq m_i \leq n_i (i = 1, 2)$.

That is, the rectangle $C : PQRS$ on $G_{n_1 \times n_2}(V, E)$ with minimum eccentricity is to be characterized .

Let $O(0,0), X(n_1,0), Y(n_1,n_2), Z(0,n_2)$ be vertices of $G_{n_1 \times n_2}$.
Then $d(v,C)(v = O,X,Y,Z)$ are the following:
$d(O,C) = d(O,P) = p_1 + p_2$,
$d(X,C) = d(X,Q) = n_1 - (p_1 + m_1) + p_2$,
$d(Y,C) = d(Y,R) = n_1 - (p_1 + m_2) + n_2 - (p_2 + m_2)$, and
$d(Z,C) = d(Z,S) = p_1 + n_2 - (p_2 + m_2)$.
Without loss of generality, we may assume that

$$p_1 \leq n_1 - (p_1 + m_1), \quad p_2 \leq n_2 - (p_2 + m_2). \tag{2.1}$$

Thus
$\max d(v,C) = d(Y,C) = n_1 - (p_1 + m_1) + n_2 - (p_2 + m_2)$.
Let $e_{in}(C) = \max\{d(x,C) : x \text{ is inside } C\}$. Note that $e_{in}(C)$ is independent of the location of C and is constant.
Let $e_{out}(C) = \max\{d(x,C) : x \text{ is outside } C\}$.
Note that $e_{out}(C) = \max\{d(O,P), d(X,Q), d(Y,R), d(Z,S)\}$.
By symmetry, $e_{out}(C) = d(O,P) = p_1 + p_2$.
This is so because, by (2.1):

$$\begin{aligned} d(O,P) &= p_1 + p_2 \\ &\geq d(X,Q) = n_1 - (p_1 + m_1) + p_2 \\ &\geq d(Y,R) = n_1 - (p_1 + m_1) + n_2 - (p_2 + m_2) \\ &\geq d(Z,S) = p_1 + n_2 - (p_2 + m_2). \end{aligned}$$

By (2.1) again, $p_i \leq \lfloor (n_i - m_i)/2 \rfloor (i = 1,2)$.(Here and in the following $\lfloor x \rfloor$ is the largest integer smaller than or equal to x.)
Then the minimum of $p_1 + p_2$ is $\lfloor (n_1 - m_1)/2 \rfloor + \lfloor (n_2 - m_2)/2 \rfloor$, which is $e_{out}(C)$. Then,

$$e(C) = \begin{cases} e_{in}(C), & \text{if } e_{in}(C) \geq e_{out}(C) \\ e_{out}(C), & \text{if } e_{in}(C) < e_{out}(C). \end{cases}$$

That is,

$$e(C) = \max\{e_{in}(C), e_{out}(C)\}.$$

Thus
$e_{in}(C) = \min\{\lfloor m_1/2 \rfloor, \lfloor m_2/2 \rfloor\}$.

Theorem 2
When $e_{in}(C) \geq e_{out}(C)$, $e(C)$ does not depend on the position of C. When $e_{in}(C) < e_{out}(C)$, the minimization of $e(C)$ is realized when $p_i = \lfloor n_i/2 - m_i/2 \rfloor (i = 1,2)$.

3 Cycle medians in grid-graphs

Problem 2
Characterize the rectangular cycle with minimum distance.
Consider a grid-graph $G_{n_1 \times n_2}(V,E)$ with cycle C of size $m_1 \times m_2$. Let:

$V = \{0,1,...,n_1\} \times \{0,1,...,n_2\}$,
$P(p_1,p_2)$; $0 \leq p_i \leq n_i (i = 1,2)$, and
$C_P = \{(p_1 + i, p_2 + j) : 0 \leq i \leq m_1, 0 \leq j \leq m_2\}$ (The inside of C).

In order to treat the distance of cycles in more detail, we define:
$d_{in}(C;P) = d_{in}(C) = \sum_{u \in C_P} d(u,C)$, and
$d_{out}(C;P) = d_{out}(C) = \sum_{u \in V - C_P} d(u,C)$

Using these we have:

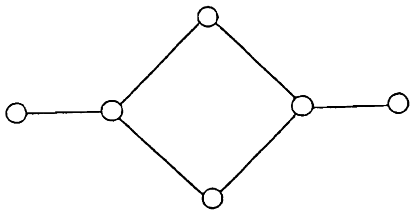

Figure 1: An extremal graph for inequality (4.1).

$d(C;P) = d_{in}(C;P) + d_{out}(C;P)$, and
$d(C) = \min_P d(C;P)$.

Here, $d_{in}(C;P)$ is independent of the position of C. So it is enough to consider the term $d_{out}(C;P)$. We first consider the grid-graph $G_{x \times y}$.
Let $d(x,y,y')$ be $\sum_{(i,j) \in V'} d((i,j),P')$, where V' is the vertex set of $G_{x \times y}$ and P' is the boundary path : $\{(x,j) | y' \leq j \leq y\}$. By an elementary calculation, we obtain:
$d(x,y,y') = (x+1)S(y') + (y+1)S(x)$, where $S(t) = t(t+1)/2$.
Let $d_i = n_i - m_i, q_i = d_i - p_i (i = 1,2)$.
Then
$d_{out}(C) = d(p_1, p_2 + m_2 - 1, p_2) + d(q_2, p_1 + m_1 - 1, p_1) + d(q_1, n_2 - p_2 - 1, q_2) + d(p_2, n_1 - p_1 - 1, q_1)$.
Thus,

$$d_{out}(C) = n_2\{S(p_1) + S(q_1)\} + n_1\{S(p_2) + S(q_2)\}.$$

Theorem 3
The minimization of $d_{out}(C)$ is also realized when $p_i = \lfloor n_i/2 - m_i/2 \rfloor (i = 1,2)$.

4 Centrality measures of general graphs

In this section, we present some properties concerning with the eccentricity, the distance and the weight of a cycle in general graphs. Let $L(C)$ be the length of any cycle C and $diam(C) = \max\{d(x,y) : x,y \in C\}$.
First we state a simple property.

Theorem 4

$$diam(G) \leq 2e(C) + diam(C). \tag{4.1}$$

The above bound is sharp, as shown for the graph in Figure 1, where $diam(G) = 4$, $diam(C) = 2$, and $e(C) = 1$.

Corollary
If $L(C) \leq l$ then $e(C) \geq \{diam(G) - l/2\}/2$. (4.2)

That is, a cycle C, for which $L(C) = l, diam(C) = l/2$, and $e(C) = \{diam(G) - l/2\}/2$, is a cycle centre.
Of course, such a cycle does not exist in certain graphs.

We now state a relationship between $e(C)$ and $d(C)$. Recall that the number of vertices in the underlying graph G, is denoted by n.

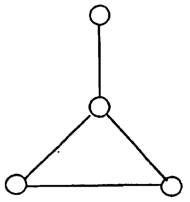

Figure 2: An extremal graph for inequality (4.3).

Theorem 5

$$d(C) \leq (n - L(C))e(C). \quad (4.3)$$

The above bound is sharp, as shown for the graph in Figure 2, where $d(C) = 1, n = 4, L(C) = 3$, and $e(C) = 1$.

Let $w(G) = \min\{w(C) : C$ is a cycle of $G\}$. A cycle C of G is called a dominating cycle when the vertex set of $G - C$ (the resultant of removing C from G) is a stable (independent) set of G. More generally, a cycle C of G is called a $k-$dominating cycle when the cardinality of vertices of each component of $G - C$ is less or equal to k.

Theorem 6
(1) If a graph G has a dominating cycle, then $w(G) \leq 1$.
(2) If a graph G has a $k-$dominating cycle, then $w(G) \leq k$.

We get the following:

Theorem 7
For any integers k, l such that $0 \leq k < l$, there exists a graph G which has a cycle C with $w(C) = k$ and $L(C) = l$.

5 A tabu search heuristic

The following two sections are concerned with algorithmic aspects about central cycles defined by Definition 2. We now present a heuristic (which does not neccessarily find the best possible central cycle in any given graph) based on the tabu search(TS) metaheuristic of Glover(1997). The same general TS procedure can be used to attempt to find either the cycle centre, the cycle median or the cycle centroid of any given graph G. We provide a simple overview of the procedure first, followed by a more rigorous description later.

The procedure requires, as input, a spanning tree T of G. Next the fundamental cycles in G with respect to T are identified by adding each of the chords in G singly to T. If a cycle is known or suspected to be central, or somewhat close to being central, then we suggest that T should be chosen so that this cycle is fundamental with respect to it. If no such information is known then T is chosen at random.

The method proceeds by transforming T at each iteration by successively dropping one of its edges and adding a new chord to create a new spanning tree. $z(C)$ (with the subscript E or D, or W according to the type of centrality desired) is termed the *cost* of any cycle in G and is to be minimized.The edge swap is performed in such a way so as to produce a fundamental cycle C in the new spanning tree which is as central as possible, in the sense that $z(C)$ is as low as possible. The best cycle identified at each iteration is recorded so as to prevent cycling. Once an edge is dropped from T, it is tabu (for a certain number of iterations) to make it part of T again.

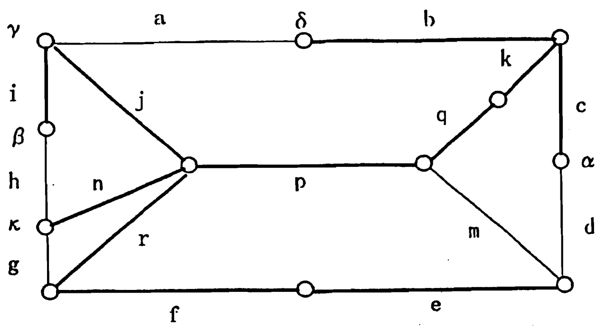

Figure 3: The initial graph for the Tabu Search : T^1.

The objective functions are:
$z_E(C) = Me(C) + L(C)$ [centre],
$z_D(C) = Md(C) + L(C)$ [median],
$z_W(C) = Mw(C) + L(C)$ [centroid],
where M is a suitably large number. For example, M can be conveniently chosen as : $M = 10|V|$.

The least cost cycle found so far is recorded and updated.

The Choice Rule:
Make the edge swap in T that produces the *new* fundamental cycle with least cost. The edge dropped must be part of the best cycle identified in the last iteration. Thus adding the chord that induced the current best fundamental cycle in the last iteration cannot be added.

Tabu Restriction:
Make tabu adding to T an edge e, dropped from T for m iterations, where m is chosen appropriately. It is often convenient to begin by setting $m = 2$.

Aspiration Criterion:
Override the tabu restriction if a different edge swap produces a new least cost cycle which is the best cycle found so far.

The TS heuristic will now be illustrated on the graph G shown in Figures 3 and on. Let $M = 100$ and $m = 2$ throughout. We begin with the objective of attempting to find the cycle centre of the graph, i.e. minimizing $z_E(C)$.

The initial spanning tree T^1 which was chosen at random, is shown by the heavy edges: $b, c, e, f, i, j, k, n, p, q$, and r in Figure 3. T^1 induces five fundamental cycles of G, each constructed by adding exactly one of the chords: a, d, g, h, and m to T^1. We now denote T^1 by T_E^1. For example the chord h induces the fundamental cycle $C_h^1 =< h, i, j, n >$, with eccentricity 3, due to the fact that vertex α is of distance 3 from C_h^1. The superscript 1 indicates that this cycle is fundamental with respect to T_E^1. As C_h^1 has length 4, $z_E(C_h^1) = 100 \times 3 + 4 = 304$. However $C_m^1 =< m, e, f, r, p >$, with $z_E(C_m^1) = 205$, is the fundamental cycle with respect to T_E^1 with the lowest z_E value. Thus it is recorded as the best cycle found so far. The next step is to drop one of the edges in $T_E^1 \cap C_m^1 = \{e, f, r, p\}$ and add another edge to T^1 to produce T_E^2. This edge swap must be selected so as to minimize the z_E value of the best of fundamental cycles with respect to T_E^2. The least cost

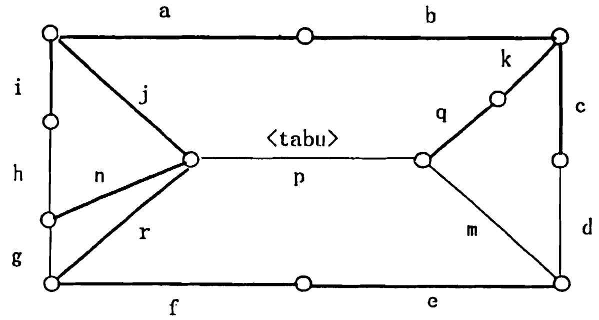

Figure 4: Central cycle search : T_E^2, T_D^2, and T_W^2.

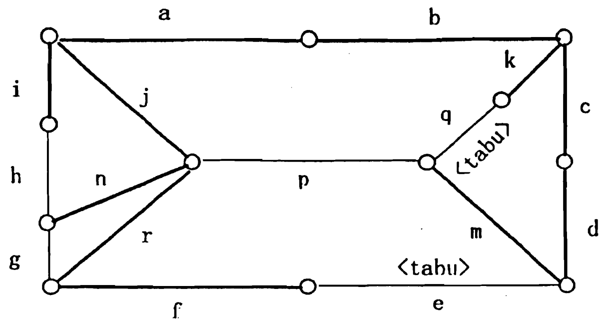

Figure 6: T_E^4 and the cycle centre C_p^4.

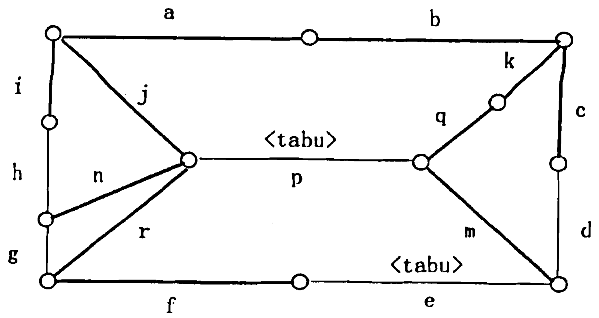

Figure 5: Cycle centre search : T_E^3.

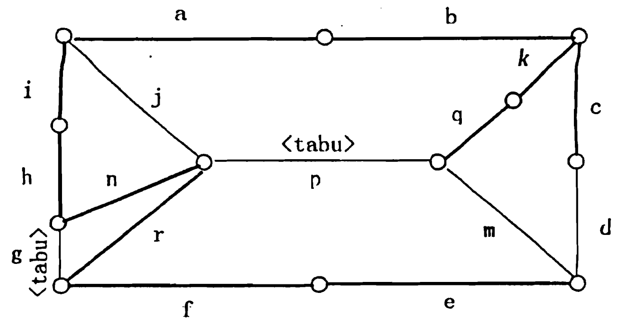

Figure 7: T_D^3 and the cycle median C_m^3.

edge swap involves dropping edge p and adding edge a. This creates the spanning tree T_E^2 shown with heavy lines in Figure 4.

The least cost cycle is $C_d^2 =< d, e, f, r, j, a, b, c >$ with $z_E(C_d^2) = 108$. This cycle is recorded as the best (least cost) cycle found so far. We make it tabu to add edge p to any spanning tree for 2 iterations.

This above process is now repeated. The least cost edge swap involves dropping edge e and adding edge m. We make the addition of edge e tabu for 2 iterations. This creates the spanning tree T_E^3, shown in Figure 5. The least cost new cycle is $C_e^3 =< e, f, r, j, a, b, k, q, m >$, with $z_E(C_e^3) = 109 (> 108)$. This is a result of the above choice rule, which forbids the addition of edge d, as doing this would merely reproduce cycle C_d^2.

Note that this objective function value is higher than that found in the previous iteration. Thus the search has moved to an (intermediate) inferior solution. We now start the third iteration and begin by removing the tabu ban on adding edge p to the tree.

The least cost edge swap involves dropping edge q and adding edge d. We make the addition of edge q tabu for 2 iterations. This creates the spanning tree T_E^4 shown in Figure 6. The least cost cycle is $C_p^4 =< p, j, a, b, c, d, m >$, with $z_E(C_p^4) = 107$. This cycle has the lowest possible objective function value of all the cycles in G. Thus we have identified cycle C_p^4 as the (unique) cycle centre C_p^4, and the search is terminated.

We now go on to illustrate the same general tabu search procedure used above for cycle centre identification to attempt to find the cycle median for the graph in Figure 3, starting once again with the spanning tree T^1, now denoted by T_D^1. In this case, the least cost fundamental cycle is $C_d^1 =< d, e, f, r, p, q, k, c >$. The value $z_D(C_d^1)$ is calculated by summing the distances to C_d^1 of all the vertices of G not on C_d^1. These are (with their distances in parentheses): $\kappa(1), \beta(2), \gamma(1)$, and $\delta(1)$. As C_d^1 has length 8: $z_D(C_d^1) = 100(1 + 2 + 1 + 1) + 8 = 508$.

The least cost edge swap involves dropping edge p and adding edge a. We make the addition of edge p tabu for 2 iterations. This creates the spanning tree T_D^2 shown in Figure 4. The least cost cycle is $C_m^2 =< m, e, f, r, j, a, b, k, q >$, with $z_D(C_m^2) = 309$. This cycle is recorded as the best found so far.

This process is now repeated. The least cost edge swap involves dropping edge j and adding edge h. We make the addition of edge j tabu for 2 iterations. This produces a new least cost cycle which is the best found so far, as shown in Figure 7, with T_D^3 and $C_m^3 =< m, e, f, r, n, h, i, a, b, k, q >$, and $z_D(C_m^3) = 111$. As C_m^3 is the cycle median, the search is terminated.

We now go on to illustrate the same general tabu search procedure, used twice before, to attempt to find the cycle centroid for the graph in Figure 3, starting once again with the spanning tree T^1, now denoted by T_W^1. In this case, the least cost cycle is: $C_d^1 =< d, e, f, r, p, q, k, c >$. The value $z_W(C_d^1)$ is calculated by first identifying the number of vertices in the largest component of $G - C_d^1$. This is the component comprising the path $< a, i, h >$, which has 4 vertices. As C_d^1 has length 8: $z_W(C_d^1) = 100 \times 4 + 8 = 408$. The least cost edge swap involves dropping edge p and adding edge a. We make the addition of edge p tabu for 2 iterations. This creates spanning tree T_W^2, shown in Figure 4. (We have performed the same edge swap as used previously for centres and medians.) The least cost cycle is $C_d^2 =< d, e, f, r, j, a, b, c >$, with $z_W(C_d^2) = 208$. This cycle is recorded as the best cycle found so far. This process is now repeated. The least cost edge swap involves drop-

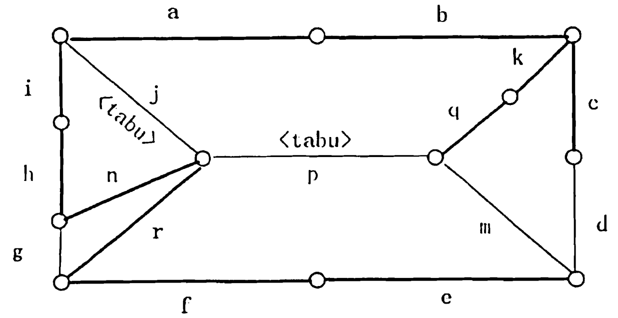

Figure 8: Cycle centroid search : T_W^3.

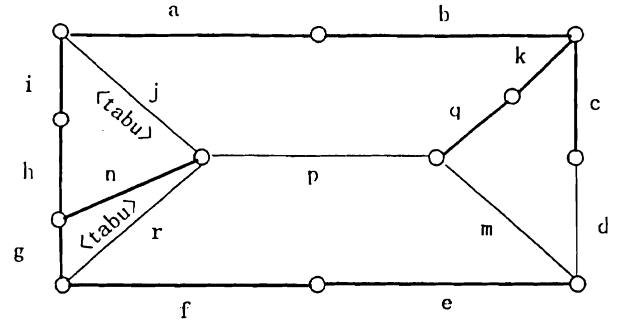

Figure 9: T_W^4 and the cycle centroid C_m^4.

ping edge j and adding edge h. We make the addition of edge j tabu for 2 iterations. This creates the spanning tree T_W^3, shown in Figure 8. The least cost cycle is : $C_m^3 =< m, e, f, r, n, h, i, a, b, k, q >$, with $z_W(C_m^3) = 111$. This cycle is recorded as the best found so far. We now repeat the process (as it turns out for the last time) to identify the cycle centroid. First we must remove the tabu ban on adding edge p to the tree. The least cost edge swap involves dropping edge r and adding edge g. We make the addition of edge r tabu for 2 iterations. This creates the spanning tree T_W^4 shown in Figure 9. The least cost cycle is: $C_m^4 =< m, e, f, g, h, i, a, b, k, q >$, with $z_W(C_m^4) = 110$. This cycle has the lowest possible objective function value of all cycles in G. Thus we have identified the cycle centroid C_m^4, and the search is terminated.

We now go on to discuss an alternative approach to identifying central cycles, via randomized edge swapping.

6 Randomized edge swapping

We have also experimented with another procedure for identifying central cycles, based on random edge swapping in a spanning tree of G. The process involves the same fundamentals as for the tabu search heuristic described in the previous section. Once again we begin with a spanning tree of G, and also transform it, at each iteration, by a single edge swap, identifying the induced fundamental cycles.

However, unlike tabu search, the edge swap is selected at random at each iteration. This approach was tested on many large graphs. These graphs were generated by choosing a number n of vertices and choosing edges independently so that each possible edge is contained in the graph with the same probability p. The method identified a cycle centre in every case for which it was possible to check the result exactly. However three points should be noted.

1. If a Hamiltonian cycle is found then clearly it is a cycle centre. If the graph was not 2-connected, then in every case it turned out to be non-Hamiltonian.
2. Hamiltonian cycles appear very suddenly in random graphs generated by this method. It has been shown (see Bollobás(1985)) that if $\omega(n) \to \infty$ as $n \to \infty$, then almost every graph generated will be Hamiltonian if $p = (1/n)(\log n + \log\log n + \omega(n))$ and amost every graph will be non-Hamiltonian if $p = (1/n)(\log n + \log\log n)$.
3. The process is wasteful in that it creates the same spanning tree rather frequently.
4. In real applications, this process is not likely to represent the kind of networks of interest, which may be planar or nearly planar, or may not have edges appearing independently.

We have also tested the same method on graphs generated by the following random process, described in Bollobás(1985). Let $V = \{1, \ldots, n\}$ be a a set of vertices and define a graph process to be a sequence $\tilde{G} = (G_t)_{t=0}^N (N = \binom{n}{2})$ of graphs on V such that:
(i) each graph G_t has t edges, and
(ii) $G_0 \subset G_1 \subset G_2 \subset \ldots$

We choose a graph process uniformly and randomly and choose the first t for which G_t has minimum degree 2. Bollobás and Thomason(1985) showed that, as $n \to \infty$, almost every G_t chosen by this method is 2-connected. Also Bollobás(1984) showed that as $n \to \infty$ almost every G_t chosen by this method is Hamiltonian. Clearly a graph is 2-connected if it is Hamiltonian, and clearly we can restrict the search for cycle centres to maximal 2-connected subgraphs of the graph G we are investigating. This random graph process was useful for generating 2-connected graphs that were not Hamiltonian, and for these the randomised edge swapping heuristic was very poor at finding cycle centres for the cases that could be checked exhaustively.

7 Conclusions

Slater(1980,1981,1982 and 1983) analyzed the problems of finding central restricted trees and central paths in graphs. We have extended this to analyzing the problem of finding central cycles in graphs. The problems are related to the covering salesman problem (see Mesa and Boffey(1996) and Current and Schilling(1989,1994)). These authors formulated the problem as a zero-one integer programming model and solved it by a heuristic based on solution procedures for the set covering problem. Note that the set covering problem for graphs of a special type (i.e. a kind of chordal graph) can be solved in polynomial time (Kolen and Tamir(1990)).

We have produced some techniques for the identification of central cycles. However much more needs to be done in this area in order to provide useful results to underpin techniques which are effective. The importance of this is underlined by our efforts to construct an integer programming method upon which to base algorithms which guarantee optimal central cycles. We plan to publish this work elsewhere.

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グラフにおける中心的サイクル

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概要

本論文では，グラフにおけるサイクル配置問題, 即ちグラフにおけるサイクルの位置についてのある評価基準（中心性，平均距離等）のもとでの最適配置を扱っている．与えられた単純連結グラフにおけるセンター，メディアン又はセントロイドという部分グラフを求める問題である．部分グラフが離散的な頂点集合は今までよく扱われている．これをサイクルのような構造をもった部分グラフへ拡張している．まず，格子グラフでのサイクル配置，一般のグラフでの関係する評価基準の性質を調べ, 次いで，タブ探索の適用について述べ, 最後にランダム探索での近似解についての考察をしている．

キーワード：グラフ, ネットワーク，サイクル，配置問題

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