カルタン空間におけるC-分解可能性について

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On C-Reducibility in Cartan Space

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We introduce the conception of C-reducibility in Cartan space, and show that *a Cartan space endowed with $(\alpha,\beta)-$ metric is C-reducible, if and only if the metric is only of Randers' or Kropina's type.* This is the main theorem in this paper. Moreover, we consider C-reducibility in Cartan space endowed with a generalized Randers metric which derived from a usual Cartan metric by $\beta-$change. We prove that the space with the generalized metric is C-reducible if the space with original metric is C-reducible, and that the converse is true.

Keywords : Cartan space, C-reducibility, $(\alpha,\beta)-$metric

1. PRELIMINARIES

In this paper,we consider two remarkable classes of Cartan spaces, that is, a C-reducible type and a generalized Randers type.

Let M be a real n-dimensional manifold, (T^*M,π^*,M) the cotangent bundle of M and $H : U^* \to R$ a regular Hamiltonian (*i.e.* real smooth function on a domain $U^* \subset T^*M$ positively homogeneous of degree two in p). The pair of $\mathcal{H}^n = (M, H(x,p))$ is called a *Hamilton space* and the function $H(x,p)$ is *fundamental* (*metric*) *function* such that the nondegenerate matrix with the entries

$$g^{ij}(x,p) = \dot{\partial}^i\dot{\partial}^j H \qquad (1)$$

is defined on U^*, hereafter in this paper, we denote $\dot{\partial}^i = \dfrac{\partial}{\partial p_i}$, while $\partial_i = \dfrac{\partial}{\partial x^i}$ and indices $i,j,\cdots$ run over $1,2,\cdots n$. Of course, g^{ij} is component of a contravariant d-tensor field, named as *metric d-tensor* of $\mathcal{H}^n$, which is symmetric, positive definite and its reciprocal component $g_{ij}(x,p)$ is given by $g^{ij}g_{jk} = \delta^i_k$.

In this paper, the term *d-tensor field* $T(x,p)$ of $\mathcal{H}^n$, for instance, of type $(1,1)$, means a collection of n^2 functions $T^i_j(x,p)$ of variables x^i and p_i which obey the usual transformation law of components of tensor of M such that

$$\bar{T}^a_b(\bar{x},\bar{p}) = \frac{\partial \bar{x}^a}{\partial x^i}\frac{\partial x^j}{\partial \bar{x}^b}T^i_j(x,p)$$

under the coordinate transformation $x^i \to \bar{x}^a$.

We can construct the operators $\delta_i = \partial_i + N_{ji}\dot{\partial}^j$ which form a local basis of horizontal distribution N supplementary to the vertical distribution V of T^*M, *i.e.* $T_u(T^*M) = N_u \oplus V_u$, $u \in T^*M$.

We can obtain on a domain U of the tangent bundle (TM,π,M) of M, a regular Lagrangian $L(x,y)$ by *Legendre transformation*

$$L(x,y) = p_i y^i - H(x,p)$$

of H, where p_i is the solution of the system $y^i = \dot{\partial}^i H(x,p)$. $L(x,y)$ is positively homogeneous of degree two and behaves as the fundamental function of *Lagrange space* $\mathcal{L}^n = (M, L(x,y))$, which is seemed the dual of Hamilton space (ref. to [4]). More precisely, the geometric structure of the cotangent bundle T^*M or of Hamilton space is reffered to our previous papers[1][2] or R. Miron's[6].

A *Cartan space* $\mathcal{C}^n = (M, F(x,p))$ is a special Hamilton space with $H = \dfrac{1}{2}F^2$, where F is called (*fundamental*) *metric function* of $\mathcal{C}^n$ and positively homogeneous of degree one in $p = (p_i)$.

A *canonical nonlinear connection* in C_n is given by

$$N_{ij} = \gamma^o_{ij} - \frac{1}{2}\gamma^o_{ro}\dot{\partial}^r g_{ij}$$

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where

$$\gamma^o_{ij} = \gamma^h_{ij} o_h, \quad \gamma^o_{ro} = \gamma^o_{rs} g^{sh} p_h$$

and

$$\gamma^i_{jk} = \frac{1}{2} g^{ih}(\partial_j g_{hk} + \partial_k g_{jh} - \partial_h g_{jk}).$$

A *canonical d-connection* $C\Gamma(N) = (H^i_{jk}, C_i^{jk})$ in C_n is defined by [7] as follows:

$$H^i_{jk} = \frac{1}{2} g^{ih}(\delta_j g^{hk} + \delta_k g_{jh} - \delta_h g_{jk}),$$

$$C_i^{jk} = -\frac{1}{2} g_{ih}(\dot{\partial}^j g^{hk} + \dot{\partial}^k g^{jh} - \dot{\partial}^h g^{jk}).$$

By the differentiability and homogeneity of $\check{F}(x,p)$, we can easily show properties about the d-tensor $C^{ijk} = g^{ih} C_h{}^{jk}$ such that

$$C^{ijk} = -\frac{1}{2} \dot{\partial}^i \dot{\partial}^j \dot{\partial}^k \left(\frac{1}{2} F^2\right), \quad C^{ojk} = C^{ijk} p_i = 0. \qquad (2)$$

The torsion tensor C^{ijk} is completely symmetric in indices, so we consider the case it splits in the form:

$$C^{ijk} = A^{ij} Q^k + A^{jk} Q^i + A^{ki} Q^j \qquad (3)$$

for some symmetric tensor A^{ij} and vector Q^i. By reason of $C^{ijk} p_k = 0$, which follows from the homogeneity of fundamental function $F(x,p)$, we have

$$A^{io} = 0, \quad Q^o = 0$$

for non-Riemannian ($C^{ijk} \neq 0$) Cartan space, where the index $_o$ means the result of contracting by p. Thus, we put the *angular metric tensor*

$$h^{ij} = F \dot{\partial}^i \dot{\partial}^j F = g^{ij} - l^i l^j, \quad l^i = \dot{\partial}^i F \qquad (4)$$

as most suitable one for A^{ij}, because this tensor is symmetric and has the properties such that

$$h^{ij} p_j = 0, \quad h^{ij} g_{ij} = n - 1, \quad i.e. \ \mathrm{rank}(h^{ij}) = 0. \qquad (5)$$

Therefore, on account of (5), we can easily prove;

Proposition 1. If the tensor C^{ijk} satisfies the relation

$$C^{ijk} = h^{ij} Q^k + h^{jk} Q^i + h^{ki} Q^j,$$

then the vector Q^i has the form $Q^i = \frac{1}{n+1} C^i$, where $C^i = g_{jk} C^{ijk}$ is non-zero vector.

And we set;

Definition 1. A Cartan space of dimensin $n > 2$ is called *C-reducible*, if the torsion tensor C^{ijk} is expressed in the form

$$C^{ijk} = \frac{1}{n+1}(h^{ij} C^k + h^{jk} C^i + h^{ki} C^j)$$

Cartan space of dimension two is always C-reducible, because there exists a scalar I such that $FC^{ijk} = I m^i m^j m^k$, where m^i is orthogonal to the unit vector l^i.

In the previous paper[3], we introduced an (α, β) metric in Cartan space, as analogous one in Finsler space (ref. to [5]). Its outline is as follows:

Definition 2. We call a Cartan space $C^n = (M, F(x,p))$ *endowed with* (α,β)*−metric*, if its metric function $F(x,p)$ is a function of $\alpha(x,p), \beta(x,p)$ only;

$$F(x,p) = \check{F}(\alpha(x,p), \beta(x,p)),$$

with $\qquad \alpha(x,p) = (a^{ij}(x) p_i p_j)^{1/2}, \quad \beta(x,p) = b^i(x) p_i,$

where $a^{ij}(x)$ is a Riemannian metric on the base manifold M and $b^i(x)$ is a vector field on M such that $\beta \neq 0$ on a domain of $T^*M \backslash \{0\}$. The space (M, α) is called *associated Riemannian space of* C^n.

We need in this paper two special Cartan space with $(\alpha,\beta)-$ metric, that is as follows:

Definition 3. A metric function $F(x,p)$ of Cartan space is called *of Randers' type* if it is given by

$$F(x,p) = \sqrt{a^{ij}(x) p_i p_j} + b^i p_j = \alpha + \beta \qquad (6)$$

and *of Kropina's type* if given by

$$F(x,p) = \frac{a^{ij}(x) P_i p_j}{b^i b_i} = \frac{\alpha^2}{\beta}. \qquad (7)$$

These metrics were introduced by R. Miron[7] into Cartan space analogously to the case of Finsler space (ref. to [5]). Clearly, the function $\check{F}(\alpha,\beta)$ in Definition 2 should satisfy the conditions imposed to the function $F(x,p)$ as a fundamental one for C^n, so it is positively homogeneous of degree one in α and β. By this reason, there would be no confusion if we adopt the notation $F(\alpha,\beta)$ instead of $\check{F}(\alpha,\beta)$. Putting $H = F^2/2$, for convenience, we remark the following homogeneities of $H(x,p)$:

$$\left.\begin{array}{l} \alpha H_\alpha + \beta H_\beta = 2H \\ \alpha H_{\alpha\alpha} + \beta H_{\alpha\beta} = H_\alpha, \quad \alpha H_{\alpha\beta} + \beta H_{\beta\beta} = H_\beta \\ \alpha^2 H_{\alpha\alpha} + 2\alpha\beta H_{\alpha\beta} + \beta^2 H_{\beta\beta} = 2H \\ \alpha H_{\alpha\alpha\alpha} + \beta H_{\beta\alpha\alpha} = \alpha H_{\alpha\alpha\beta} + \beta H_{\alpha\beta\beta} \\ \qquad = \alpha H_{\alpha\beta} + \beta H_{\beta\beta\beta} = 0 \end{array}\right\} \qquad (8)$$

where the subscripts α, β of H mean its partial derivatives with respect to them. In the previous paper[3], we obtained the concrete expression for the metric tensor $g^{ij}(x,p)$ and its reciprocal component g_{ij} in Cartan space with (α,β) metric such that

$$g^{ij} = \rho a^{ij} + \rho_0 b^i b^j + \rho_1 (b^i P^j + b^j P^i) + \rho_2 P^i P^j$$

$$g_{ij} = \sigma a_{ij} - \sigma_0 B_i B_j + \sigma_1 (B_i p_j + B_j p_i) + \sigma_2 p_i p_j$$

and also for the tensor C^{ijk},

$$C^{ijk} = -\frac{1}{2}[r_1 b^i b^j b^k + \mathfrak{S}_{ijk}\{\rho_1 a^{ij} b^k + \rho_2 a^{ij} P^k + r_2 b^i b^j P^k + r_3 b^i P^j P^k\} + r_4 P^i P^j P^k].$$

In the above three expressions, the notation

$$\left.\begin{array}{l} P^i(x,p) = a^{ij}(x) p_j, \quad B_i(x) = a^{ij}(x) b^j(x) \\ B^2(x) = a_{ij} b^i b^j = a^{ij} B_i B_j \end{array}\right\} \qquad (9)$$

are used and the coefficients ρ's, σ's and r's are certain functions of partial derivatives of $H(\alpha,\beta)$, at most, of third order, and the operator $\mathfrak{S}_{ijk}$ plays the rôle of abbreviation of completely symmetric summation with respect to the indices i,j,k for each term in the brackets, for example,

$$A^{ij}Q^k + A^{jk}Q^i + A^{ki}Q^j = \mathfrak{S}_{ijk}\{A^{ij}Q^k\} \qquad (10)$$

in place of the right hand of (3).

In the next chapter, however, we use more direct calculation without above formulas for g^{ij}, g_{ij} and C^{ijk}.

2. MAIN THEOREM

Partial differentiation of $\alpha(x,p)$ and $\beta(x,p)$ yields

$$\dot\partial^i\alpha = \alpha^{-1}P^i, \quad \dot\partial^i\beta = b^i(x).$$

The vector $P^i(x,p)$ in (9) satisfies the relation

$$P^ip_i = \alpha^2, \quad \dot\partial^j P^i = a^{ij}(x), \quad \dot\partial^k\dot\partial^j P^i = 0.$$

$$\dot\partial^j\dot\partial^i\alpha = \alpha^{-1}(a^{ij}(x) - \alpha^{-2}P^iP^j).$$

By means of the quantitie

$$k^{ij} = a^{ij} - \alpha^{-1}P^iP^j,$$

the partial derivative of alpha of third order is expressed such that

$$\dot\partial^j\dot\partial^i\dot\partial^k\alpha = -\alpha^{-3}\mathfrak{S}_{ijk}\{k^{ij}P^k\}. \qquad (11)$$

Direct differentiation of $H(x,p)$ gives expressions for the metric tensor such that

$$\begin{aligned} g^{ij} &= \dot\partial^j\dot\partial^i H \\ &= H_{\alpha\alpha}(\dot\partial^i\alpha)(\dot\partial^j\alpha) + H_{\alpha\beta}\{(\dot\partial^i\alpha)b^j + (\dot\partial^j\alpha)b^i\} \\ &\quad + H_{\beta\beta}b^ib^j + H_\alpha(\dot\partial^i\dot\partial^j\alpha) \\ &= \alpha^{-1}H_\alpha k^{ij} + H_{\beta\beta}b^ib^j + \alpha^{-1}H_{\alpha\beta}(b^iP^j + b^jP^i) \\ &\quad + \alpha^{-2}H_{\alpha\alpha}P^iP^j \end{aligned} \qquad (12)$$

using (11) and for the torsion tensor C^{ijk} such that

$$\begin{aligned} -2C^{ijk} &= \dot\partial^k g^{ij} \\ &= \mathfrak{S}_{ijk}\{(\alpha^{-2}H_{\alpha\alpha} - \alpha^{-3}H_\alpha)k^{ij}P^K + \alpha^{-1}H_{\alpha\beta}k^{ij}b^k \\ &\quad + \alpha^{-1}H_{\alpha\beta\beta}P^ib^jb^k + \alpha^{-2}H_{\alpha\alpha\beta}P^iP^jb^k\} \\ &\quad + \alpha^{-3}H_{\alpha\alpha\alpha}P^iP^jP^k + H_{\beta\beta\beta}b^ib^jb^k . \end{aligned}$$

Substituting the new quantity $Q^i = b^i - \alpha^{-2}\beta P^i$ and using the homogeneity (8) for the above expression, we have a very simple form;

$$-2C^{ijk} = \alpha^{-1}H_{\alpha\beta}\mathfrak{S}_{ijk}\{k^{ij}Q^k\} + H_{\beta\beta\beta}Q^iQ^jQ^k. \qquad (13)$$

If we put here

$$\alpha^{-1}H_{\alpha\beta}k^{ij} + 3H_{\beta\beta\beta}Q^iQ^j = -2A^{ij},$$

then (13) yields the equation

$$C^{ijk} = A^{ij}Q^k + A^{jk}Q^i + A^{ki}Q^j$$

which is nothing but (3), hence we get

Proposition 2. In the Cartan space with (α,β)–metric, the torsion tensor C^{ijk} splits in the form (3).

On the other hand, if we use the unit vector

$$l^i = \dot\partial^i F = (\sqrt{2H})^{-1}(\alpha^{-1}H_\alpha P^i + H_\beta b^i) \qquad (14)$$

of $\mathcal{C}_n$, the angular metric tensor in (4) is rewritten as

$$h^{ij} = \alpha^{-1}H_\alpha k^{ij} + (H_{\beta\beta} - (2H)^{-1}H_\beta^2)Q^iQ^j, \qquad (15)$$

where we also substituted the above quantity Q^i and using the homogeneity (8). Therefore, by means of

$$k^{ij} = \alpha^{-1}H_\alpha\{h^{ij} - (H_{\beta\beta} - (2H)^{-1}H_\beta{}^2)Q^iQ^j\}$$

which follows from (15), the expression (11) is changed to the final form;

$$\begin{aligned} -2C^{ijk} = \frac{H_{\alpha\beta}}{H_\alpha}\mathfrak{S}_{ijk}\{h^{ij}Q^k\} + \{H_{\beta\beta\beta} - \frac{3H_{\alpha\beta}}{H_\alpha} \times \\ (H_{\beta\beta} - \frac{1}{2H}H_\beta{}^2)\}Q^iQ^jQ^k. \end{aligned} \qquad (16)$$

Picking up the last term, we put

$$Q = H_{\beta\beta\beta} - \frac{3H_{\alpha\beta}}{H_\alpha}(H_{\beta\beta} - \frac{1}{2H}H_\beta{}^2). \qquad (17)$$

We are interested in the case $Q = 0$, that is,

$$C^{ijk} = \mathfrak{S}_{ijk}\{h^{ij}A^k\}$$

with $A^kp_k = 0$. Contracting g_{jk} to the bothside of the above equation, we have

$$C^i = g_{jk}C^{ijk} = (n+1)A^i, \quad i.e., \quad A^i = \frac{1}{n+1}C^i$$

by means of (5), which implies

$$C^{ijk} = \frac{1}{n+1}\mathfrak{S}_{ijk}\{h^{ij}C^k\}.$$

On account of Proposition 2, we can conclude

Proposition 3. When the quantity Q in (17) vanishes the space is C-reducible.

Obviously, for the cases of Randers' and Kropina's metric presented in last section the quantity Q vanishes, because, for the former $H = (\alpha+\beta)^2/2$,
$H_\alpha = H_\beta = \alpha+\beta, \quad H_{\alpha\beta} = 0, \quad H_{\beta\beta} = 1, \quad H_{\beta\beta\beta} = 0,$
and for the later metric $H = \alpha^4/(2\beta^2)$,
$H_\alpha = 2\alpha^3\beta^{-2}, \quad H_\beta = -\alpha^4\beta^{-3}, \quad H_{\alpha\beta} = -4\alpha^3\beta^{-3},$
$H_{\beta\beta} = 3\alpha^4\beta^{-4}, \quad H_{\beta\beta\beta} = -12\alpha^4\beta^{-5},$
hence in both cases $Q = 0$ holds.

Proposition 4. The Cartan spaces with Randers' and Kropina's type metric are both C-reducible.

Let us obtain the necessary condition for Q to vanish. In order to replace H and its partial derivatives in the quantity Q by F and its ones, we use the relations

$$H_{\alpha\beta} = FF_{\alpha\beta} + F_\alpha F_\beta, \quad H_{\beta\beta} = FF_{\beta\beta} + F_\beta{}^2$$

$$H_{\beta\beta\beta} = -\alpha^3\beta^{-3}H_{\alpha\alpha\alpha} = -\alpha^3\beta^{-3}(FF_{\alpha\alpha\alpha} + 3F_\alpha F_{\alpha\alpha})$$

and the homogeneity about $F(x,p)$ such that

$$\alpha F_\alpha + \beta F_\beta = F$$

$$\alpha F_{\alpha\alpha} + \beta F_{\alpha\beta} = \alpha F_{\alpha\beta} + \beta F_{\beta\beta} = 0.$$

Therefore, (17) is rewritten as

$$Q = -\frac{\alpha^3}{\beta^3}F\left(\frac{3F_{\alpha\alpha}}{\alpha} + F_{\alpha\alpha\alpha} - 3\frac{F_{\alpha\alpha}{}^2}{F_\alpha}\right).$$

We start from $Q = 0$, then case
(i) $F_{\alpha\alpha} = 0$ reduces to the metric of Randers' type, because $F_\alpha = c_1$ and $F = c_1 + \phi(\beta) = c_1\alpha + c_2\beta$, by the homogeneity of $\phi(\beta)$ of degree one in β. The case
(ii) $F_{\alpha\alpha} \neq 0$ follows a differential equation such that

$$\frac{3}{\alpha} + \frac{F_{\alpha\alpha\alpha}}{F_{\alpha\alpha}} - \frac{3F_{\alpha\alpha}}{F_\alpha} = 0.$$

Paying attention to logarithmic differentiation, we obtain at first,

$$\frac{\alpha^3 F_{\alpha\alpha}}{F_\alpha{}^3} = e^{\psi(\beta)}, \quad \alpha^3 F_{\alpha\alpha} = k\beta^2 F_\alpha{}^3,$$

where the coefficient $e^{\psi(\beta)} = c\beta^2$, $c = \text{const.} \neq 0$ is caused by that the left side of the first expression is homogeneous of degree two in p, then we have

$$\frac{F_{\alpha\alpha}}{F_\alpha{}^3} = \frac{c\beta^2}{\alpha^3}$$

Integration this by α yields

$$\frac{1}{F_\alpha{}^2} = \frac{c\beta^2 + c_1\alpha^2}{\alpha^2}, \quad F_\alpha{}^2 = \frac{\alpha^2}{c\beta^2 + c_1\alpha^2}, \quad c_1 = \text{const.}$$

where we used the homogeneity of F again. Hence, the following two cases occur.
(iia) If $c_1 \neq 0$, then

$$F_\alpha = \pm\frac{\alpha}{\sqrt{c_1\alpha^2 + c\beta^2}}, \quad F = \pm\frac{1}{c_1}\sqrt{c_1\alpha^2 + c\beta^2} + c_0\beta,$$

($c_0 = \text{const.}$), F is rewriten as

$$\begin{aligned} F(x,p) &= \pm\sqrt{\left(\frac{1}{c_1}a^{ij} + \frac{c}{c_1^2}b^1 b^j\right)p_i p_j} + (c_0 b^k)p_k \\ &= \tilde{\alpha} + \tilde{\beta} \end{aligned}$$

which is the metric of Randers' type.
(iib) If $c_1 = 0$, then

$$F_\alpha{}^2 = \frac{\alpha^2}{k\beta}, \quad F_\alpha = \pm\frac{1}{k}\frac{\alpha}{\beta}, \quad F = \pm\frac{1}{2k}\frac{\alpha^2}{\beta} + k_1\beta$$

($k, k_1 = \text{const.}$). F is rewriten as

$$F(x,p) = \frac{(\frac{1}{2k}a^{ij} + k_1 b^i b^j)p_i p_j}{\pm b^k p_k} = \frac{\tilde{\alpha}^2}{\tilde{\beta}}$$

that is, the metric of Kropina's type is obtained.

Summerizing the consideration in this chapter, we conclude

Theorem 1. *The Cartan space with (α,β)-metric is C-reducible if and only if the metric is of Randers' or Kropina's type.*

3. C-REDUCIBILITY IN THE GENERALIZED SPACE

In this chapter, we need to generalize the metric of Randers' type as follows:

Definition 4. For a given Cartan metric $F(x,p)$ in C^n, a metric function of $\tilde{F}(x,p)$ of the Cartan space $\tilde{C}^n$ is called *of generalized Randers' type* if it has the form such that

$$\tilde{F}(x,p) = F(x,p) + \beta(x,p), \quad F^2 = g^{ij}(x,p)p_i p_j.$$

It should be remarked here that $F(x,p)$ is not Riemannian metric as α in Chapter 2 and $\beta(x,p)$ is the same one. We call the original space C^n as *associated Cartan space of* $\tilde{C}^n$ and the deformation of the metric $F \to \tilde{F}$ as *β-change* of metric.
Actually, $\tilde{F}(x,p)$ given above is homogeneous of degree one in p. We ditinguish the quantities in this section by attaching notation "$\sim$" to the top of ones of $\tilde{C}^n$, if there are the same ones appear in the associated space C^n. We take the quantities such as

$$\begin{aligned} g^{ij} &= \dot\partial^j\dot\partial^i(F^2/2) = h^{ij} + l^i l^j, \quad l^i = \dot\partial^i F \quad (\text{in } \tilde{C}^n) \\ \tilde{g}^{ij} &= \dot\partial^j\dot\partial^i(\tilde{F}^2/2) = \tilde{h}^{ij} + \tilde{l}^i\tilde{l}^j \quad (\text{in } C^n) \\ \tilde{l}^i &= \dot\partial^i\tilde{F} = l^i + b^i, \quad \dot\partial^j\tilde{l}^i = \dot\partial^j l^i = F^{-1}h^{ij}. \end{aligned} \tag{18}$$

And denoting

$$\tau = F^{-1}\tilde{F} = F^{-1}(F + \beta) = 1 + F^{-1}\beta, \tag{19}$$

$$\tilde{h}^{ij} = \tau h^{ij}, \quad \dot\partial^i\tau = F^{-1}(b^i - F^{-1}\beta\, l^i), \quad (\dot\partial^i\tau)p_i = 0 \tag{20}$$

are obtained. And for the metric tensor $\tilde{g}^{ij}$ in (18) is given by two manners such that of $\tilde{C}^n$,

$$\tilde{g}^{ij} = \tau g^{ij} + b^i b^j + b^i l^j + b^j l^i - F^{-1}\beta\, l^i l^j \tag{21}$$

$$= \tau g^{ij} + \tilde{l}^i\tilde{l}^j - \tau\, l^i l^j = \tau\, h^{ij} + \tilde{l}^i\tilde{l}^j\,. \tag{22}$$

On account of the useful Proposition 30.1 of Matsumoto[5], we can obtain the reciprocal component $\tilde{g}_{ij}$ of $\tilde{g}^{ij}$ in (22) such that

$$\tilde{g}_{ij} = \frac{1}{\tau}\{g_{ij} - B_i\tilde{l}_j - B_j\tilde{l} + (B^2 + \beta F^{-1})\tilde{l}^i\tilde{l}^j\} \tag{23}$$

where g_{ij} is the reciprocal component of g^{ij} in C^n of (18) and notations

$$\tilde{l}_i = \tilde{F}^{-1}p_i, \quad B_i = g_{ij}b^j, \quad B^2 = g_{ij}b^i b^j = g^{ij}B^i B^j$$

are used instead of (9). In fact, we can verify $\tilde{g}^{ij}\tilde{g}_{jk} = \delta^i_k$ using the relations

$$\tilde{l}^i l_i = \tau, \quad \tilde{l}_i l^i = 0, \quad \tilde{l}_i b^i = \beta\tilde{F}^{-1},$$

$$\tilde{l}^i B_i = B^2 + \beta F^{-1}, \quad b_i l^i = B_i l^i = \beta F^{-1}.$$

We are concerned with the torsion tensor $\tilde{C}^{ijk}$ (resp. C^{ijk}) in the space $\tilde{C}^n$ (resp. C^n). Differentiating the both side of (22), as the followings;

$$\begin{aligned} \dot\partial^k\tilde{g}^{ij} &= -2\tilde{C}^{ijk} \\ &= F^{-1}(b^k - \beta F^{-1}l^k)g^{ij} - 2\tau C^{ijk} + b^i F^{-1}h^{jk} \\ &\quad + b^j F^{-1}h^{ik} + F^{-2}l^k\beta\, l^i l^j - F^{-1}b^k l^i l^j \\ &\quad - F^{-2}\beta\, h^{ik}l^j - F^{-2}\beta\, l^i h^{jk} \\ &= -2\tau C^{ijk} + \mathfrak{S}_{ijk}\{h^{ij}F^{-1}(b^k - \beta F^{-1}l^k)\}, \end{aligned}$$

we obtain the relation between $\tilde{C}^{ijk}$ and C^{ijk} such that

$$\tilde{C}^{ijk} = \tau C^{ijk} + \mathfrak{S}_{ijk}\{h^{ij}\gamma^k\} \tag{24}$$

with $\quad \gamma^k = -\frac{1}{2}(b^k - \beta F^{-1}l^k), \quad \gamma^o = \gamma^k p_k = 0$

where quantities (18) and (19) are used.

At last of this chapter, we consider C-reducibility in this generalized space. Transvecting the expression (24) by (23), we have

$$\tilde{C}^i = \tilde{g}^{jk}\tilde{C}^{ijk} = g^{jk}C^{ijk} + \frac{n+1}{\tau}\gamma^i = C^i + \frac{n+1}{\tau}\gamma^i$$

because of (5) and $h^{ik}\tilde{l}_k = \tilde{F}^{-1}h^{i0} = 0$.
Furthermore, we know $\tilde{C}^i p_i = 0$ as well as $C_i p_i = 0$.

In the other words, if there exists a vector C^k in C^n such that $C^{ijk} = \mathfrak{S}_{ijk}\{h^{ij}C^k\}/(n+1)$, then there also exists a vector

$$\tilde{C}^k = C^k - \frac{n+1}{2\tau}(b^k - \beta\, F^{-1}l^k)$$

in $\tilde{C}^n$ such that $\tilde{C}^{ijk} = \mathfrak{S}_{ijk}\{\tilde{h}^{ij}C^k\}/(n+1)$, and conversely. Hence, we conclude

Theorem 2. Let $n > 2$. *Cartan space $\tilde{C}^n$ with the generalized Randers' metric is C-reducible if and only if its associated Cartan space C^n is C-reducible.*

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カルタン空間におけるＣ－分解可能性について

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概要

カルタン空間に「Ｃ－分解可能性」という概念を導入し，(α,β)－計量を与えられたカルタン空間がＣ－分解可能であるのは，その計量がランダース型かクロピナ型のときであり，かつ，そのときに限る」ことを明らかにする．これが本論文の主定理である．さらに，通常のカルタン空間の計量から導出された一般ランダース計量をもつ空間を考える．そして，一般化された計量をもつ空間がＣ－分解可能なのは最初の空間がＣ－分解可能なときであり，また，逆も成り立つことを示す．

キーワード：カルタン空間，Ｃ－分解可能性，(α,β)－計量

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