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2-adic properties for the numbers of involutions in the alternating groups.

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Abstract

We study the 2-adic properties for the numbers of involutions in the alternative groups, and give an affirmative answer to a conjecture of D. Kim and J. S. Kim [14]. Some analogous and general results are also presented.

1 Introduction

Let S_n be the symmetric group of degree n, and let A_n be the alternating group of degree n. Let ϵ be the identity of a group. Given a positive integer m, we denote by $a_n(m)$ the number of permutations $\sigma \in S_n$ such that $\sigma^m = \epsilon$. Let p be a prime. By definition and Wilson's theorem, $a_p(p) = 1 + (p-1)! \equiv 0 \pmod{p}$. Moreover, $a_n(m) \equiv 0 \pmod{\gcd(m, n!)}$ by a theorem of Frobenius (see, *e.g.*, [10]).

Let u be a positive integer. There exist remarkable p-adic properties of $a_n(p^u)$ (cf. Theorems 4.2–4.4). The beginning of them is due to H. Ochiai [16] and K. Conrad [4]. For each integer a, $\mathrm{ord}_p(a)$ denotes the exponent of p in the decomposition of a into prime factors. As a pioneer work, the formula

$$\mathrm{ord}_p(a_n(p)) \geq \left[\frac{n}{p}\right] - \left[\frac{n}{p^2}\right]$$

(cf. Corollary 4.5) was given in [6, 7, 9], which was also shown by various methods (cf. [4, 11, 13, 14]); moreover, the equality holds for all n such that $n - [n/p^2]p^2 \leq p - 1$ (see, *e.g.*, [6, 11, 13]). When $p = 2$, this formula was found by S. Chowla, I. N.

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Herstein, and W. K. Moore [2]. The precise formula for $\mathrm{ord}_2(a_n(2))$ is known as

$$\mathrm{ord}_2(a_n(2)) = \begin{cases} \left[\dfrac{n}{2}\right] - \left[\dfrac{n}{4}\right] + 1 & \text{if } n \equiv 3 \pmod 4, \\ \left[\dfrac{n}{2}\right] - \left[\dfrac{n}{4}\right] & \text{otherwise} \end{cases}$$

(cf. Example 4.6). The value of $\mathrm{ord}_2(a_n(4))$ is also determined (cf. Proposition 4.7).

We denote by $t_n(m)$ the number of even permutations $\sigma \in A_n$ such that $\sigma^m = \epsilon$. Recently, D. Kim and J. S. Kim [14] proved that for any nonnegative integer y,

$$\mathrm{ord}_2(t_{4y}(2)) = y + \chi_o(y),\ \mathrm{ord}_2(t_{4y+2}(2)) = \mathrm{ord}_2(t_{4y+3}(2)) = y,$$

where $\chi_o(y) = 1$ if y is odd, and $\chi_o(y) = 0$ if y is even. They also conjectured that for any nonnegative integer y, there exists a 2-adic integer α satisfying

$$\mathrm{ord}_2(t_{4y+1}(2)) = y + \chi_o(y) \cdot (\mathrm{ord}_2(y + \alpha) + 1)$$

(see [14, Conjecture 5.6]). According to [14], $\alpha = 1 + 2 + 2^3 + 2^8 + 2^{10} + \cdots$ satisfies the condition for all $y \leq 1000$. In this paper, we solve affirmatively their conjecture (cf. Theorem 5.1), and present some analogous and general results, including the result for $\mathrm{ord}_2(t_n(4))$ (cf. Theorems 5.4). We adapt K. Conrad's methods presented in [4] to the case of $t_n(2^u)$.

Sections 2–5 are devoted to the study of $\mathrm{ord}_p(a_n(p^u))$ and $\mathrm{ord}_2(t_n(2^u))$. In addition to the above results, we also show that, if $r = 0$ or $r = 1$, then there exists a 2-adic integer α_r such that

$$\mathrm{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \chi_o(y) \cdot (\mathrm{ord}_2(y + \alpha_r) + u)$$

for any nonnegative integer y (cf. Theorem 5.6).

Let $C_p \wr S_n$ be the wreath product of C_p by S_n, where C_p is a cyclic group of order p, and let $C_2 \wr A_n$ be the wreath product of C_2 by A_n. We are also interested in the number of elements x of these wreath products such that $x^m = \epsilon$. Let $b_n(p^u)$ be the number of elements x of $C_p \wr S_n$ such that $x^{p^u} = \epsilon$, and let $q_n(2^u)$ be the number of elements x of $C_2 \wr A_n$ such that $x^{2^u} = \epsilon$. In Sections 6–8, we focus on the p-adic properties of $b_n(p^u)$ and the 2-adic properties of $q_n(2^u)$. When $u = 1$, we are successful in finding the fact that

$$\mathrm{ord}_p(b_n(p)) = n - \left[\frac{n}{p}\right] \quad \text{and} \quad \mathrm{ord}_2(q_n(2)) = \left[\frac{n+1}{2}\right] + \chi_o\left(\left[\frac{n}{2}\right]\right)$$

(cf. Examples 7.4 and 8.2). The former fact with $p = 2$ is due to T. Yoshida [20]. The results for $\mathrm{ord}_p(b_n(p^u))$ and $\mathrm{ord}_2(q_n(2^u))$ with $u \geq 2$ are similar to those for $\mathrm{ord}_p(a_n(p^{u-1}))$ and $\mathrm{ord}_2(t_n(2^{u-1}))$, while there are slight differences between the proofs (cf. Example 7.5, Proposition 7.6, Theorems 8.3, 8.5, and 8.7).

2 Generating functions

For each $\sigma \in S_n$, $\sigma^{p^u} = \epsilon$ if and only if the cycle type of σ is of the form

$$(1^{j_0},\, p^{j_1}, \dots,\, (p^u)^{j_u}),$$

where $j_0, j_1, \dots, j_u$ are nonnegative integers satisfying $\sum_k j_k p^k = n$. Since the number of such a permutations is $n!/\prod_{k=0}^{u} p^{kj_k} j_k!$ (see, *e.g.*, [12, Lemma 1.2.15] or [18, Chap. 4 §2]), it follows that

$$a_n(p^u) = \sum_{j_0+j_1p+\cdots+j_up^u=n} \frac{n!}{\prod_{k=0}^{u} p^{kj_k} j_k!}. \tag{1}$$

Set $a_n^0(p^u) = a_n(p^u)$, and define

$$a_n^1(p^u) = \sum_{j_0+j_1p+\cdots+j_up^u=n} \frac{(-1)^{j_0+j_1+\cdots+j_u} n!}{\prod_{k=0}^{u} p^{kj_k} j_k!}. \tag{2}$$

Then we have

$$t_n(p^u) = \frac{a_n^0(p^u) + (-1)^n a_n^1(p^u)}{2}. \tag{3}$$

(Obviously, $a_n(p^u) = t_n(p^u)$ if $p \neq 2$.) Let $\natural$ denotes both 0 and 1. We always assume that $a_0^\natural(p^u) = 1$. By Eqs. (1)–(3), we have

$$\sum_{n=0}^{\infty} \frac{a_n^\natural(p^u)}{n!} X^n = \exp\left((-1)^\natural \sum_{k=0}^{u} \frac{1}{p^k} X^{p^k}\right) \tag{4}$$

and

$$\sum_{n=0}^{\infty} \frac{t_n(2^u)}{n!} X^n = \frac{1}{2}\exp\left(\sum_{k=0}^{u} \frac{1}{2^k} X^{2^k}\right) + \frac{1}{2}\exp\left(X - \sum_{k=1}^{u} \frac{1}{2^k} X^{2^k}\right)$$

(see also [3] and [18, Chap. 4, Problem 22]). Let $\{c_n^\natural\}_{n=0}^{\infty}$ be a sequence given by

$$\sum_{n=0}^{\infty} c_n^\natural X^n = \exp\left((-1)^\natural \sum_{k=0}^{\infty} \frac{1}{p^k} X^{p^k}\right). \tag{5}$$

Then by [5, Proposition 1] (see also [15, p. 97, Exercise 18]), $c_n^\natural \in \mathbb{Z}_p \cap \mathbb{Q}$, where $\mathbb{Z}_p$ is the ring of p-adic integers. When $\natural = 0$, this formal power series is called the Artin-Hasse exponential (cf. [5], [15, Chap. IV §2], [19, §48]). We write $c_n = c_n^0$ for the sake of simplicity. By definition, $c_r = a_r(p^u)/r!$ for any nonnegative integer r less than p^{u+1}. According to Mathematica, we have the following lemma.

Lemma 2.1 *If $p=2$, then the values of $c_r^\natural$ for integers r with $0 \le r \le 17$ are as follows* :

r	0	1	2	3	4	5	6	7	8	9	10	11
c_r^0	1	1	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{7}{15}$	$\frac{16}{45}$	$\frac{67}{315}$	$\frac{88}{315}$	$\frac{617}{2835}$	$\frac{2626}{14175}$	$\frac{18176}{155925}$
c_r^1	1	-1	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{5}$	$\frac{1}{45}$	$-\frac{5}{63}$	$-\frac{8}{105}$	$\frac{43}{405}$	$-\frac{74}{14175}$	$-\frac{559}{17325}$

r	12	13	14	15	16	17
c_r^0	$\frac{6949}{66825}$	$\frac{423271}{6081075}$	$\frac{2172172}{42567525}$	$\frac{19151162}{638512875}$	$\frac{58438907}{638512875}$	$\frac{899510224}{10854718875}$
c_r^1	$\frac{697}{18711}$	$-\frac{13232}{552825}$	$-\frac{30727}{14189175}$	$\frac{450991}{49116375}$	$-\frac{5519014}{91216125}$	$\frac{8250311}{144729585}$

For any nonnegative integer r less than p^{u+1}, we set

$$H_{u,r}^\natural(X) = \sum_{y=0}^{\infty} \frac{a_{p^{u+1}y+r}^\natural(p^u)}{(p^{u+1}y+r)!}(-(-1)^\natural p^{u+1})^y X^y,$$

and define a sequence $\{d_{n,r}^\natural\}_{n=0}^\infty$ by

$$\sum_{n=0}^{\infty} d_{n,r}^\natural X^n = \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^\natural(-(-1)^\natural p^{u+1})^j X^j\right) \exp\left(\sum_{i=1}^{\infty} \frac{\varepsilon^\natural p^{p^i(u+1)}}{p^{u+i+1}} X^{p^i}\right),$$

where $\varepsilon^\natural = -1$ if $p=2$ and $\natural = 0$, and $\varepsilon^\natural = +1$ otherwise.

Lemma 2.2 *Let r be a nonnegative integer less than p^{u+1}. Then*

$$H_{u,r}^\natural(X) = \exp(X)\sum_{n=0}^{\infty} d_{n,r}^\natural X^n.$$

Proof. Using Eqs. (4) and (5), we have

$$\sum_{n=0}^{\infty} \frac{a_n^\natural(p^u)}{n!} X^n = \left(\sum_{n=0}^{\infty} c_n^\natural X^n\right) \exp\left(-(-1)^\natural \sum_{k=u+1}^{\infty} \frac{1}{p^k} X^{p^k}\right).$$

This formula yields

$$\sum_{y=0}^{\infty} \frac{a_{p^{u+1}y+r}^\natural(p^u)}{(p^{u+1}y+r)!} X^{p^{u+1}y+r} = \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^\natural X^{p^{u+1}j+r}\right) \times \exp\left(-(-1)^\natural \sum_{i=0}^{\infty} \frac{1}{p^{u+i+1}} X^{p^{u+i+1}}\right).$$

Omit X^r and substitute $(-(-1)^\natural p^{u+1})X$ for $X^{p^{u+1}}$. Then we have

$$\sum_{y=0}^{\infty}\frac{a^\natural_{p^{u+1}y+r}(p^u)}{(p^{u+1}y+r)!}(-(-1)^\natural p^{u+1})^y X^y = \left(\sum_{j=0}^{\infty} c^\natural_{p^{u+1}j+r}(-(-1)^\natural p^{u+1})^j X^j\right) \times \exp\left(-(-1)^\natural \sum_{i=0}^{\infty}\frac{(-(-1)^\natural p^{u+1})^{p^i}}{p^{u+i+1}} X^{p^i}\right).$$

This completes the proof. $\square$

Remark 2.3 In [4], Conrad has given the equation in Lemma 2.2 with $\natural = 0$.

3 Fundamental facts

In this section, we provide four fundamental facts for the study of $\mathrm{ord}_p(a^\natural_n(p^u))$ and $\mathrm{ord}_p(t_n(p^u))$. The next lemma is well-known (cf. [8, Problems 164 and 165], [15, p. 7, Exercise 14], [19, Lemma 25.5]).

Lemma 3.1 *Suppose that $n = n_0 + n_1 p + n_2 p^2 + \cdots \neq 0$, where n_i, $i = 0, 1, \ldots$, are nonnegative integers less than p. Then*

$$\mathrm{ord}_p(n!) = \sum_{j=1}^{\infty}\left[\frac{n}{p^j}\right] = \frac{n - n_0 - n_1 - n_2 - \cdots}{p-1} \leq \frac{n-1}{p-1}.$$

For each non-zero p-adic integer $x = \sum_{i=0}^{\infty} x_i p^i$ with $0 \leq x_i \leq p-1$, we denote by $\mathrm{ord}_p(x)$ the first index i such that $x_i \neq 0$. The p-adic absolute vale of a p-adic integer x is given by

$$|x|_p = \begin{cases} p^{-\mathrm{ord}_p(x)} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We define a subring $\mathbb{Z}_p\langle X\rangle$ of $\mathbb{Z}_p[[X]]$ by

$$\mathbb{Z}_p\langle X\rangle = \left\{ \sum_{n=0}^{\infty} m_n X^n \in \mathbb{Z}_p[[X]] \;\middle|\; \lim_{n\to\infty} |m_n|_p = 0 \right\}.$$

For each $g(X) = \sum_{n=0}^{\infty} g_n X^n \in \mathbb{Z}_p[[X]]$, $g(X) + p^{k_1}X^{k_2}\mathbb{Z}_p\langle X\rangle$ denotes the set of all formal power series $f(X) = \sum_{n=0}^{\infty} f_n X^n$ such that $f(X) - g(X) \in p^{k_1}X^{k_2}\mathbb{Z}_p\langle X\rangle$, where k_1 and k_2 are nonnegative integers.

Lemma 3.2 *Let k be a positive integer, and let a be a p-adic integer such that $\mathrm{ord}_p(a) = k$. Excepting the case where $p = 2$ and $k = 1$,*

$$\exp(aX) \in 1 + aX + \frac{a^2}{2}X^2 + \frac{a^3}{6}X^3 + p^{2k+1}X^4\mathbb{Z}_p\langle X\rangle.$$

Proof. Observe that

$$\exp(aX) - 1 - aX - \frac{a^2}{2}X^2 - \frac{a^3}{6}X^3 = p^{2k}X^3 \sum_{n=1}^{\infty} p^{-2k} \frac{a^{n+3}}{(n+3)!} X^n.$$

Then it follows from Lemma 3.1 that

$$\mathrm{ord}_p\left(p^{-2k}\frac{a^{n+3}}{(n+3)!}\right) \geq k(n+1) - \frac{n+2}{p-1} = \left(k - \frac{1}{p-1}\right)n + \left(k - \frac{2}{p-1}\right).$$

This completes the proof. □

The next lemma is essentially due to K. Conrad [4] (see also [19, Theorem 54.4]).

Lemma 3.3 *Let $\sum_{n=0}^{\ell} m_n X^n$ be a polynomial of degree ℓ with coefficients in $\mathbb{Z}_p$, and let $\sum_{n=1}^{\infty} w_n X^n \in p^k X\mathbb{Z}_p\langle X\rangle$, k a nonnegative integer. Define a sequence $\{d_n\}_{n=0}^{\infty}$ by $d_0 = m_0$ and $d_n = m_n + w_n$ for $n = 1, 2, \ldots$. Then there exists a p-adic analytic function $g(X) \in \mathbb{Z}_p\langle X\rangle$ such that*

$$\sum_{n=0}^{\infty} \frac{g(n)}{n!} X^n = \exp(X) \sum_{n=0}^{\infty} d_n X^n \quad \textit{and} \quad g(X) \in \sum_{i=0}^{\ell} m_i i! \binom{X}{i} + p^k X\mathbb{Z}_p\langle X\rangle,$$

where

$$\binom{X}{i} = \frac{X(X-1)\cdots(X-i+1)}{i!}, \quad i = 1, 2, \ldots, \quad \textit{and} \quad \binom{X}{0} = 1.$$

Proof. Define a formal series

$$f(X) = \sum_{i=0}^{\infty} d_i i! \binom{X}{i}.$$

For any nonnegative integer i, we have

$$\sum_{n=0}^{\infty} \frac{i!\binom{n}{i}}{n!} X^n = \exp(X) \cdot X^i,$$

which is extended to the formula

$$\sum_{n=0}^{\infty} \frac{f(n)}{n!} X^n = \exp(X) \sum_{n=0}^{\infty} d_n X^n$$

by $\mathbb{Z}_p$-linearly. For each positive integer i, let $\{k_{in}\}_{n=1}^{\infty}$ be a sequence given by

$$\sum_{n=1}^{\infty} k_{in} X^n = i! \binom{X}{i}.$$

Then $k_{in} \in \mathbb{Z}$, and $k_{in} = 0$ if $n \geq i+1$. Since $\lim_{n\to\infty} |w_n|_p = 0$, it follows that

$$f(x) - \sum_{i=0}^{\ell} m_i i! \binom{x}{i} = \sum_{i=1}^{\infty} w_i i! \binom{x}{i} = \sum_{i=1}^{\infty}\sum_{n=1}^{i} w_i k_{in} x^n = \sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty} w_i k_{in}\right) x^n$$

for any $x \in \mathbb{Z}_p$. In particular, $\sum_{i=n}^{\infty} w_i k_{in} \in p^k\mathbb{Z}_p$ for any positive integer n. Moreover, $\lim_{n\to\infty} \left|\sum_{i=n}^{\infty} w_i k_{in}\right|_p = 0$. Now define a formal power series

$$g(X) = \sum_{i=0}^{\ell} m_i i! \binom{X}{i} + \sum_{n=1}^{\infty}\left(\sum_{i=n}^{\infty} w_i k_{in}\right) X^n.$$

Then $f(n) = g(n)$ for $n = 0, 1, 2, \ldots$. This completes the proof. □

The following theorem is part of [8, Theorem 6.2.6] (see also [15, Chap. IV Theorem 14]).

Theorem 3.4 (p-adic Weierstrass Preparation Theorem) *Let*

$$f(X) = \sum f_n X^n$$

be a power series with coefficients in the field $\mathbb{Q}_p$ of p-adic numbers such that $\lim_{n\to\infty} |f_n|_p = 0$. Let N be the number defined by

$$|f_N|_p = \max |f_n|_p \quad \textit{and} \quad |f_n|_p < |f_N|_p \textit{ for all } n > N.$$

Then there exists a polynomial

$$k_0 + k_1 X + k_2 X^2 + \cdots + k_N X^N$$

of degree N with coefficients in $\mathbb{Q}_p$, and a formal power series

$$1 + m_1 X + m_2 X^2 + \cdots$$

with coefficients in $\mathbb{Q}_p$, satisfying

(i) $f(X) = (k_0 + k_1 X + k_2 X^2 + \cdots + k_N X^N)(1 + m_1 X + m_2 X^2 + \cdots)$,

(ii) $|k_N|_p = \max |k_n|_p$,

(iii) $\lim\limits_{n\to\infty} |m_n|_p = 0$,

(iv) $|m_n|_p < 1$ *for all* $n \geq 1$.

4 p-adic properties of $a_n(p^u)$

We define a sequence $\{e_n^\natural\}_{n=0}^\infty$ by

$$\sum_{n=0}^{\infty} e_n^\natural X^n = \exp\left(\sum_{i=2}^{\infty} \frac{\varepsilon^\natural p^{p^i(u+1)}}{p^{u+i+1}} X^{p^i}\right),$$

so that for any nonnegative integer r less than p^{u+1},

$$\sum_{n=0}^{\infty} d_{n,r}^\natural X^n = \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}^\natural (-(-1)^\natural p^{u+1})^j X^j\right) \exp\left(\frac{\varepsilon^\natural p^{p(u+1)}}{p^{u+2}} X^p\right) \sum_{n=0}^{\infty} e_n^\natural X^n.$$

To give p-adic properties of $a_n(p^u)$, we need the following.

Lemma 4.1 $\sum_{n=0}^{\infty} e_n^\natural X^n \in 1 + p^{3u+1} X \mathbb{Z}_p\langle X\rangle$.

Proof. If $i \geq 2$, then $p^i = (1+p-1)^i \geq i(p-1)+p \geq i+2 \geq 4$, and thereby,

$$\begin{aligned} \operatorname{ord}_p\left(\frac{p^{p^i(u+1)}}{p^{u+i+1}}\right) &= p^i(u+1) - (u+i+1) \\ &= p^i u + p^i - (u+i+1) \\ &\geq 4u + (i+2) - (u+i+1) \\ &= 3u+1. \end{aligned}$$

Hence the assertion follows from Lemma 3.2. This completes the proof. □

The results are divided into three theorems, which generalize part of the results proved by K. Conrad [4] (see also [11, 16]).

Theorem 4.2 *Suppose that $p \geq 3$. Let r be a nonnegative integer less than p^{u+1}. Then there exists a p-adic analytic function $g_r(X) \in \mathbb{Z}_p\langle X\rangle$ such that*

$$g_r(y) = \frac{a_{p^{u+1}y+r}(p^u)}{(p^{u+1}y+r)!}(-p^{u+1})^y y!$$

for any nonnegative integer y and

$$g_r(X) \in c_r - c_{p^{u+1}+r} p^{u+1} X + p^{2u+1} X \mathbb{Z}_p\langle X\rangle.$$

Proof. Using Lemmas 3.2 and 4.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} d_{n,r}^0 X^n &= \left(\sum_{j=0}^{\infty} c_{p^{u+1}j+r}(-p^{u+1})^j X^j\right) \exp\left(\frac{p^{p(u+1)}}{p^{u+2}} X^p\right) \sum_{n=0}^{\infty} e_n^0 X^n \\ &\in c_r - c_{p^{u+1}+r} p^{u+1} X + p^{2u+1} X \mathbb{Z}_p\langle X\rangle. \end{aligned}$$

Hence the assertion follows from Lemmas 2.2 and 3.3. This completes the proof. □

Theorem 4.3 *Suppose that $p = 2$ and $u \geq 2$. Let r be a nonnegative integer less than 2^{u+1}. Then there exists a 2-adic analytic function $g_r^\natural(X) \in \mathbb{Z}_2\langle X\rangle$ such that*

$$g_r^\natural(y) = \frac{a^\natural_{2^{u+1}y+r}(2^u)}{(2^{u+1}y+r)!}(-(-1)^\natural 2^{u+1})^y y!$$

for any nonnegative integer y and

$$g_r^\natural(X) \in c_r^\natural(1 - (-1)^\natural 2^u X(X-1) + 2^{2u-1}X(X-1)(X-2)(X-3)) \\ -(-1)^\natural c^\natural_{2^{u+1}+r}2^{u+1}X + 2^{2u+1}X\mathbb{Z}_2\langle X\rangle.$$

Proof. By definition,

$$\sum_{n=0}^{\infty} d^\natural_{n,r}X^n = \left(\sum_{j=0}^{\infty} c^\natural_{2^{u+1}j+r}(-(-1)^\natural 2^{u+1})^j X^j\right)\exp(-(-1)^\natural 2^u X^2)\sum_{n=0}^{\infty} e_n^\natural X^n.$$

(Note that $\varepsilon^\natural = -(-1)^\natural$ if $p = 2$.) Using Lemma 3.2, we have

$$\exp(-(-1)^\natural 2^u X^2) \in 1 - (-1)^\natural 2^u X^2 + 2^{2u-1}X^4 + 2^{2u+1}X^6\mathbb{Z}_2\langle X\rangle.$$

Moreover, it follows from Lemma 4.1 that

$$\sum_{i=0}^{\infty} d^\natural_{n,r}X^n \in c_r^\natural(1 - (-1)^\natural 2^u X^2 + 2^{2u-1}X^4) \\ -(-1)^\natural c^\natural_{2^{u+1}+r}2^{u+1}X + 2^{2u+1}X\mathbb{Z}_2\langle X\rangle.$$

Hence the assertion follows from Lemmas 2.2 and 3.3. This completes the proof. □

Theorem 4.4 *Suppose that $p = 2$ and $u = 1$. Let r be a nonnegative integer less than 4. Then there exists a 2-adic analytic function $g_r^\natural(X) \in \mathbb{Z}_2\langle X\rangle$ such that*

$$g_r^\natural(y) = \frac{a^\natural_{4y+r}(2)}{(4y+r)!}((-1)^\natural 4)^y y!$$

for any nonnegative integer y and

$$g_r^\natural(X) \in c_r^\natural(1 - 2X + 4\delta_{\natural 1}X(X-1) - 4X(X-1)(X-2)(X-3)) \\ +(-1)^\natural 4c^\natural_{4+r}X + 8X\mathbb{Z}_2\langle X\rangle,$$

where δ is the Kronecker delta.

Proof. Substituting $-X$ for X in Lemma 2.2, we have

$$\begin{aligned}\sum_{y=0}^{\infty} \frac{a^{\natural}_{4y+r}(2)}{(4y+r)!}((-1)^{\natural}4)^y X^y &= \exp(X)\exp(-2X-(-1)^{\natural}2X^2) \\ &\quad \times \left(\sum_{j=0}^{\infty} c^{\natural}_{4j+r}((-1)^{\natural}4)^j X^j\right) \sum_{n=0}^{\infty} e^{\natural}_n (-1)^n X^n.\end{aligned} \tag{6}$$

Moreover, it follows from Eq. (4) with $p=2$ and $u=2$ that

$$\begin{aligned}\exp(-2X-(-1)^{\natural}2X^2) &= \exp(-2X+2X^2+4X^4)\exp(-4\delta_{\natural 0}X^2-4X^4) \\ &= \left(\sum_{n=0}^{\infty} \frac{a_n(4)}{n!}(-2X)^n\right)\exp(-4\delta_{\natural 0}X^2-4X^4).\end{aligned}$$

By Lemma 3.1 and Theorem 4.3,

$$\mathrm{ord}_2\left(\frac{a_n(4)}{n!}(-2)^n\right) = \mathrm{ord}_2(a_n(4)) + \mathrm{ord}_2\left(\frac{(-2)^n}{n!}\right) \geq \left[\frac{n}{2}\right] + \left[\frac{n}{4}\right] - 2\left[\frac{n}{8}\right] + 1$$

if $n \geq 1$ (see also Proposition 4.7). Observe that

$$\mathrm{ord}_2\left(\frac{a_n(4)}{n!}(-2)^n\right) \geq 4$$

if $n \geq 4$. Then, since $a_0(4) = a_1(4) = 1$, $a_2(4) = 2$, and $a_3(4) = 4$, we have

$$\sum_{n=0}^{\infty} \frac{a_n(4)}{n!}(-2X)^n \in 1 - 2X + 4X^2 + 16X\mathbb{Z}_2\langle X\rangle.$$

This, combined with Lemma 3.2, yields

$$\exp(-2X-(-1)^{\natural}2X^2) \in (1-2X+4X^2)(1-4\delta_{\natural 0}X^2-4X^4) + 8X\mathbb{Z}_2\langle X\rangle.$$

Hence it follows from Lemma 4.1 that

$$\begin{aligned}&\exp(-2X-(-1)^{\natural}2X^2)\left(\sum_{j=0}^{\infty} c^{\natural}_{4j+r}((-1)^{\natural}4)^j X^j\right)\sum_{n=0}^{\infty} e^{\natural}_n(-1)^n X^n \\ &\qquad \in c^{\natural}_r(1-2X+4\delta_{\natural 1}X^2-4X^4) + (-1)^{\natural}4c^{\natural}_{4+r}X + 8X\mathbb{Z}_2\langle X\rangle.\end{aligned}$$

The assertion now follows from Lemma 3.3 and Eq. (6). □

Let r be a nonnegative integer less than p^{u+1}. By Lemma 3.1,

$$\mathrm{ord}_p\left(\frac{(p^{u+1}y+r)!}{p^{(u+1)y}y!}\right) = \sum_{j=1}^{u}\left[\frac{p^{u+1}y+r}{p^j}\right] - uy = \left\{\frac{p^{u+1}-1}{p-1} - (u+1)\right\}y + \mathrm{ord}_p(r!)$$

for any nonnegative integer y. Combining this fact with Theorems 4.2, 4.3, and 4.4, we obtain the following.

Corollary 4.5 ([13]) *Let r be a nonnegative integer less than p^{u+1}. Then*

$$\begin{aligned}\mathrm{ord}_p\left(a_{p^{u+1}y+r}(p^u)\right) &\geq \sum_{j=1}^{u}\left[\frac{p^{u+1}y+r}{p^j}\right] - uy \\ &= \left\{\frac{p^{u+1}-1}{p-1} - (u+1)\right\} y + \mathrm{ord}_p(r!)\end{aligned}$$

for any nonnegative integer y. Moreover, if $\mathrm{ord}_p(c_r) \leq u$, then

$$\begin{aligned}\mathrm{ord}_p(a_{p^{u+1}y+r}(p^u)) &= \sum_{j=1}^{u}\left[\frac{p^{u+1}y+r}{p^j}\right] - uy + \mathrm{ord}_p(c_r) \\ &= \left\{\frac{p^{u+1}-1}{p-1} - (u+1)\right\} y + \mathrm{ord}_p(r!) + \mathrm{ord}_p(c_r)\end{aligned}$$

for any nonnegative integer y.

Example 4.6 ([6, 13, 14, 16]) Suppose that $p = 2$ and $u = 1$. By Lemma 2.1 and Corollary 4.5,

$$\mathrm{ord}_2(a_n(2)) = \begin{cases} \left[\dfrac{n}{2}\right] - \left[\dfrac{n}{4}\right] + 1 & \text{if } n \equiv 3 \pmod 4, \\ \left[\dfrac{n}{2}\right] - \left[\dfrac{n}{4}\right] & \text{otherwise.} \end{cases}$$

Proposition 4.7 *Suppose that $p = 2$ and $u = 2$, and let r be a nonnegative integer less than 8. For any nonnegative integer y,*

$$\begin{aligned}\mathrm{ord}_2(a_{8y+r}(4)) &= \left[\frac{8y+r}{2}\right] + \left[\frac{8y+r}{4}\right] - 2y + \mathrm{ord}_2(c_r) \\ &= 4y + \mathrm{ord}_2(r!) + \mathrm{ord}_2(c_r),\end{aligned}$$

that is, the values of $\mathrm{ord}_2(a_{8y+r}(4)) - 4y$, $0 \leq r \leq 7$, are as follows :

r	0	1	2	3	4	5	6	7
$\mathrm{ord}_2(a_{8y+r}(4)) - 4y$	0	0	1	2	4	3	8	4

Proof. If $r \neq 6$, then the proposition follows from Lemma 2.1 and Corollary 4.5. By Theorem 4.3, there exists a 2-adic analytic function $g_6^0(X) \in \mathbb{Z}_2\langle X\rangle$ such that

$$g_6^0(y) = \frac{a_{8y+6}(4)}{(8y+6)!}(-8)^y y!$$

for any nonnegative integer y and

$$g_6^0(X) \in c_6(1 - 4X(X-1) + 8X(X-1)(X-2)(X-3)) - 8c_{14}X + 2^5 X\mathbb{Z}_2\langle X\rangle.$$

Let y be a nonnegative integer. We have $\mathrm{ord}_2(a_{8y+6}(4)) = 4y + 4 + \mathrm{ord}_2(g_6^0(y))$. Since $c_6 = 16/45$ and $c_{14} = 2172172/42567525$, it follows that $\mathrm{ord}_2(g_6^0(y)) = 4$. Hence $\mathrm{ord}_2(a_{8y+6}(4)) = 4y + 8$. This completes the proof. □

5 2-adic properties of $t_n(2^u)$

The first statement of the following theorem is due to D. Kim and J. S. Kim [14], and the second one is an affirmative answer to a conjecture of them.

Theorem 5.1 *Suppose that $p = 2$ and $u = 1$. Then the following statements hold for any nonnegative integer y.*

(a) $\mathrm{ord}_2(t_{4y}(2)) = y + \chi_o(y)$, $\mathrm{ord}_2(t_{4y+2}(2)) = \mathrm{ord}_2(t_{4y+3}(2)) = y$.

(b) *There exists a 2-adic integer α such that*

$$\mathrm{ord}_2(t_{4y+1}(2)) = y + \chi_o(y) \cdot (\mathrm{ord}_2(y + \alpha) + 1).$$

Proof. Keep the notation of Theorem 4.4, and let y be a nonnegative integer. Then by Eq. (3), we have

$$t_{4y+r}(2) = \frac{(4y + r)!}{4^y \cdot y!} \cdot \frac{g_r^0(y) + (-1)^{r+y} g_r^1(y)}{2}.$$

Now set $L_{r,y}(X) = (g_r^0(X) + (-1)^{r+y} g_r^1(X))/2$. Then there exists a 2-adic analytic function $M_{r,y}(X) \in \mathbb{Z}_2\langle X\rangle$ such that

$$\begin{aligned} L_{r,y}(X) = c_r^0 & \frac{1 - 2X - 4X(X-1)(X-2)(X-3)}{2} \\ & + (-1)^{r+y} c_r^1 \frac{1 - 2X + 4X(X-1) - 4X(X-1)(X-2)(X-3)}{2} \\ & + 2(c_{4+r}^0 - (-1)^{r+y} c_{4+r}^1)X + 4XM_{r,y}(X). \end{aligned}$$

Moreover, it follows from Lemma 2.1 that

$$\begin{aligned} & L_{0,y}(y) \equiv L_{1,y}(y) \equiv 1 \pmod 4, \\ & L_{2,y}(y) \equiv \frac{1}{2} \pmod 2, \quad L_{3,y}(y) \equiv \frac{1}{6} \pmod 2 \end{aligned}$$

if y is even, and

$$\begin{aligned} & L_{0,y}(y) \equiv -2y^2 \pmod 4, \quad L_{1,y}(y) \equiv \frac{38}{15} y - 2y^2 \pmod 4, \\ & L_{2,y}(y) \equiv \frac{1}{2} - y \pmod 2, \quad L_{3,y}(y) \equiv \frac{1}{2} - y \pmod 4 \end{aligned}$$

if y is odd. Since $\mathrm{ord}_2((4y + r)!/4^y \cdot y!) = y + \mathrm{ord}_2(r!)$, it follows that

$$\mathrm{ord}_2(t_{4y+r}(2)) = \begin{cases} y + \chi_o(y) & \text{if } r = 0, \\ y & \text{if } r = 1 \text{ and } y \text{ is even}, \\ y & \text{if } r = 2 \text{ or } r = 3. \end{cases}$$

Assume that y is odd. Then by Lemma 2.1,

$$L_{1,y}(X) = -2X(X-1) + \frac{8}{15}X + 4XM_{1,y}(X) = \frac{38}{15}X - 2X^2 + 4XM_{1,y}(X).$$

Hence it follows from Theorem 3.4 that there exists a polynomial

$$k_0 + k_1X + k_2X^2$$

of degree 2 with coefficients in $\mathbb{Q}_2$, and a power series

$$1 + m_1X + m_2X^2 + \cdots$$

with coefficients in $\mathbb{Q}_2$, satisfying the conditions (i)–(iv) with $f(X) = L_{1,y}(X)$, $N = 2$, and $p = 2$. We have $k_0 = 0$, $k_1 \equiv 38/15 \pmod 4$, and $k_2 \equiv -2 - k_1m_1 \pmod 4$. Now set $\lambda = 2^{-1}k_2$. Then $\mathrm{ord}_2(\lambda) = 0$, because $\mathrm{ord}_2(m_1) > 0$. Observe that $\alpha := 2^{-1}k_1\lambda^{-1} \in \mathbb{Z}_2$ and

$$L_{1,y}(X) = 2\lambda X(X+\alpha)(1 + m_1X + m_2X^2 + \cdots).$$

Then we have

$$\mathrm{ord}_2(t_{4y+1}) = y + 1 + \mathrm{ord}_2(y+\alpha).$$

This completes the proof. □

Remark 5.2 According to Mathematica,

$$\alpha \equiv 1 + 2 + 2^3 + 2^8 + 2^{10} + 2^{12} \pmod{2^{14}}.$$

The following lemma is an immediate consequence of Eq. (3) and Theorem 4.3.

Lemma 5.3 *Suppose that $p = 2$ and $u \geq 2$. Let r be a nonnegative integer less than 2^{u+1}, and let y be a nonnegative integer. Then there exists a 2-adic analytic function $M_{r,y}(X) \in \mathbb{Z}_2\langle X\rangle$ such that*

$$t_{2^{u+1}y+r}(2^u) = \frac{(2^{u+1}y+r)!}{2^{(u+1)y}\cdot y!}\cdot L_{r,y}(y)$$

with

$$\begin{aligned} L_{r,y}(X) = \; & (-1)^y c_r^0 \frac{1 - 2^uX(X-1) + 2^{2u-1}X(X-1)(X-2)(X-3)}{2} \\ & + (-1)^r c_r^1 \frac{1 + 2^uX(X-1) + 2^{2u-1}X(X-1)(X-2)(X-3)}{2} \\ & + 2^u(-(-1)^y c^0_{2^{u+1}+r} + (-1)^r c^1_{2^{u+1}+r})X + 2^{2u}XM_{r,y}(X). \end{aligned}$$

Moreover, $\mathrm{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \mathrm{ord}_2(r!) + \mathrm{ord}_2(L_{r,y}(y))$.

We set $\chi_e(y) = 1 - \chi_o(y)$ for any nonnegative integer y.

Theorem 5.4 *Suppose that $p = 2$ and $u = 2$. Then the following statements hold for any nonnegative integer y.*

(a) $\mathrm{ord}_2(t_{8y+2}(4)) = \mathrm{ord}_2(t_{8y+3}(4)) = 4y$, $\mathrm{ord}_2(t_{8y+4}(4)) = 4y + 2$,
$\mathrm{ord}_2(t_{8y+5}(4)) = 4y + 3 + \chi_e(y)$, $\mathrm{ord}_2(t_{8y+6}(4)) = 4y + 3$,
$\mathrm{ord}_2(t_{8y+7}(4)) = 4y + 4 + \chi_e(y)$.

(b) *If $r = 0$ or $r = 1$, then there exists a 2-adic integer α_r such that*

$$\mathrm{ord}_2(t_{8y+r}(4)) = 4y + \chi_o(y) \cdot (\mathrm{ord}_2(y + \alpha_r) + 2).$$

Proof. Keep the notation of Lemma 5.3 with $u = 2$. Then by Lemma 2.1,

$$\begin{aligned}
&L_{0,y}(y) \equiv L_{1,y}(y) \equiv 1 \pmod 8, \quad L_{2,y}(y) \equiv \frac{1}{2} \pmod 4,\\
&L_{3,y}(y) \equiv L_{4,y}(y) \equiv \frac{1}{6} \pmod 4, \quad L_{5,y}(y) \equiv \frac{2}{15} \pmod 8,\\
&L_{6,y}(y) \equiv \frac{17}{90} \pmod 4, \quad L_{7,y}(y) \equiv \frac{46}{315} \pmod 8
\end{aligned}$$

if y is even, and

$$\begin{aligned}
&L_{0,y}(y) \equiv 4y\left(y - \frac{251}{315}\right) \pmod{16}, \quad L_{1,y} \equiv 4y\left(y - \frac{2519}{2835}\right) \pmod{16},\\
&L_{2,y}(y) \equiv L_{3,y}(y) \equiv L_{4,y}(y) \equiv -\frac{1}{2} \pmod 4, \quad L_{5,y}(y) \equiv -\frac{1}{3} \pmod 4,\\
&L_{6,y}(y) \equiv -\frac{1}{6} \pmod 4, \quad L_{7,y}(y) \equiv -\frac{1}{15} \pmod 4
\end{aligned}$$

if y is odd. This, combined with Lemma 5.3, yields the statement (a). The proof of the statement (b) is analogous to that of Theorem 5.1, while the assertion is a special case of Theorem 5.6. This completes the proof. □

Remark 5.5 According to Mathematica,

$$\alpha_0 \equiv 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + 2^7 + 2^9 + 2^{10} + 2^{12} + 2^{13} + 2^{14} + 2^{15} \pmod{2^{17}}$$

and

$$\alpha_1 \equiv 1 + 2 + 2^4 + 2^7 + 2^8 \pmod{2^{12}}.$$

The statement (b) of Theorem 5.4 is extended to a result for $\mathrm{ord}_2(t_{2^{u+1}y+r}(2^u))$ with $u \geq 3$ and $r = 0$ or $r = 1$.

Theorem 5.6 *Suppose that $p = 2$ and $u \geq 2$. Let y be a nonnegative integer. If $r = 0$ or $r = 1$, then there exists a 2-adic integer α_r such that*

$$\mathrm{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \chi_o(y) \cdot (\mathrm{ord}_2(y + \alpha_r) + u).$$

Moreover, if $\mathrm{ord}_2(c^0_{2^{u+1}+r} + (-1)^r c^1_{2^{u+1}+r}) = 0$ with $r = 0$ or $r = 1$, then

$$\mathrm{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \chi_o(y) \cdot u.$$

Proof. Keep the notation of Lemma 5.3. Since $c^0_0 = c^1_0 = c^0_1 = 1$ and $c^1_1 = -1$ by Lemma 2.1, it follows from Lemma 5.3 that the assertion holds if y is even. Assume that y is odd. Then

$$L_{r,y}(X) = 2^u(-1 + \hat{c}_{2^{u+1}+r})X + 2^u X^2 + 2^{2u} X M_{r,y}(X),$$

where $\hat{c}_{2^{u+1}+r} = c^0_{2^{u+1}+r} + (-1)^r c^1_{2^{u+1}+r}$. In each of the cases where $r = 0$ and $r = 1$, it follows from Theorem 3.4 that there exists a polynomial

$$k_0 + k_1 X + k_2 X^2$$

of degree 2 with coefficients in $\mathbb{Q}_2$, and a power series

$$1 + m_1 X + m_2 X^2 + \cdots$$

with coefficients in $\mathbb{Q}_2$, satisfying the conditions (i)–(iv) with $f(X) = L_{r,y}(X)$, $N = 2$, and $p = 2$. We have $k_0 = 0$, $k_1 \equiv 2^u(-1 + \hat{c}_{2^{u+1}+r}) \pmod{2^{2u}}$, and $k_2 \equiv 2^u - k_1 m_1 \pmod{2^{2u}}$. Now set $\lambda_r = 2^{-u} k_2$. Then $\mathrm{ord}_2(\lambda_r) = 0$, because $\mathrm{ord}_2(m_1) > 0$. Observe that $\alpha_r := 2^{-u} k_1 \lambda_r^{-1} \in \mathbb{Z}_2$ and

$$L_{r,y}(X) = 2^u \lambda_r X(X + \alpha_r)(1 + m_1 X + m_2 X^2 + \cdots).$$

Combining this fact with Lemma 5.3, we conclude that

$$\mathrm{ord}_2(t_{2^{u+1}y+r}(2^u)) = (2^{u+1} - u - 2)y + \mathrm{ord}_2(y + \alpha_r) + u$$

Moreover, if $\mathrm{ord}_2(\hat{c}_{2^{u+1}+r}) = 0$, then $\mathrm{ord}_2(\alpha_r) > 0$, and thereby, $\mathrm{ord}_2(y + \alpha_r) = 0$. This completes the proof. □

6 Wreath products

Let G be a finite group, and let K be a subgroup of S_n. The wreath product $G \wr K$ of G by K is defined to be the set

$$G \wr K = \{(g_1, \ldots, g_n)\sigma \mid (g_1, \ldots, g_n) \in G^{(n)} \quad \text{and} \quad \sigma \in K\},$$

where $G^{(n)}$ is the direct product of n copies of G, with multiplication given by

$$(g_1, \ldots, g_n)\sigma(h_1, \ldots, h_n)\tau = (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)})\sigma\tau.$$

Let m be a positive integer. We set

$$a(G \wr K, m) = \sharp\{(g_1, \ldots, g_n)\sigma \in G \wr K \mid ((g_1, \ldots, g_n)\sigma)^m = \epsilon\}.$$

Lemma 6.1 *Let $\tau \in S_n$ be a cycle of length ℓ. Then $((g_1, \dots, g_n)\tau)^m = \epsilon$ if and only if ℓ divides m and $(g_i g_{\tau^{-1}(i)} \cdots g_{\tau^{-\ell+1}(i)})^{m/\ell} = \epsilon$ for all $i = 1, 2, \dots, n$.*

Proof. The proof is straightforward. □

Let $\{\ell_0, \ell_1, \dots, \ell_s\}$ be the set of divisors of a positive integer m. We quote the following (cf. [12, Lemma 4.2.10]).

Lemma 6.2 *The number of elements $(g_1, \dots, g_n)\sigma$ of $G \wr S_n$ such that the cycle type of σ is $(\ell_0^{j_0}, \ell_1^{j_1}, \dots, \ell_s^{j_s})$ and $((g_1, \dots, g_n)\sigma)^m = \epsilon$ is*

$$n! \prod_{k=0}^{s} \frac{|G|^{(\ell_k-1)j_k} a(G, m/\ell_k)^{j_k}}{\ell_k{}^{j_k} j_k!},$$

where $a(G, m/\ell_k) = \sharp\{g \in G \mid g^{m/\ell_k} = \epsilon\}$.

Proof. Let k be a nonnegative integer less than or equal to s, and let $\tau = (i_1 \ \cdots \ i_{\ell_k})$ be a cycle of length ℓ_k. Then it follows from Lemma 6.1 that the number of elements $(g_1, \dots, g_n)$ of $G^{(n)}$ such that $((g_1, \dots, g_n)\tau)^m = \epsilon$ and $g_i = \epsilon$ for all $i \neq i_1, \dots, i_{\ell_k}$ is $|G|^{\ell_k - 1} a(G, m/\ell_k)$. Thus the lemma holds. □

By Lemma 6.2, we have

$$b_n(p^u) = a(C_p \wr S_n, p^u) = \sum_{j_0 + j_1 p + \cdots + j_u p^u = n} n! \left(\prod_{k=0}^{u} \frac{p^{p^k j_k}}{p^{k j_k} j_k!} \right) \frac{1}{p^{j_u}}. \tag{7}$$

Set $b_n^0(p^u) = b_n(p^u)$, and define

$$b_n^1(p^u) = \sum_{j_0 + j_1 p + \cdots + j_u p^u = n} (-1)^{j_0 + j_1 + \cdots + j_u} n! \left(\prod_{k=0}^{u} \frac{p^{p^k j_k}}{p^{k j_k} j_k!} \right) \frac{1}{p^{j_u}}. \tag{8}$$

Then by Lemma 6.2, we have

$$q_n(p^u) = a(C_p \wr A_n, p^u) = \frac{b_n^0(p^u) + (-1)^n b_n^1(p^u)}{2}. \tag{9}$$

(Obviously, $b_n(p^u) = q_n(p^u)$ if $p \neq 2$.) Let $\natural$ denotes both 0 and 1. We always assume that $b_0^\natural(p^u) = 1$. By Eqs. (7)–(9), we have

$$\sum_{n=0}^{\infty} \frac{b_n^\natural(p^u)}{n!} X^n = \exp\left((-1)^\natural \sum_{k=0}^{u-1} \frac{p^{p^k}}{p^k} X^{p^k} + (-1)^\natural \frac{p^{p^u}}{p^{u+1}} X^{p^u} \right), \tag{10}$$

$$\sum_{n=0}^{\infty} \frac{q_n(2^u)}{n!} X^n = \frac{1}{2} \exp\left(\sum_{k=0}^{u-1} \frac{2^{2^k}}{2^k} X^{2^k} + \frac{2^{2^u}}{2^{u+1}} X^{2^u} \right)$$
$$+ \frac{1}{2} \exp\left(2X - \sum_{k=1}^{u-1} \frac{2^{2^k}}{2^k} X^{2^k} - \frac{2^{2^u}}{2^{u+1}} X^{2^u} \right)$$

(cf. [1], [17, Proposition 3.4]). Moreover, by Eq. (5), we have

$$\sum_{n=0}^{\infty} c_n^{\natural}(pX)^n = \exp\left((-1)^{\natural}\sum_{k=0}^{\infty}\frac{p^{p^k}}{p^k}X^{p^k}\right). \tag{11}$$

Recall that $\varepsilon^{\natural} = -1$ if $p = 2$ and $\natural = 0$, and $\varepsilon^{\natural} = +1$ otherwise. For any nonnegative integer r less than p^u, we set

$$\widetilde{H}_{u,r}^{\natural}(X) = \sum_{y=0}^{\infty}\frac{b_{p^u y+r}^{\natural}(p^u)}{(p^u y+r)!}\left(-(-1)^{\natural}\frac{p^{u+1}}{p^{p^u}(p-1)}X\right)^y,$$

and define a sequence $\{\widetilde{d}_{n,r}^{\natural}\}_{n=0}^{\infty}$ by

$$\sum_{n=0}^{\infty}\widetilde{d}_{n,r}^{\natural}X^n = \left(\sum_{j=0}^{\infty}c_{p^u j+r}^{\natural}p^r\left(-(-1)^{\natural}\frac{p^{u+1}}{p-1}X\right)^j\right)\exp\left(\sum_{i=1}^{\infty}\frac{\varepsilon^{\natural}p^{p^i(u+1)}}{p^{u+i}(p-1)^{p^i}}X^{p^i}\right).$$

Lemma 6.3 *Let r be a nonnegative integer less than p^u. Then*

$$\widetilde{H}_{u,r}^{\natural}(X) = \exp(X)\sum_{n=0}^{\infty}\widetilde{d}_{n,r}^{\natural}X^n.$$

Proof. Using Eqs. (10) and (11), we have

$$\sum_{n=0}^{\infty}\frac{b_n^{\natural}(p^u)}{n!}X^n = \left(\sum_{n=0}^{\infty}c_n^{\natural}(pX)^n\right)\exp\left(-(-1)^{\natural}\frac{p^{p^u}}{p^u}X^{p^u}\right)$$
$$\times\exp\left((-1)^{\natural}\frac{p^{p^u}}{p^{u+1}}X^{p^u}\right)\exp\left(-(-1)^{\natural}\sum_{k=u+1}^{\infty}\frac{p^{p^k}}{p^k}X^{p^k}\right).$$

This formula yields

$$\sum_{y=0}^{\infty}\frac{b_{p^u y+r}^{\natural}(p^u)}{(p^u y+r)!}X^{p^u y+r} = \left(\sum_{j=0}^{\infty}c_{p^u j+r}^{\natural}p^{p^u j+r}X^{p^u j+r}\right)\exp\left(-(-1)^{\natural}\frac{p^{p^u}}{p^u}X^{p^u}\right)$$
$$\times\exp\left((-1)^{\natural}\frac{p^{p^u}}{p^{u+1}}X^{p^u}\right)\exp\left(-(-1)^{\natural}\sum_{i=1}^{\infty}\frac{p^{p^{u+i}}}{p^{u+i}}X^{p^{u+i}}\right).$$

Omit X^r and substitute $(-(-1)^{\natural}p^{u+1}X/p^{p^u}(p-1))^{1/p^u}$ for X. Then we have

$$\sum_{y=0}^{\infty}\frac{b_{p^u y+r}^{\natural}(p^u)}{(p^u y+r)!}\left(-(-1)^{\natural}\frac{p^{u+1}}{p^{p^u}(p-1)}X\right)^y = \left(\sum_{j=0}^{\infty}c_{p^u j+r}^{\natural}p^r\left(-(-1)^{\natural}\frac{p^{u+1}}{p-1}X\right)^j\right)$$
$$\times\exp(X)\exp\left(\sum_{i=1}^{\infty}\frac{-(-1)^{\natural}\cdot(-(-1)^{\natural})^{p^i}p^{p^i(u+1)}}{p^{u+i}(p-1)^{p^i}}X^{p^i}\right).$$

This completes the proof. □

7 p-adic properties of $b_n(p^u)$

In order to analyze $\sum_{n=0}^{\infty} \tilde{d}^{\natural}_{n,r} X^n$, we define a sequence $\{\tilde{e}^{\natural}_n\}_{n=0}^{\infty}$ by

$$\sum_{n=0}^{\infty} \tilde{e}^{\natural}_n X^n = \exp\left(\sum_{i=2}^{\infty} \frac{\varepsilon^{\natural} p^{p^i(u+1)}}{p^{u+i}(p-1)^{p^i}} X^{p^i} \right).$$

The proof of the following lemma is analogous to that of Lemma 4.1.

Lemma 7.1 $\displaystyle\sum_{n=0}^{\infty} \tilde{e}^{\natural}_n X^n \in 1 + p^{3u+2} X \mathbb{Z}_p\langle X \rangle.$

We are now in position to state a p-adic property of $b_n(p^u)$.

Theorem 7.2 *Let r be a nonnegative integer less than p^u. Then there exists a p-adic analytic function $g^{\natural}_r(X) \in \mathbb{Z}_p\langle X \rangle$ such that*

$$g^{\natural}_r(y) = \frac{b^{\natural}_{p^u y + r}(p^u)}{(p^u y + r)!} \left(-(-1)^{\natural} \frac{p^{u+1}}{p^{p^u}(p-1)} \right)^y y!$$

for any nonnegative integer y and

$$g^{\natural}_r(X) \in c^{\natural}_r p^r \left\{ 1 + \varepsilon^{\natural} \frac{p^{(u+1)(p-1)}}{(p-1)^p} X(X-1)(X-2)\cdots(X-p+1) \right\} - (-1)^{\natural} c^{\natural}_{p^u+r} \frac{p^{u+1+r}}{p-1} X + p^{2u+1+r} X \mathbb{Z}_p\langle X \rangle.$$

Proof. Using Lemmas 3.2 and 7.1, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \tilde{d}^{\natural}_{n,r} X^n &= \left(\sum_{j=0}^{\infty} c^{\natural}_{p^u j + r} p^r \left(-(-1)^{\natural} \frac{p^{u+1}}{p-1} X \right)^j \right) \\ &\qquad \times \exp\left(\frac{\varepsilon^{\natural} p^{p(u+1)}}{p^{u+1}(p-1)^p} X^p \right) \sum_{n=0}^{\infty} \tilde{e}^{\natural}_n X^n \\ &\in c^{\natural}_r p^r \left\{ 1 + \varepsilon^{\natural} \frac{p^{(u+1)(p-1)}}{(p-1)^p} X^p \right\} \\ &\qquad - (-1)^{\natural} c^{\natural}_{p^u+r} \frac{p^{u+1+r}}{p-1} X + p^{2u+1+r} X \mathbb{Z}_p\langle X \rangle. \end{aligned}$$

Hence the assertion follows from Lemmas 3.3 and 6.3. This completes the proof. □

This theorem, together with Lemma 3.1, yields the following.

Corollary 7.3 *Let r be a nonnegative integer less than p^u. Then*

$$\begin{aligned}\mathrm{ord}_p\left(b_{p^u y+r}(p^u)\right) &\geq \sum_{j=0}^{u-1}\left[\frac{p^u y+r}{p^j}\right]-uy\\ &=\left\{\frac{p^u-1}{p-1}+p^u-(u+1)\right\}y+r+\mathrm{ord}_p(r!)\end{aligned}$$

for any nonnegative integer y. If $\mathrm{ord}_p(c_r)\leq u$*, then*

$$\begin{aligned}\mathrm{ord}_p(b_{p^u y+r}(p^u)) &= \sum_{j=0}^{u-1}\left[\frac{p^u y+r}{p^j}\right]-uy+\mathrm{ord}_p(c_r)\\ &=\left\{\frac{p^u-1}{p-1}+p^u-(u+1)\right\}y+r+\mathrm{ord}_p(r!)+\mathrm{ord}_p(c_r)\end{aligned}$$

for any nonnegative integer y.

Example 7.4 Suppose that $u=1$. Then for any nonnegative integer r less than p, we have $\mathrm{ord}_p(c_r)=0$. Hence

$$\mathrm{ord}_p(b_n(p))=n-\left[\frac{n}{p}\right]\quad\text{and}\quad \mathrm{ord}_2(b_n(2))=\left[\frac{n+1}{2}\right].$$

Example 7.5 Suppose that $p=2$ and $u=2$. By Lemma 2.1 and Corollary 7.3,

$$\mathrm{ord}_2(b_n(4))=\begin{cases} n+\left[\dfrac{n}{2}\right]-2\left[\dfrac{n}{4}\right]+1 & \text{if } n\equiv 3 \pmod 4,\\ n+\left[\dfrac{n}{2}\right]-2\left[\dfrac{n}{4}\right] & \text{otherwise.}\end{cases}$$

Proposition 7.6 *Suppose that $p=2$ and $u=3$, and let r be a nonnegative integer less than 8. For any nonnegative integer y,*

$$\begin{aligned}\mathrm{ord}_2(b_{8y+r}(8)) &= 8y+r+\left[\frac{8y+r}{2}\right]+\left[\frac{8y+r}{4}\right]-3y+\mathrm{ord}_2(c_r)\\ &= 11y+r+\mathrm{ord}_2(r!)+\mathrm{ord}_2(c_r),\end{aligned}$$

that is, the values of $\mathrm{ord}_2(b_{8y+r}(8))-11y-r$, $0\leq r\leq 7$, *are the following :*

r	0	1	2	3	4	5	6	7
$\mathrm{ord}_2(b_{8y+r}(8))-11y-r$	0	0	1	2	4	3	8	4

Proof. If $r\neq 6$, then the theorem follows from Lemma 2.1 and Corollary 7.3. By Lemma 2.1 and Theorem 7.2, there exists a 2-adic analytic function $g_6^0(X)\in\mathbb{Z}_2\langle X\rangle$ such that

$$g_6^0(y)=\frac{b_{8y+6}(8)}{(8y+6)!}\left(-\frac{1}{16}\right)^y y!$$

for any nonnegative integer y and

$$g_6^0(X) \in 2^6 \cdot \frac{16}{45}(1-16X(X-1)) - 2^{10} \cdot \frac{2172172}{42567525} X + 2^{13} X \mathbb{Z}_2\langle X\rangle.$$

Hence Lemma 3.1 implies that $\mathrm{ord}_2(b_{8y+6}(8)) = 11y + 4 + \mathrm{ord}_2(g_6^0(y)) = 11y + 14$ for any nonnegative integer y. This completes the proof. $\square$

8 2-adic properties of $q_n(2^u)$

The following lemma is an immediate consequence of Eq. (9) and Theorem 7.2.

Lemma 8.1 *Suppose that $p = 2$. Let r be a nonnegative integer less than 2^u, and let y be a nonnegative integer. Then there exists a 2-adic analytic function $M_{r,y}(X) \in \mathbb{Z}_2\langle X\rangle$ such that*

$$q_{2^u y+r}(2^u) = \frac{(2^u y + r)!}{2^{uy} \cdot y!} \cdot 2^{(2^u-1)y} \cdot L_{r,y}(y)$$

with

$$\begin{aligned} L_{r,y}(X) = (-1)^y 2^r c_r^0 \frac{1-2^{u+1}X(X-1)}{2} + (-2)^r c_r^1 \frac{1+2^{u+1}X(X-1)}{2} \\ +2^{u+r}(-(-1)^y c_{2^u+r}^0 + (-1)^r c_{2^u+r}^1)X + 2^{2u+r} X M_{r,y}(X). \end{aligned}$$

Moreover, $\mathrm{ord}_2(q_{2^u y+r}(2^u)) = (2^{u+1} - u - 2)y + \mathrm{ord}_2(r!) + \mathrm{ord}_2(L_{r,y}(y))$.

Example 8.2 Suppose that $p = 2$ and $u = 1$. Let r be a nonnegative integer less than 2, and let y be a nonnegative integer. By Lemma 2.1 and Lemma 8.1, we have

$$\mathrm{ord}_2(q_{2y+r}(2)) = y + \left[\frac{r+1}{2}\right] + \chi_o(y) = \begin{cases} y & \text{if } y \text{ is even and if } r = 0, \\ y+1 & \text{if } y \text{ is even and if } r = 1, \\ y+1 & \text{if } y \text{ is odd and if } r = 0, \\ y+2 & \text{if } y \text{ is odd and if } r = 1. \end{cases}$$

We conclude this paper with the following three results for $\mathrm{ord}_2(q_n(2^u))$.

Theorem 8.3 *Suppose that $p = 2$ and $u = 2$. Then the following statements hold for any nonnegative integer y.*

(a) $\mathrm{ord}_2(q_{4y}(4)) = 4y + 2\chi_o(y)$, $\mathrm{ord}_2(q_{4y+2}(4)) = 4y + 2$, $\mathrm{ord}_2(q_{4y+3}(4)) = 4y + 3$.

(b) *There exists a 2-adic integer β such that*

$$\mathrm{ord}_2(q_{4y+1}(4)) = 4y + 1 + \chi_o(y) \cdot (\mathrm{ord}_2(y + \beta) + 3).$$

Proof. Keep the notation of Lemma 8.1 with $u = 2$. Set $h_{r,y} = \mathrm{ord}_2(L_{r,y}(y))$. Then by Lemma 2.1,

$$h_{0,y}(y) = 0, \quad h_{1,y} = h_{2,y} = 1, \quad h_{3,y} = 2$$

if y is even, and

$$h_{0,y} = 2, \quad L_{1,y}(y) \equiv 16y\left(y - \frac{13}{15}\right) \pmod{32}, \quad h_{2,y} = 1, \quad h_{3,y} = 2$$

if y is odd. Thus the statement (a) follows from Lemma 8.1. The proof of the statement (b) is analogous to that of Theorem 5.1(b), while the assertion is a special case of Theorem 8.7. This completes the proof. □

Remark 8.4 According to Mathematica,

$$\beta \equiv 1 + 2^2 + 2^3 + 2^4 + 2^6 + 2^7 + 2^8 \pmod{2^{13}}.$$

Theorem 8.5 *Suppose that $p = 2$ and $u = 3$. Then the following statements hold for any nonnegative integer y.*

(a) $\mathrm{ord}_2(q_{8y+2}(8)) = 11y + 2$, $\mathrm{ord}_2(q_{8y+3}(8)) = 11y + 3$, $\mathrm{ord}_2(q_{8y+4}(8)) = 11y + 6$, $\mathrm{ord}_2(q_{8y+5}(8)) = 11y + 8 + \chi_e(y)$, $\mathrm{ord}_2(q_{8y+6}(8)) = 11y + 9$, $\mathrm{ord}_2(q_{8y+7}(8)) = 11y + 11 + \chi_e(y)$.

(b) *If $r = 0$ or $r = 1$, then there exists a 2-adic integer β_r such that*

$$\mathrm{ord}_2(q_{8y+r}(8)) = 11y + r + \chi_o(y) \cdot (\mathrm{ord}_2(y + \beta_r) + 4).$$

Proof. Keep the notation of Lemma 8.1 with $u = 3$. Set $h_{r,y} = \mathrm{ord}_2(L_{r,y}(y))$. Then by Lemma 2.1,

$$h_{0,y} = 0, \quad h_{1,y} = h_{2,y} = 1, \quad h_{3,y} = 2, \quad h_{4,y} = 3, \quad h_{5,y} = 6, \quad h_{6,y} = 5, \quad h_{7,y} = 8$$

if y is even, and

$$L_{0,y}(y) \equiv 16y\left(y - \frac{283}{315}\right) \pmod{64}, \quad L_{1,y}(y) \equiv 32y\left(y - \frac{2677}{2835}\right) \pmod{128},$$
$$h_{2,y} = 1, \quad h_{3,y} = 2, \quad h_{4,y} = 3, \quad h_{5,y} = h_{6,y} = 5, \quad h_{7,y} = 7$$

if y is odd. Thus the statement (a) follows from Lemma 8.1. The proof of the statement (b) is analogous to that of Theorem 5.1(b), while the assertion is a special case of Theorem 8.7. This completes the proof. □

Remark 8.6 According to Mathematica,

$$\beta_0 \equiv 1 + 2 + 2^2 + 2^3 + 2^4 + 2^6 + 2^8 + 2^9 \pmod{2^{12}}$$

and

$$\beta_1 \equiv 1 + 2^3 + 2^4 + 2^5 + 2^6 + 2^8 + 2^{10} + 2^{11} + 2^{12} \pmod{2^{14}}.$$

The statement (b) both of Theorems 8.3 and 8.5 is extended to a result for $\mathrm{ord}_2(q_{2^u y+r}(2^u))$ with $u \geq 4$ and $r = 0$ or $r = 1$.

Theorem 8.7 *Suppose that $p = 2$ and $u \geq 2$. Let y be a nonnegative integer. If $r = 0$ or $r = 1$, then there exists a 2-adic integer β_r such that*

$$\mathrm{ord}_2(q_{2^u y+r}(2^u)) = (2^{u+1} - u - 2)y + r + \chi_o(y) \cdot (\mathrm{ord}_2(y + \beta_r) + u + 1).$$

Moreover, if $\mathrm{ord}_2(c^0_{2^u+r} + (-1)^r c^1_{2^u+r}) = 0$ *with $r = 0$ or $r = 1$, then*

$$\mathrm{ord}_2(q_{2^u y+r}(2^u)) = (2^{u+1} - u - 2)y + r + \chi_o(y) \cdot u.$$

Proof. Keep the notation of Lemma 8.1. Since $c^0_0 = c^1_0 = c^0_1 = 1$ and $c^1_1 = -1$ by Lemma 2.1, it follows from Lemma 8.1 that the assertion holds if y is even. Assume that y is odd. Then

$$L_{0,y}(X) = 2^{u+1}X(X-1) + 2^u(c^0_{2^u} + c^1_{2^u})X + 2^{2u}XM_{0,y}(X),$$
$$L_{1,y}(X) = 2^{u+2}X(X-1) + 2^{u+1}(c^0_{2^u+1} - c^1_{2^u+1})X + 2^{2u+1}XM_{1,y}(X).$$

Hence, if $\mathrm{ord}_2(c^0_{2^u+r} + (-1)^r c^1_{2^u+r}) = 0$, then the assertion follows from Lemma 8.1. Suppose that $\mathrm{ord}_2(c^0_{2^u+r} + (-1)^r c^1_{2^u+r}) > 0$. Then by an argument analogous to that in the proof of Theorem 5.6, we have

$$\mathrm{ord}_2(L_{r,y}(y)) = r + \mathrm{ord}_2(y + \beta_r) + u + 1$$

for some $\beta_r \in \mathbb{Z}_2$. Hence the assertion follows from Lemma 8.1. This completes the proof. □

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