Induction formulae for Mackey functors with applications to representations of the twisted quantum double of a finite group

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Induction formulae for Mackey functors with applications to representations of the twisted quantum double of a finite group

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Abstract

In the theory of canonical induction formulae for Mackey functors, Boltje [4] demonstrated that the plus constructions, together with the mark morphism, are useful for the study of canonical versions of induction theorems analogous to those in representation theory of finite groups. In this paper, we present a short exact sequence for the plus constructions derived from Cauchy-Frobenius lemma, and apply it to the proof of Boltje's integrality result for canonical induction formulae. The methods appearing in Boltje's theory, combined with the Dress construction for Mackey functors, are applicable to induction theorems on representations of the twisted quantum double of a finite group. As a sequel to such a research, we describe canonical versions of two induction theorems whose origins are Artin's induction theorem and Brauer's induction theorem on $\mathbb{C}$-characters of a finite group.

Contents

1 Introduction **2**

2 The plus constructions **5**

3 The Burnside ring functor **10**

4 The crossed Mackey functor **16**

5 Induction formulae for Mackey functors **23**

6 Induction formulae for crossed Mackey functors **28**

7 The twisted group algebra $\mathbb{C}^\alpha G$ **31**

8 The twisted quantum double $D^\omega(G)$ of a finite group **33**

9 Fundamental theorems for the plus constructions **44**

10 Integral canonical induction formulae 50

11 Induction formulae for representations of $\mathbb{C}^{\alpha}G$ 54

12 Induction formulae for representations of $D^{\omega}(G)$ 56

1 Introduction

The theory of canonical induction formulae for Mackey functors due to Boltje [4] has been developed from Brauer's induction theorem, which states that every $\mathbb{C}$-character of a finite group G can be expressed as a $\mathbb{Z}$-linear combination of induced linear $\mathbb{C}$-characters from subgroups of G (cf. [7]), and its canonical versions (cf. [3, 33]). A Mackey functor for G over a commutative ring k, denoted by a quadruple $X = (X, \mathrm{con}, \mathrm{res}, \mathrm{ind})$, is defined to be a family of k-modules $X(H)$, $H \le G$, together with conjugation maps $\mathrm{con}_H^g : X(H) \to X({}^gH)$, where $g \in G$, restriction maps $\mathrm{res}_K^H : X(H) \to X(K)$, and induction maps $\mathrm{ind}_K^H : X(K) \to X(H)$, where $K \le H$ in both cases, satisfying certain axioms, which is a G-functor over k introduced by Green [19]. A restriction functor and a conjugation functor, denoted by a triple $A = (A, \mathrm{con}, \mathrm{res})$ and a couple $A = (A, \mathrm{con})$, respectively, are defined in similar fashion. Considering the corresponding categories, Boltje [4] has introduced two functors $-_+ : \mathbf{Res}(G)_k \to \mathbf{Mack}(G)_k$ and $-^+ : \mathbf{Con}(G)_k \to \mathbf{Mack}(G)_k$ arising from adjoints of forgetful functors; these functors are called the lower and upper plus constructions. A canonical induction formula for a Mackey functor X from a restriction subfunctor A is a morphism $\Psi : X \to A_+$ of restriction functors such that $\Theta^{X,A} \circ \Psi = \mathrm{id}_X$ for a morphism $\Theta^{X,A} : A_+ \to X$ of Mackey functors called the induction morphism (cf. [4]). A canonical choice of Brauer's induction theorem comes from a certain canonical induction formula for the character ring functor R from a restriction subfunctor R^{ab} defined by the $\mathbb{Z}$-span of all linear $\mathbb{C}$-characters (cf. [3, 4]). In this case $R_+^{\mathrm{ab}}(G)$ is isomorphic to the ring of monomial representations of G introduced by Dress [15].

If X is a Mackey functor for G over $\mathbb{Z}$ (or the localization of $\mathbb{Z}$ at a prime p), then one may attempt to find an induction theorem on X analogous to Brauer's induction theorem. Concerning the existence of such a theorem, Boltje [4] has given an integrality criterion for canonical induction formulae. In this paper, we establish a new fundamental theorem for the plus constructions, which ensures the existence of a short exact sequence derived from Cauchy-Frobenius lemma (cf. Theorem 9.4), and successfully apply it to an argument of the integrality of canonical induction formulae under a suitable condition given in [4] (cf. Theorem 10.1).

For a normalized 3-cocycle $\omega : G \times G \times G \to \mathbb{C}^{\times}$, Dijkgraaf, Pasquier, and Roche

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[14] have introduced a quasi-triangular quasi-Hopf algebra $D^\omega(G)$ with underlying vector space $(\mathbb{C}G)^*\otimes_{\mathbb{C}}\mathbb{C}G$, where $(\mathbb{C}G)^*$ is the Hopf algebra dual to the group algebra $\mathbb{C}G$. The algebra $D^\omega(G)$ is called the twisted quantum double of G. If ω is trivial, then it is the quantum double of G and is denoted by $D(G)$. Given $H \le G$, we denote by $D^\omega_G(H)$ the subalgebra $(\mathbb{C}G)^*\otimes_{\mathbb{C}}\mathbb{C}H$ of $D^\omega(G)$. The representation group $R(D^\omega_G(H))$ of $D^\omega_G(H)$ is defined to be the additive group consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of finitely generated left $D^\omega_G(H)$-modules with direct sum for addition. With the standard definition of conjugation, restriction, and induction maps (see, *e.g.*, [2, 37]), the family of representation groups $R(D^\omega_G(H))$, $H \le G$, becomes a Mackey functor for G over $\mathbb{Z}$, which is denoted by RD^ω_G and is called the $D^\omega(G)$-representation functor. If ω is trivial, then RD^ω_G is a Green functor (cf. [37]), which is denoted by RD_G. As for applications of the methods given in [4], it is worth studying the existence of nice induction formulae for RD^ω_G. The main purpose of this paper is to present a canonical induction formula for RD^ω_G from a certain restriction subfunctor which brings Brauer's induction theorem on representations of $D^\omega(G)$ (cf. Theorem 12.2, Corollary 12.3).

If $\alpha : G\times G \to \mathbb{C}^\times$ is a normalized 2-cocycle, then for each $H \le G$, the representation group $R(\mathbb{C}^\alpha H)$ of the twisted group algebra $\mathbb{C}^\alpha H$ is defined to be the additive group consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of finitely generated left $\mathbb{C}^\alpha H$-modules with direct sum for addition. The family of representation groups $R(\mathbb{C}^\alpha H)$, $H \le G$, together with suitable conjugation, restriction, and induction maps, defines a Mackey functor for G over $\mathbb{Z}$, which is denoted by R_α and is called the $\mathbb{C}^\alpha G$-representation functor. If α is trivial, then R_α is a Green functor, which is called the $\mathbb{C}G$-representation functor. For each $s \in G$, there exists a normalized 2-cocycle $\theta_s : G_s \times G_s \to \mathbb{C}^\times$ given by $\theta_s(g,r) = \omega(s,g,r)\omega(g,r,s)/\omega(g,s,r)$ for all g, $r \in G_s$, where G_s is the centralizer of s in G, and then the $\mathbb{C}^{\theta_s}G_s$-representation functor R_{θ_s} is a Mackey functor for G_s over $\mathbb{Z}$ assigning $R(\mathbb{C}^{\theta_s}H)$ to each $H \le G_s$. Every finitely generated left $D^\omega(G)$-module is characterized by a family of certain left $\mathbb{C}^{\theta_s}G_s$-modules, s running over the elements of G (cf. [38]). We introduce a new concept, namely, the Mackey bundle composed of $\mathbb{C}^{\theta_s}G_s$-representation functors R_{θ_s}, $s \in G$, and employ it to investigate RD^ω_G. This concept, which adapts successfully the Dress construction for Mackey functors (cf. [5, 30]), defines a crucial Mackey functor for the study of $R(D^\omega(G))$ (cf. Theorem 8.4, Corollary 8.6).

If $X = (X, \mathrm{con}, \mathrm{res}, \mathrm{ind})$ is a Mackey functor for G over k, then $\overline{X}$ denotes the conjugation functor for G over k such that

$$\overline{X}(H) = \overline{X(H)} := X(H)/\sum_{K<H} \mathrm{ind}^H_K(X(K))$$

for all $H \le G$, and the conjugation maps are determined by those of X. The twin functor TX of X introduced by Thévenaz [35] is just the Mackey functor $\overline{X}^+$ (cf. [9]). Under the assumption that $|G|$ is invertible in k, Thévenaz [35] has given an induction formula for X based on a result of Puig [32], which is deduced from the inverse of an isomorphism $\beta : X \to TX$ of Mackey functors defined to be the family of k-module isomorphisms $\beta_H : X(H) \to TX(H)$, $x \mapsto (\mathrm{res}^H_K(x))_{K\le H}$ for $H \le G$ (cf. Remark 5.7). In this context, we emphasize that

$$\mathbb{Q}\otimes_{\mathbb{Z}} R(D^\omega(G)) \cong \prod_{H\in \mathrm{Cl}(G,Cyc)} \mathbb{Q}\otimes_{\mathbb{Z}} \left(\prod_{s\in C_G(H)} \overline{R(\mathbb{C}^{\theta_s}H)}\right)^{N_G(H)} \tag{I}$$

as $\mathbb{Q}$-spaces, where $\mathrm{Cl}(G,Cyc)$ is a full set of nonconjugate cyclic subgroups of G and the action of $N_G(H)$ is defined by the conjugation maps of the $D^\omega(G)$-representation functor (cf. Corollary 8.8). If ω is trivial, then this is a $\mathbb{Q}$-algebra isomorphism (cf. [37]). Using idempotent formulae for the crossed Burnside ring, Oda [28] has shown Artin's induction theorem on representations of $D(G)$. Regarding such a result, we present a canonical choice of Artin's induction theorem on representations of $D^\omega(G)$ (cf. Corollary 8.9), which is concerned with (I) and is described by using a canonical induction formula of a minimal type due to Boltje [4].

In Section 2, we recall the lower and upper plus constructions, together with the mark morphism and the induction morphism, from [4]. Section 3 contains the study of the Burnside ring functor and the crossed Burnside ring functor associated to a finite G-monoid S, which are Green functors obtained by the lower plus construction. In Section 4, we introduce the notion of a crossed Mackey functor on a Mackey bundle composed of $X_s \in \mathbf{Mack}(G_s)_k$, $s \in S$, which generalizes the Dress construction for Mackey functors associated to S or the crossing by S. The Green functor obtained by the ordinary Dress construction from the Burnside ring functor is isomorphic to the crossed Burnside ring functor. This fact is worth examining in our research, because we see that the isomorphism is deduced from a certain induction morphism. In Section 5, we recall a fundamental fact for canonical induction formulae from [4], and explain Thévenaz's results on the twin functor of a Mackey functor. Section 6 is devoted to some results for the crossed Mackey functors.

In Section 7, we turn to the study of the $\mathbb{C}^\alpha G$-representation functor R_α, and then provide two lemmas about finitely generated $\mathbb{C}^\alpha G$-modules, which are essential to a canonical choice of Brauer's induction theorem on representations of $\mathbb{C}^\alpha G$. Section 8 is devoted to representation theory of $D^\omega(G)$. We show that the $D^\omega(G)$-representation functor RD^ω_G is isomorphic to the crossed Mackey functor on the Mackey bundle composed of $\mathbb{C}^{\theta_s}G_s$-representation functors R_{θ_s}, $s \in G$, and then show that the Green functor RD_G is isomorphic to the Green functor obtained by the ordinary Dress construction from the $\mathbb{C}G$-representation functor associated to the G-monoid G on which G acts by conjugation. Some important consequences of such results are also given, including (I) and a canonical choice of Artin's induction theorem on representations of $D^\omega(G)$. Section 9 contains two fundamental theorems for the plus constructions, which are generalizations of fundamental theorems for the Burnside ring of a finite group. In Section 10, we give an alternative proof of Boltje's integrality result for canonical induction formulae, and show an integrality condition for the crossed Mackey functors, too. Section 11 describes a canonical choice of Brauer's induction theorem on representations of $\mathbb{C}^{\alpha}G$. In Section 12, we study canonical induction formulae for $RD^{\omega}(G)$, and present a canonical choice of Brauer's induction theorem on representations of $D^{\omega}(G)$.

Notation Throughout the paper, let G be a finite group, k a commutative ring with unity, $\mathbb{Z}$ the rational integers, $\mathbb{Q}$ the rational numbers, and $\mathbb{C}$ the complex numbers. We denote by ϵ the identity of G. The subgroup generated by an element g of G is denoted by $\langle g \rangle$. We write $K \leq H$ if H and K are subgroups of G with $K \subseteq H$. Let $H \leq G$. Given $K \leq H$, we write $K < H$ if $K \neq H$, and write $K \trianglelefteq H$ if K is a normal subgroup of H. The Möbius function of the poset $(\mathfrak{S}(H), \leq)$ of all subgroups of H is denoted by μ (see, *e.g.*, [1]). We set ${}^{r}H = rHr^{-1}$ and ${}^{r}g = rgr^{-1}$ for all g, $r \in G$, and denote by $\mathrm{Cl}(H)$ a full set of nonconjugate subgroups of H. For each $K \leq H$, $N_H(K)$ denotes the normalizer of K in H, and $C_H(K)$ denotes the centralizer of K in H. Given $K \leq H$, we denote by H/K the set of left cosets hK, $h \in H$, of K in H. For each pair (K, U) of subgroups K and U of H, $K\backslash H/U$ denotes the set of (K, U)-double cosets KhU, $h \in H$, in H. We denote by G-**set** the category of finite left G-sets and G-maps. Let $S \in G$-**set**. Given $g \in G$ and $s \in S$, ${}^{g}s$ denotes the effect of g on s. We view S as an H-set via the restriction of operations from G to H, and denote by $C_S(H)$ the set of all elements s of S such that ${}^{h}s = s$ for all $h \in H$. For each $s \in S$, H_s denotes the stabilizer of s in H. We set $\mathrm{Stab}(G; S) = \{G_s \mid s \in S\}$. A semigroup with identity is called a monoid. A monoid on which G acts as monoid homomorphisms is called a G-monoid. We denote by G-**mon** the category of finite G-monoids and G-maps. For an object M of a category, $[M]$ denotes the isomorphism class containing M. Given a ring R, we denote by R-**mod** the category of finitely generated left R-modules, and set $R\text{-}\overline{\mathbf{mod}} = \{[M] \mid M \in R\text{-}\mathbf{mod}\}$. The identity map on a set Σ is denoted by id_{Σ}. We denote by $\Lambda(G)$ the set of all primes dividing $|G|$, and denote by Λ the set consisting of all primes and the symbol ∞. Let p be a prime. For each $\mathbb{Z}$-module M, we set $M_{(p)} = \mathbb{Z}_{(p)} \otimes_{\mathbb{Z}} M$, where $\mathbb{Z}_{(p)}$ is the localization of $\mathbb{Z}$ at p, and set $M_{(\infty)} = M$. The expression ' ∞-group' means only 'group'. We denote by $O^p(H)$ the smallest normal subgroup of H such that $H/O^p(H)$ is a p-group, and set $O^{\infty}(H) = \{\epsilon\}$. For each natural number n, n_p denotes the p-part of n, and n_{∞} denotes n.

2 The plus constructions

We start with the following definition which is given in [4] (see also [19, 35, 40]).

Definition 2.1 (a) A conjugation functor for G over k is a couple $A = (A, \mathrm{con})$ consisting of a family of k-modules $A(H)$, $H \leq G$, and a family of k-module homomorphisms

$$\mathrm{con}_H^g : A(H) \to A({}^{g}H),$$

the conjugation maps, for $H \leq G$ and $g \in G$, satisfying the axioms

$$\text{(G.1)}\ \ \mathrm{con}^g_{{}^rH} \circ \mathrm{con}^r_H = \mathrm{con}^{gr}_H, \quad \mathrm{con}^h_H = \mathrm{id}_{A(H)}$$

for all $H \leq G$, g, $r \in G$, and $h \in H$. An algebra conjugation functor for G over k is a conjugation functor $A = (A, \mathrm{con})$ for G over k such that $A(H)$, $H \leq G$, are k-algebras and the conjugation maps are k-algebra homomorphisms.

(b) A restriction functor for G over k is a triple $A = (A, \mathrm{con}, \mathrm{res})$ consisting of a conjugation functor (A, con) for G over k and a family of k-module homomorphisms
$$\mathrm{res}^H_K : A(H) \to A(K),$$
the restriction maps, for $K \leq H \leq G$, satisfying the axioms

$$\begin{aligned} &\text{(G.2)}\ \ \mathrm{res}^K_L \circ \mathrm{res}^H_K = \mathrm{res}^H_L, \quad \mathrm{res}^H_H = \mathrm{id}_{A(H)}, \\ &\text{(G.3)}\ \ \mathrm{con}^g_K \circ \mathrm{res}^H_K = \mathrm{res}^{\,{}^gH}_{\,{}^gK} \circ \mathrm{con}^g_H \end{aligned}$$

for all $L \leq K \leq H \leq G$ and $g \in G$. An algebra restriction functor for G over k is a restriction functor $A = (A, \mathrm{con}, \mathrm{res})$ for G over k such that (A, con) is an algebra conjugation functor and the restriction maps are k-algebra homomorphisms.

(c) A Mackey functor for G over k is a quadruple $A = (A, \mathrm{con}, \mathrm{res}, \mathrm{ind})$ consisting of a restriction functor $(A, \mathrm{con}, \mathrm{res})$ for G over k and a family of k-module homomorphisms
$$\mathrm{ind}^H_K : A(K) \to A(H),$$
the induction maps, for $K \leq H \leq G$, satisfying the axioms

$$\begin{aligned} &\text{(G.4)}\ \ \mathrm{ind}^H_K \circ \mathrm{ind}^K_L = \mathrm{ind}^H_L, \quad \mathrm{ind}^H_H = \mathrm{id}_{A(H)}, \\ &\text{(G.5)}\ \ \mathrm{con}^g_H \circ \mathrm{ind}^H_K = \mathrm{ind}^{\,{}^gH}_{\,{}^gK} \circ \mathrm{con}^g_K, \\ &\text{(G.6)}\ \ \text{(Mackey axiom)} \\ &\qquad \mathrm{res}^H_K \circ \mathrm{ind}^H_U = \sum_{KhU \in K\backslash H/U} \mathrm{ind}^K_{K\cap {}^hU} \circ \mathrm{res}^{{}^hU}_{K \cap {}^hU} \circ \mathrm{con}^h_U \end{aligned}$$

for all $L \leq K \leq H \leq G$, $U \leq H$, and $g \in G$. A Green functor for G over k is a Mackey functor $A = (A, \mathrm{con}, \mathrm{res}, \mathrm{ind})$ for G over k such that $(A, \mathrm{con}, \mathrm{res})$ is an algebra restriction functor and

$$\begin{aligned} &\text{(G.7)}\ \ \text{(Frobenius axioms)} \\ &\qquad \sigma \cdot \mathrm{ind}^H_K(\tau) = \mathrm{ind}^H_K(\mathrm{res}^H_K(\sigma) \cdot \tau), \quad \mathrm{ind}^H_K(\tau) \cdot \sigma = \mathrm{ind}^H_K(\tau \cdot \mathrm{res}^H_K(\sigma)) \end{aligned}$$

for all $K \leq H$, $\sigma \in A(H)$, and $\tau \in A(K)$.

A morphism $f : X \to Y$ of Green functors for G over k is a family of k-algebra homomorphisms $f_H : X(H) \to Y(H)$, $H \leq G$, commuting with conjugation, restriction, and induction maps. A morphism of conjugation, algebra conjugation, restriction, algebra restriction, or Mackey functors for G over k is defined in similar fashion. For a morphism $f : X \to Y$ of Mackey functors for G over k, we require that $f_H : X(H) \to Y(H)$, $H \leq G$, are k-module homomorphisms. The others are defined by omitting unnecessary terminology. We now obtain the categories of conjugation, algebra conjugation, restriction, algebra restriction, Mackey, and Green functors for G over k, denoted by $\mathbf{Con}(G)_k$, $\mathbf{Con}_{\mathrm{alg}}(G)_k$, $\mathbf{Res}(G)_k$, $\mathbf{Res}_{\mathrm{alg}}(G)_k$, $\mathbf{Mack}(G)_k$, and $\mathbf{Green}(G)_k$, respectively. The sets of morphisms $f : X \to Y$ of conjugation, restriction, Mackey, and Green functors are denoted by $\mathbf{Con}(G)(X, Y)_k$, $\mathbf{Res}(G)(X, Y)_k$, $\mathbf{Mack}(G)(X, Y)_k$, and $\mathbf{Green}(G)(X, Y)_k$, respectively.

Following [4], we define plus constructions $-_+ : \mathbf{Res}(G)_k \to \mathbf{Mack}(G)_k$ and $-^+ : \mathbf{Con}(G)_k \to \mathbf{Mack}(G)_k$, and state some basic facts concerned with them.

Let $A \in \mathbf{Con}(G)_k$. For each $H \leq G$, set

$$M(H) = \prod_{U \leq H} A(U), \tag{II}$$

and view it as a left kH-module with the action given by

$$h.(x_U)_{U \leq H} = (\mathrm{con}^h_U(x_U))_{{}^hU \leq H}$$

for all $h \in H$ and $(x_U)_{U \leq H} \in M(H)$. We define

$$A^+ = (A^+, \mathrm{con}^+, \mathrm{res}^+, \mathrm{ind}^+) \in \mathbf{Mack}(G)_k$$

by

$$\begin{aligned}
&A^+(H) = \{(x_U)_{U \leq H} \in M(H) \mid h.(x_U)_{U \leq H} = (x_U)_{U \leq H} \text{ for all } h \in H\},\\
&{\mathrm{con}^+}^g_H((x_U)_{U \leq H}) = (\mathrm{con}^g_H(x_U))_{{}^gU \leq {}^gH},\\
&{\mathrm{res}^+}^H_K((x_U)_{U \leq H}) = (x_U)_{U \leq K},\\
&{\mathrm{ind}^+}^H_K((y_U)_{U \leq K}) = \sum_{hK \in H/K} (c^h_L)_{L \leq H}
\end{aligned}$$

for all $K \leq H \leq G$, $g \in G$, $(x_U)_{U \leq H} \in A^+(H)$, and $(y_U)_{U \leq K} \in A^+(K)$, where

$$c^h_L = \begin{cases} \mathrm{con}^h_U(y_U) & \text{if } L = {}^hU \text{ with } U \leq K,\\ 0 & \text{otherwise.} \end{cases}$$

In short, $A(H)^+$ is just the set of H-invariants on $M(H)$.

If A is an algebra conjugation functor, then $A^+(H)$, $H \leq G$, are k-algebras with obvious multiplication and A^+ is a Green functor.

Given $H \le G$, we define $I(M(H))$ to be the smallest kH-submodule of $M(H)$ such that H acts trivially on the factor module $M(H)/I(M(H))$, and denote by $\overline{(x_U)_{U\le H}}$ an element $(x_U)_{U\le H} + I(M(H))$ of $M(H)/I(M(H))$.

Suppose next that $A \in \mathbf{Res}(G)_k$. We define

$$A_+ = (A_+, \mathrm{con}_+, \mathrm{res}_+, \mathrm{ind}_+) \in \mathbf{Mack}(G)_k$$

by

$$\begin{aligned}
&A_+(H) = M(H)/I(M(H)),\\
&\mathrm{con}_{+H}^{\,g}(\overline{(x_U)_{U\le H}}) = \overline{(\mathrm{con}_H^g(x_U))_{{}^gU\le {}^gH}},\\
&\mathrm{res}_{+K}^{\,H}(\overline{(x_U)_{U\le H}}) = \sum_{U\le H}\ \sum_{KhU\in K\backslash H/U} \overline{(d_L^h)_{L\le K}},\\
&\mathrm{ind}_{+K}^{\,H}(\overline{(y_U)_{U\le K}}) = \overline{(y'_U)_{U\le H}}
\end{aligned}$$

for all $K \le H \le G$, $g \in G$, $(x_U)_{U\le H} \in M(H)$, and $(y_U)_{U\le K} \in M(K)$, where

$$d_L^h = \begin{cases} \mathrm{res}^{{}^hU}_{K\cap {}^hU} \circ \mathrm{con}_U^h(x_U) & \text{if } L = K \cap {}^hU,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$y'_U = \begin{cases} y_U & \text{if } U \le K,\\ 0 & \text{otherwise.}\end{cases}$$

In short, $A(H)_+$ is just the set of H-coinvariants on $M(H)$.

Given $K \le H \le G$ and $\sigma \in A(K)$, we set

$$[K, \sigma] = \overline{(\delta_{KU}\sigma)_{U\le H}} \in A_+(H),$$

where $\delta_{KU}\sigma = 0$ if $K \ne U$ and $\delta_{KK}\sigma = \sigma$.

If A is an algebra restriction functor, then multiplication on $A_+(H)$ with $H \le G$ is defined by

$$[K,\sigma]\cdot[U,\tau] = \sum_{KhU\in K\backslash H/U} [K\cap {}^hU, \mathrm{res}^K_{K\cap {}^hU}(\sigma)\cdot \mathrm{res}^{{}^hU}_{K\cap {}^hU}\circ \mathrm{con}_U^h(\tau)],$$

extended to $A_+(H)$ by k-linearly. This k-algebra structure of $A_+(H)$ forces A_+ to be a Green functor.

Let $H \le G$. The mark homomorphism $\rho_H^A : A_+(H) \to A^+(H)$ is defined by

$$\rho_H^A(\overline{(x_U)_{U\le H}}) = \sum_{U\le H}\left(\sum_{hU\in H/U,\, K\le {}^hU} \mathrm{res}_K^{{}^hU}\circ \mathrm{con}_U^h(x_U)\right)_{K\le H}$$

for all $(x_U)_{U\le H} \in M(H)$, where the sum $\sum_{hU\in H/U,\, K\le {}^hU}$ is taken over all cosets hU, $h\in H$, of U in H such that $K \le {}^hU$. We define a morphism $\rho^A : A_+ \to A^+$ of Mackey functors to be the family of mark homomorphisms ρ^A_H, $H\le G$, and call it the mark morphism. If A is an algebra restriction functor, then ρ^A is a morphism of Green functors. We define a map $\eta^A_H : A^+(H) \to A_+(H)$ by

$$\eta^A_H((y_K)_{K\le H}) = \sum_{K\le H}\sum_{U\le K} |U|\mu(U,K)[U, \mathrm{res}^K_U(y_K)]$$

for all $(y_K)_{K\le H} \in A^+(H)$.

The following proposition is [4, Proposition 2.4].

Proposition 2.2 *Let $A \in \mathbf{Res}(G)_k$. For each $H\le G$,*

$$\eta^A_H \circ \rho^A_H = |H|\mathrm{id}_{A_+(H)} \quad \textit{and} \quad \rho^A_H \circ \eta^A_H = |H|\mathrm{id}_{A^+(H)}.$$

A stable k-basis $\mathcal{B}$ of A is defined to be a family of k-bases $\mathcal{B}(H)$ of $A(H)$, $H\le G$, such that

$$\mathcal{B}({}^gH) = \{\mathrm{con}^g_H(\sigma) \mid \sigma \in \mathcal{B}(H)\}$$

for all $H\le G$ and $g\in G$ (see [4, Definition 7.1]). Suppose that $\mathcal{B}$ is a stable k-basis of A. Let $H\le G$, and set

$$\mathfrak{S}(H,\mathcal{B}) = \{(K,\sigma) \mid K\le H \quad \text{and} \quad \sigma\in\mathcal{B}(K)\}.$$

Then $\mathfrak{S}(H,\mathcal{B})$ is a left H-set with the action given by

$$h.(K,\sigma) = ({}^hK, \mathrm{con}^h_K(\sigma))$$

for all $h\in H$ and $(K,\sigma)\in\mathfrak{S}(H,\mathcal{B})$. We denote by $\mathfrak{R}(H,\mathcal{B})$ a complete set of representatives of H-orbits in $\mathfrak{S}(H,\mathcal{B})$ such that $K\in \mathrm{Cl}(H)$ for all $(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})$.

The following lemma is the second statement of [4, Lemma 7.2].

Lemma 2.3 *Let $A\in\mathbf{Res}(G)_k$, and let $\mathcal{B}$ be a stable k-basis of A. For each $H\le G$, the elements $[K,\sigma]$ for $(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})$ form a k-basis of $A_+(H)$.*

Suppose that $X = (X, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Mack}(G)_k$. Let A be a restriction subfunctor of X, that is, each $A(H)$ with $H\le G$ is a submodule of the k-module $X(H)$, and the conjugation and restriction maps of A are the restriction of con^g_H and res^H_K for $K\le H\le G$ and $g\in G$. We define $\Theta^{X,A} : A_+ \to X$ to be a family of k-module homomorphisms $\Theta^{X,A}_H : A_+(H)\to X(H)$, $H\le G$, such that

$$\Theta^{X,A}_H([K,\sigma]) = \mathrm{ind}^H_K(\sigma)$$

for all $[K,\sigma]\in A_+(H)$, and call it the induction morphism (cf. [4, 3.1]).

The next lemma is due to Boltje [4].

Lemma 2.4 *Let $X \in \mathbf{Mack}(G)_k$, and let A be a restriction subfunctor of X. Then $\Theta^{X,A} \in \mathbf{Mack}(G)(A_+, X)_k$. If X is a Green functor and if each $A(H)$ with $H \leq G$ is a subalgebra of the k-algebra $X(H)$, then $\Theta^{X,A} \in \mathbf{Green}(G)(A_+, X)_k$.*

Proof. Obviously, $\Theta^{X,A} \in \mathbf{Con}(G)(A_+, X)_k$. By the Mackey axiom,

$$\begin{aligned}\Theta^{X,A}_K \circ {\mathrm{res}_+}^H_K([U,\tau]) &= \sum_{KhU \in K\backslash H/U} \mathrm{ind}^K_{K\cap {}^hU} \circ \mathrm{res}^{{}^hU}_{K\cap {}^hU} \circ \mathrm{con}^h_U(\tau) \\ &= \mathrm{res}^H_K \circ \mathrm{ind}^H_U(\tau)\end{aligned}$$

for all $K \leq H \leq G$ and $[U, \tau] \in A_+(H)$. Moreover,

$$\Theta^{X,A}_H \circ {\mathrm{ind}^+}^H_K([U,\tau]) = \mathrm{ind}^H_U(\tau) = \mathrm{ind}^H_K \circ \Theta^{X,A}_K([U,\tau])$$

for all $K \leq H \leq G$ and $[U, \tau] \in A_+(K)$. Thus $\Theta^{X,A} \in \mathbf{Mack}(G)(A_+, X)_k$. Suppose that X is a Green functor and each $A(H)$ with $H \leq G$ is a subalgebra of the k-algebra $X(H)$. Using the Mackey and Frobenius axioms, we have

$$\begin{aligned}\mathrm{ind}^H_K(\sigma) \cdot \mathrm{ind}^H_U(\tau) &= \mathrm{ind}^H_K(\sigma \cdot \mathrm{res}^H_K \circ \mathrm{ind}^H_U(\tau)) \\ &= \mathrm{ind}^H_K\left(\sigma \cdot \sum_{KhU \in K\backslash H/U} \mathrm{ind}^K_{K\cap {}^hU} \circ \mathrm{res}^{{}^hU}_{K\cap {}^hU} \circ \mathrm{con}^h_U(\tau)\right) \\ &= \sum_{KhU \in K\backslash H/U} \mathrm{ind}^H_{K\cap {}^hU}(\mathrm{res}^K_{K\cap {}^hU}(\sigma) \cdot \mathrm{res}^{{}^hU}_{K\cap {}^hU} \circ \mathrm{con}^h_U(\tau))\end{aligned}$$

for all $K \leq H \leq G$, $U \leq H$, $\sigma \in X(K)$, and $\tau \in X(U)$ (cf. [19, Proposition 1.84], [35, Proposition 1.10]). Hence the k-module homomorphisms $\Theta^{X,A}_H$ for $H \leq G$ are k-algebra homomorphisms, and thereby, $\Theta^{X,A} \in \mathbf{Green}(G)(A_+, X)_k$. This completes the proof. □

3 The Burnside ring functor

We explore the lower plus construction from an algebra restriction functor for G over k in terms of H-sets with $H \leq G$ (see also [27, Section 3]).

Suppose that $A \in \mathbf{Res}_{\mathrm{alg}}(G)_k$. Let $H \leq G$, and view the left kH-module $M(H)$ (see (II)) as an H-monoid with obvious multiplication. Given $K \leq H$, we regard $A(K)$ as a k-submodule of $M(H)$ via the obvious embedding $A(K) \hookrightarrow M(H)$. Given J, $J' \in H$-**set**, we denote by $\mathrm{Map}_H(J, J')$ the set of H-maps from J to J'. There exists a contravariant functor $T = T^A_H : H$-**set** $\to$ **Mon**, where **Mon** is the category of monoids, such that $T(J)$ with $J \in H$-**set** is defined to be the monoid

$$\{\pi \in \mathrm{Map}_H(J, M(H)) \mid \pi(x) \in A(H_x) \text{ for all } x \in J\}$$

with pointwise multiplication, where H_x is the stabilizer of x, and the morphism $T(f) : T(J) \to T(J')$ with J, $J' \in H$-**set** and $f \in \mathrm{Map}_H(J', J)$ is defined by

$$T(f)(\pi) : J' \to M(H), \quad x \mapsto \mathrm{res}^{H_{f(x)}}_{H_x}(\pi(f(x)))$$

for all $\pi \in T(J)$. This functor is additive, that is, for any J_1, $J_2 \in H$-**set** with inclusions $\iota_i : J_i \to J_1 \dot\cup J_2$, the induced map

$$T(\iota_1) \times T(\iota_2) : T(J_1 \dot\cup J_2) \to T(J_1) \times T(J_2)$$

is an isomorphism (cf. [21, Section 2]). Following [21], we set

$$\pi_1 \dot+ \pi_2 = (T(\iota_1) \times T(\iota_2))^{-1}(\pi_1, \pi_2)$$

for all $(\pi_1, \pi_2) \in T(J_1) \times T(J_2)$. A pair (J, π) with $J \in H$-**set** and $\pi \in T(J)$ is called an element of T. A morphism $f : (J', \pi') \to (J, \pi)$ of elements of T is defined to be an H-map $f : J' \to J$ such that $T(f)(\pi) = \pi'$. We now obtain the category $\mathbf{El}(H\text{-}\mathbf{set}, T)$ of elements of T (cf. [29, (2.10)]).

The Burnside ring $\Omega(H)$ is the commutative ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of finite left H-sets with disjoint union for addition and cartesian product for multiplication (see, *e.g.*, [11, §80]). We give a generalization of $\Omega(H)$ associated with $\mathbf{El}(H\text{-}\mathbf{set}, T)$.

For each $(J, \pi) \in \mathbf{El}(H\text{-}\mathbf{set}, T)$, we denote by $\overline{(J, \pi)}$ the isomorphism class of elements of T containing (J, π). Let $\mathbf{F}(H, T)$ be the free abelian group on the isomorphism classes of elements of T, and let $\mathbf{F}(H, T)_0$ be the subgroup of $\mathbf{F}(H, T)$ generated by all expressions $\overline{(J_1 \dot\cup J_2, \pi_1 \dot+ \pi_2)} - \overline{(J_1, \pi_1)} - \overline{(J_2, \pi_2)}$. Multiplication on $\mathbf{F}(H, T)$ is defined by

$$\overline{(J_1, \pi_1)} \cdot \overline{(J_2, \pi_2)} = \overline{(J_1 \times J_2, T(\mathrm{Pr}_1)(\pi_1) \cdot T(\mathrm{Pr}_2)(\pi_2))},$$

extended to $\mathbf{F}(H, T)$ by $\mathbb{Z}$-linearly, where $\mathrm{Pr}_i : J_1 \times J_2 \to J_i$ are projections. Then $\mathbf{F}(H, T)$ is a ring, and $\mathbf{F}(H, T)_0$ is a two sided ideal of $\mathbf{F}(H, T)$. We now define $\Omega(H, T)$ to be the quotient $\mathbf{F}(H, T)/\mathbf{F}(H, T)_0$. This ring is the F-Burnside ring with $F = T$ introduced by Jacobson [21] (see also [27]). For each $(J, \pi) \in \mathbf{El}(H\text{-}\mathbf{set}, T)$, an element $\overline{(J, \pi)} + \mathbf{F}(H, T)_0$ of $\Omega(H, T)$ is denoted by $[J, \pi]_0$. By an argument analogous to the proof of [11, Lemma 80.4], we can show that $[J_1, \pi_1]_0 = [J_2, \pi_2]_0$ if and only if $\overline{(J_1, \pi_1)} = \overline{(J_2, \pi_2)}$. By definition, addition and multiplication of two elements $[J_1, \pi_1]_0$ and $[J_2, \pi_2]_0$ of $\Omega(H, T)$ are given by

$$[J_1, \pi_1]_0 + [J_2, \pi_2]_0 = [J_1 \dot\cup J_2, \pi_1 \dot+ \pi_2]_0 \text{ and } [J_1, \pi_1]_0 \cdot [J_2, \pi_2]_0 = [J_1 \times J_2, \pi_1 \cdot \pi_2]_0$$

with

$$\pi_1 \dot+ \pi_2 : J_1 \dot\cup J_2 \to M(H), \quad x \mapsto \pi_1(x) \text{ if } x \in J_1, \quad x \mapsto \pi_2(x) \text{ if } x \in J_2$$

and

$$\pi_1 \cdot \pi_2 : J_1 \times J_2 \to M(H),$$
$$(x_1, x_2) \mapsto \mathrm{res}^{H_{x_1}}_{H_{x_1} \cap H_{x_2}}(\pi_1(x_1)) \cdot \mathrm{res}^{H_{x_2}}_{H_{x_1} \cap H_{x_2}}(\pi_2(x_2)).$$

Given $K \leq H$ and $\sigma \in A(K)$, define an H-map $\pi_\sigma : H/K \to M(H)$ by

$$\pi_\sigma(hK) = h.\sigma$$

for all $h \in H$. Then $\Omega(H, T)$ is the ring consisting of all $\mathbb{Z}$-linear combinations of the elements $[H/K, \pi_\sigma]_0$ for $K \leq H$ and $\sigma \in A(K)$. Moreover, $k \otimes_{\mathbb{Z}} \Omega(H, T)$ is the ring consisting of all k-linear combinations of the elements $1 \otimes [H/K, \pi_\sigma]_0$ for $K \leq H$ and $\sigma \in A(K)$ such that the $\mathbb{Z}$-module homomorphism

$$\Omega(H, T) \to k \otimes_{\mathbb{Z}} \Omega(H, T), \quad [H/K, \pi_\sigma]_0 \mapsto 1 \otimes [H/K, \pi_\sigma]_0$$

is a ring homomorphism. Suppose that $\pi \in T(H/K)$ and $\pi' \in T(H/U)$, where $K, U \leq H$. Then $[H/K, \pi]_0 = [H/U, \pi']_0 \in \Omega(H, T)$ if and only if there exists an element r of H such that $K = {}^rU$ and π' is the H-map

$$T(f^r_U)(\pi) : H/U \to M(H), \quad hU \mapsto \pi(hr^{-1}K),$$

where f^r_U is an H-map from H/U to H/K defined by $f^r_U(hU) = hr^{-1}K$ for all $h \in H$. From this, we know that $[H/K, \pi]_0 = [H/U, \pi']_0 \in \Omega(H, T)$ if and only if $[K, \pi(K)] = [U, \pi'(U)] \in A_+(H)$. Hence there exists a k-module epimorphism $\Upsilon = \Upsilon^A_H : k \otimes_{\mathbb{Z}} \Omega(H, T) \to A_+(H)$ given by

$$\Upsilon(1 \otimes [H/K, \pi]_0) = [K, \pi(K)]$$

for all $K \leq H$ and $\pi \in T(H/K)$. Let υ^A_H be the k-module isomorphism from $(k \otimes_{\mathbb{Z}} \Omega(H, T))/\mathrm{Ker}\,\Upsilon$ to $A_+(H)$ determined by Υ. We denote by $[H/K, \pi]$ the element $1 \otimes [H/K, \pi]_0 + \mathrm{Ker}\,\Upsilon$ of the factor module $(k \otimes_{\mathbb{Z}} \Omega(H, T))/\mathrm{Ker}\,\Upsilon$.

Let $K \leq H$. For each $J \in H$-**set**, we denote by $\mathrm{res}^H_K(J)$ the restriction of J to K. Suppose that $V \in K$-**set**. We consider the cartesian product $H \times V$ to be a left K-set with the action given by

$$r(h, x) = (hr^{-1}, rx)$$

for all $r \in K$ and $(h, x) \in H \times V$. Given $(h, x) \in H \times V$, let $h \otimes x$ denote the K-orbit containing (h, x). We denote by $\mathrm{ind}^H_K(V)$ the set of K-orbits in $H \times V$, and view it as a left H-set with the action given by

$$h(h' \otimes x) = hh' \otimes x$$

for all $h \in H$ and $(h', x) \in H \times V$. This H-set is called an induced H-set (cf. [11, §80]). We define $\mathrm{con}^h_K(V) \in {}^hK$-**set** with $h \in H$ to be the subset

$$h \otimes V := \{h \otimes x \mid x \in V\}$$

of $\mathrm{ind}_K^H(V)$ with the action given by

$$^{h}r(h \otimes x) = h \otimes rx$$

for all $r \in K$ and $x \in V$. This ^hK-set is called a conjugate ^hK-set.

We now define

$$\Omega^A = (\Omega^A, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Green}(G)_k$$

by

$$\begin{aligned}
&\Omega^A(H) = (k \otimes_{\mathbb{Z}} \Omega(H, T_H^A))/\mathrm{Ker}\,\Upsilon_H^A,\\
&\mathrm{con}_H^g([J,\pi]) = [\mathrm{con}_H^g(J),\, {}^g\pi],\\
&\mathrm{res}_K^H([J,\pi]) = [\mathrm{res}_K^H(J), \pi|_K],\\
&\mathrm{ind}_K^H([V,\varpi]) = [\mathrm{ind}_K^H(V), \varpi^H]
\end{aligned}$$

for all $K \le H \le G$, $g \in G$, $(J,\pi) \in \mathbf{El}(H\text{-}\mathbf{set}, T_H^A)$, and $(V,\varpi) \in \mathbf{El}(K\text{-}\mathbf{set}, T_K^A)$, where ${}^g\pi$, $\pi|_K$, and ϖ^H are defined by

$$({}^g\pi)(g \otimes x) = \mathrm{con}_{H_x}^g(\pi(x)), \quad \pi|_K(x) = \mathrm{res}_{K_x}^{H_x}(\pi(x)), \quad \varpi^H(h \otimes y) = \mathrm{con}_{K_y}^h(\varpi(y))$$

for all $x \in J$, $y \in V$, and $h \in H$. This Green functor is a G-functor version of the F-Burnside ring functor with $F = T_G^A$ defined in [21, 27].

Proposition 3.1 *Let $A \in \mathbf{Res}_{\mathrm{alg}}(G)_k$. Then the Green functor Ω^A is isomorphic to A_+. Really, the family of k-algebra isomorphisms $\upsilon_H^A : \Omega^A(H) \to A_+(H)$, $H \le G$, defines an isomorphism $\upsilon^A : \Omega^A \to A_+$ of Green functors.*

Proof. Let $K \le H \le G$, and let $g \in G$. Obviously, the diagrams

$$\begin{array}{ccc}
\Omega^A(H) & \xrightarrow{\upsilon_H^A} & A_+(H)\\
{\scriptstyle \mathrm{con}_H^g}\downarrow & & \downarrow{\scriptstyle \mathrm{con}_{+H}^{\,g}}\\
\Omega^A({}^gH) & \xrightarrow[\upsilon_{{}^gH}^A]{} & A_+({}^gH)
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
\Omega^A(H) & \xrightarrow{\upsilon_H^A} & A_+(H)\\
{\scriptstyle \mathrm{ind}_K^H}\uparrow & & \uparrow{\scriptstyle \mathrm{ind}_{+K}^{\,H}}\\
\Omega^A(K) & \xrightarrow[\upsilon_K^A]{} & A_+(K)
\end{array}$$

are commutative, because

$$\mathrm{con}_H^g([H/U,\pi]) = [{}^gH/{}^gU, {}^g\pi] \quad \text{and} \quad \mathrm{ind}_K^H([K/L,\varpi]) = [H/L, \varpi^H]$$

for all $U \le H$, $\pi \in T_H^A(H/U)$, $L \le K$, and $\varpi \in T_K^A(K/L)$. Let $U \le H$, and let $\pi \in T_H^A(H/U)$. For each $h \in \overline{K\backslash H/U}$, where $\overline{K\backslash H/U}$ is a complete set of representatives of $K\backslash H/U$, we define $\pi|_{(K,h)} \in T_K^A(K/K \cap {}^hU)$ by

$$\pi|_{(K,h)}(r(K \cap {}^hU)) = \mathrm{res}_{K \cap {}^{rh}U}^{{}^{rh}U}(\pi(rhU))$$

for all $r \in K$. The map

$$\mathrm{res}^H_K(H/U) \to \dot{\bigcup_{h\in\overline{K\backslash H/U}}} K/K\cap {}^hU, \quad h'U \mapsto r(K\cap {}^hU)$$

is an isomorphism of K-sets, where $h'U = rhU$ with $r \in K$ and $h \in \overline{K\backslash H/U}$. Hence

$$\mathrm{res}^H_K([H/U,\pi]) = \sum_{h\in\overline{K\backslash H/U}} [K/K\cap {}^hU, \pi|_{(K,h)}].$$

Since $\pi|_{(K,h)}(K\cap {}^hU) = \mathrm{res}^{{}^hU}_{K\cap {}^hU} \circ \mathrm{con}^h_U(\pi(U))$ for all $U \le H$ and $h \in \overline{K\backslash H/U}$, it turns out that the diagram

$$\begin{array}{ccc}
\Omega^A(H) & \xrightarrow{\upsilon^A_H} & A_+(H) \\
{\scriptstyle \mathrm{res}^H_K}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}_{+}{}^H_K} \\
\Omega^A(K) & \xrightarrow[\upsilon^A_K]{} & A_+(K)
\end{array}$$

is commutative. Thus it suffices to verify that υ^A_H is a ring homomorphism. We know that the map

$$(H/K)\times(H/U) \to \dot{\bigcup_{h\in\overline{K\backslash H/U}}} H/K\cap {}^hU, \quad (h_1K, h_2U) \mapsto h_1r(K\cap {}^hU)$$

is an isomorphism of H-sets, where $h_1^{-1}h_2U = rhU$ with $r \in K$ and $h \in \overline{K\backslash H/U}$. Suppose that $\pi_1 \in T^A_H(H/K)$ and $\pi_2 \in T^A_H(H/U)$. For each $h \in \overline{K\backslash H/U}$, we define $\pi_3 \in T^A_H(H/K\cap {}^hU)$ by

$$\pi_3(r(K\cap {}^hU)) = \mathrm{res}^{{}^rK}_{{}^rK\cap {}^{rh}U}(\pi_1(rK)) \cdot \mathrm{res}^{{}^{rh}U}_{{}^rK\cap {}^{rh}U}(\pi_2(rhU))$$

for all $r \in H$. Observe that

$$[H/K,\pi_1]_0 \cdot [H/U,\pi_2]_0 = \sum_{h\in\overline{K\backslash H/U}} [H/K\cap {}^hU, \pi_3]_0.$$

Then we have

$$\Upsilon^A_H([H/K,\pi_1]_0 \cdot [H/U,\pi_2]_0) = \Upsilon^A_H([H/K,\pi_1]_0)\cdot \Upsilon^A_H([H/U,\pi_2]_0).$$

Consequently, υ^A_H is a ring homomorphism. Hence we conclude that υ^A is an isomorphism of Green functors. This completes the proof. □

Let $S \in G$-**mon**, and set $C_S(H) = \{s \in S \mid {}^h s = s \text{ for all } h \in H\}$, where ${}^h s$ denotes the effect of h on s. We define

$$\underline{k}_{\otimes S} = (\underline{k}_{\otimes S}, \mathrm{con}_{\otimes S}, \mathrm{res}_{\otimes S}) \in \mathbf{Res}_{\mathrm{alg}}(G)_k$$

by

$$\underline{k}_{\otimes S}(H) = kC_S(H), \quad \mathrm{con}_{\otimes S}{}^g_H(s) = {}^g s, \quad \mathrm{res}_{\otimes S}{}^H_K(s) = s$$

for all $K \le H \le G$, $s \in C_S(H)$, and $g \in G$, where $kC_S(H)$ is the monoid ring. For each $H \le G$, the k-module $A_{\otimes S}(H) := A(H) \otimes_k kC_S(H)$ has an obvious k-algebra structure. The family of k-algebras $A_{\otimes S}(H)$, $H \le G$, together with the k-algebra homomorphisms

$$\begin{aligned} \mathrm{con}_{\otimes S}{}^g_H &: A_{\otimes S}(H) \to A_{\otimes S}({}^g H), \qquad && x \otimes s \mapsto \mathrm{con}^g_H(x) \otimes {}^g s, \\ \mathrm{res}_{\otimes S}{}^H_K &: A_{\otimes S}(H) \to A_{\otimes S}(K), && x \otimes s \mapsto \mathrm{res}^H_K(x) \otimes s \end{aligned}$$

for $K \le H$ and $g \in G$, defines an algebra restriction functor for G over k, which is a generalization of $\underline{k}_{\otimes S}$, and is denoted by $A_{\otimes S} = (A_{\otimes S}, \mathrm{con}_{\otimes S}, \mathrm{res}_{\otimes S})$.

Set $\mathrm{C}\Omega(-, S) = \Omega^{\underline{\mathbb{Z}}_{\otimes S}}$ and $\Omega_k = (\underline{k}_{\otimes\{\epsilon\}})_+$, where $\{\epsilon\}$ denotes the G-monoid consisting of only the identity ϵ. We consider the Green functor $\mathrm{C}\Omega(-, \{\epsilon\})$ as the Burnside ring functor Ω (cf. [35, Section 6], [40, Example 2.11]). For each $H \le G$, the element $[H/K, \epsilon] \in \mathrm{C}\Omega(H, \{\epsilon\})$ is denoted by $[H/K]$. The ring $\mathrm{C}\Omega(H, S)$ with $H \le G$ is the crossed Burnside ring defined by Oda and Yoshida [29] (see also [6]), and the Green functor $\mathrm{C}\Omega(-, S)$ is the crossed Burnside ring functor defined by Oda and Yoshida [30].

By Proposition 3.1, the family of $\mathbb{Z}$-lattice isomorphisms

$$\Omega_{\mathbb{Z}}(H) \xrightarrow{\sim} \Omega(H), \quad [K, \epsilon] \mapsto [H/K],$$

where $H \le G$, defines an isomorphism between Green functors $\Omega_{\mathbb{Z}}$ and Ω, which induces an isomorphism between Green functors Ω_k and $k \otimes \Omega$ (cf. [4, Section 2]). We identify Ω_k with $k \otimes \Omega$, and regard $[K, \epsilon] \in \Omega_k(H)$ as $[H/K] := 1 \otimes [H/K] \in k \otimes_{\mathbb{Z}} \Omega(H)$ for all $K \le H \le G$. If $|G|$ is invertible in k, then it follows from Proposition 2.2 that for any $K \le H \le G$,

$$e_K^{(H)} := \frac{1}{|H|} \eta_H^{\underline{k}_{\otimes\{\epsilon\}}}((x_K(L))_{L \le H}) = \frac{1}{|N_H(K)|} \sum_{U \le K} |U| \mu(U, K)[H/U],$$

where $x_K(L) = \epsilon$ if $L = {}^h K$ for some $h \in H$, and $x_K(L) = 0$ otherwise, is an idempotent of $\Omega_k(H)$ (cf. [4, Remark 2.5]).

Remark 3.2 The idempotents $e_K^{(H)}$, $K \in \mathrm{Cl}(H)$, of $\Omega_{\mathbb{Q}}(H)$ are the primitive idempotents of $\Omega_{\mathbb{Q}}(H)$. This fact was given by Gluck [18] and Yoshida [41].

4 The crossed Mackey functor

We introduce the crossed restriction and Mackey functors. Let $S \in G$-**set**. For each $s \in S$, G_s denotes the stabilizer of s in G. To begin with, we define a restriction bundle A for $\mathrm{Stab}(G;S) := \{G_s \mid s \in S\}$ over k to be a collection of restriction functors

$$A_s = (A_s, \mathrm{con}, \mathrm{res}) \in \mathbf{Res}(G_s)_k, \quad s \in S,$$

equipped with a family of k-module homomorphisms

$$\mathrm{con}_{s\,H}^{\;g} : A_s(H) \to A_{\,{}^g s}({}^g H),$$

the crossed conjugation maps, for $s \in S$, $H \leq G_s$, and $g \in G$, satisfying the axioms

$$\begin{aligned}
&(\mathrm{C.0})\ \ \mathrm{con}_{s\,H}^{\;t} = \mathrm{con}_H^t,\\
&(\mathrm{C.1})\ \ \mathrm{con}_{\,{}^r s\,{}^r H}^{\;g} \circ \mathrm{con}_{s\,H}^{\;r} = \mathrm{con}_{s\,H}^{\;gr},\\
&(\mathrm{C.2})\ \ \mathrm{con}_{s\,K}^{\;g} \circ \mathrm{res}_K^H = \mathrm{res}_{{}^g K}^{{}^g H} \circ \mathrm{con}_{s\,H}^{\;g}
\end{aligned}$$

for all $s \in S$, $K \leq H \leq G_s$, $g, r \in G$, and $t \in G_s$. In this case, A is called the restriction bundle composed of $A_s \in \mathbf{Res}(G_s)_k$, $s \in S$. Morphisms of restriction bundles for $\mathrm{Stab}(G;S)$ over k are defined in a usual way. We now obtain the category $\mathbf{Res}(G;S)_k$ of restriction bundles for $\mathrm{Stab}(G;S)$ over k. If $A \in \mathbf{Res}(G)_k$, then we naturally view A as a restriction functor for each $G_s \in \mathrm{Stab}(G;S)$, and identify A with the restriction bundle composed of

$$A_s := A = (A, \mathrm{con}, \mathrm{res}) \in \mathbf{Res}(G_s)_k, \quad s \in S,$$

such that the crossed conjugation maps are the conjugation maps of A.

Let $A \in \mathbf{Res}(G;S)_k$. We define

$$A_S = (A_S, \mathrm{con}_S, \mathrm{res}_S) \in \mathbf{Res}(G)_k$$

by

$$A_S(H) = \left\{ (x(s))_{s\in S} \in \prod_{s\in S} A_s(H_s) \;\middle|\; \begin{array}{l} x(s) \in A_s(H) \text{ if } s \in C_S(H), \text{ and} \\ x(s) = 0 \text{ otherwise} \end{array} \right\},$$

$$\mathrm{con}_{S\,H}^{\;g}((x(s))_{s\in S}) = (\mathrm{con}_{s\,H}^{\;g}(x(s)))_{\,{}^g s\in S},$$

$$\mathrm{res}_{S\,K}^{\;H}((x(s))_{s\in S}) = (\mathrm{res}_{K_s}^{H_s}(x(s)))_{s\in S}$$

for all $K \leq H \leq G$, $g \in G$, and $(x(s))_{s\in S} \in A_S(H)$, and call it the crossed restriction functor on A. If A is a restriction functor, then this construction of A_S is called the crossing of A by S. If A is an algebra restriction functor and if $S \in G$-**mon**, then A_S denotes the algebra restriction functor with multiplication on $A_S(H)$ given by

$$(x(s))_{s\in S}(y(t))_{t\in S} = \left(\sum_{(s,t)\in C_S(H)\times C_S(H),\, st=r} x(s)y(t) \right)_{r\in S},$$

where the sum is taken over all pairs (s,t) for s, $t \in C_S(H)$ such that $st = r$. In this case, the algebra restriction functor A_S is isomorphic to $A_{\otimes S}$.

We next define a Mackey bundle X for $\mathrm{Stab}(G;S)$ over k to be a collection of Mackey functors

$$X_s = (X_s, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Mack}(G_s)_k, \quad s \in S,$$

equipped with a family of k-module homomorphisms

$$\mathrm{con}_{s\,H}^{\;g} : X_s(H) \to X_{{}^g s}({}^g H),$$

the crossed conjugation maps, for $s \in S$, $H \le G_s$, and $g \in G$, satisfying the axioms (C.0)–(C.2) and

$$\text{(C.3)}\;\; \mathrm{con}_{s\,H}^{\;g} \circ \mathrm{ind}_K^H = \mathrm{ind}_{{}^gK}^{{}^gH} \circ \mathrm{con}_{s\,K}^{\;g}$$

for all $s \in S$, $K \le H \le G_s$, and $g \in G$. In this case, X is called the Mackey bundle composed of $X_s \in \mathbf{Mack}(G_s)_k$, $s \in S$. Morphisms of Mackey bundles for $\mathrm{Stab}(G;S)$ over k are defined in a usual way. We now obtain the category $\mathbf{Mack}(G;S)_k$ of Mackey bundles for $\mathrm{Stab}(G;S)$ over k. If $X \in \mathbf{Mack}(G)_k$, then we naturally view X as a Mackey functor for each $G_s \in \mathrm{Stab}(G;S)$, and identify X with the Mackey bundle composed of

$$X_s := X = (X, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Mack}(G_s)_k, \quad s \in S,$$

such that the crossed conjugation maps are the conjugation maps of X.

Let $X \in \mathbf{Mack}(G;S)_k$. We define

$$X_S = (X_S, \mathrm{con}_S, \mathrm{res}_S, \mathrm{ind}_S) \in \mathbf{Mack}(G)_k$$

by

$$X_S(H) = \left\{ (x(s))_{s\in S} \in \prod_{s\in S} X_s(H_s) \;\middle|\; \mathrm{con}_{s\,H_s}^{\;h}(x(s)) = x({}^h s) \text{ for all } h \in H \right\},$$

$$\mathrm{con}_{S\,H}^{\;g}((x(s))_{s\in S}) = (\mathrm{con}_{s\,H_s}^{\;g}(x(s)))_{{}^g s\in S},$$

$$\mathrm{res}_{S\,K}^{\;H}((x(s))_{s\in S}) = (\mathrm{res}_{K_s}^{H_s}(x(s)))_{s\in S},$$

$$\mathrm{ind}_{S\,K}^{\;H}((y(s))_{s\in S}) = \left(\sum_{H_s hK \in H_s\backslash H/K} \mathrm{ind}_{({}^hK)_s}^{H_s} \circ \mathrm{con}_{h^{-1}s\,K_{h^{-1}s}}^{\;h} (y({}^{h^{-1}}s)) \right)_{s\in S}$$

for all $K \leq H \leq G$, $g \in G$, $(x(s))_{s\in S} \in X_S(H)$, and $(y(s))_{s\in S} \in X_S(K)$ (cf. [30, 3.11]), and call it the crossed Mackey functor on X. If X is a Mackey functor, then this construction of X_S is the G-functor version of the Dress construction associated to S, and is called the crossing of X by S. Verification of the axioms is analogous to that in the case where X is a Mackey functor. If X is a Green functor and if $S \in G$-**mon**, then X_S denotes the Green functor with multiplication on $X_S(H)$ given by

$$(x(s))_{s\in S}(y(t))_{t\in S} = \left(\sum_{(s,t)\in \overline{H_r\backslash S\times S},\, st=r} \mathrm{ind}_{H_{s,t}}^{H_r}(\mathrm{res}_{H_{s,t}}^{H_s}(x(s)) \cdot \mathrm{res}_{H_{s,t}}^{H_t}(y(t))) \right)_{r\in S},$$

where $\overline{H_r\backslash S\times S}$ is a complete set of representatives of H_r-orbits of the diagonal action on $S\times S$, the sum is taken over all $(s,t) \in \overline{H_r\backslash S\times S}$ such that $st = r$, and $H_{s,t} = H_s \cap H_t$ (cf. [5, Theorem 6.1], [30, 3.14]).

We show the commutativity between the construction $-_+$ and the crossing $-_S$.

Proposition 4.1 *Let $S \in G$-**set**, and let $A \in \mathbf{Res}(G)_k$. Then the Mackey functor A_{S+} is isomorphic to A_{+S}.*

Proof. Define a restriction subfunctor $\widetilde{A}$ of $A_+ = (A_+, \mathrm{con}_+, \mathrm{res}_+, \mathrm{ind}_+)$ by

$$\widetilde{A}(H) = \{[H,\sigma] \in A_+(H) \mid \sigma \in A(H)\}$$

for all $H \leq G$. Then the restriction functor A_S is isomorphic to $\widetilde{A}_S$. Hence it suffices to verify that the Mackey functor $\widetilde{A}_{S+}$ is isomorphic to A_{+S}. Obviously, $\widetilde{A}_S$ is a restriction subfunctor of $A_{+S} = (A_{+S}, \mathrm{con}_{+S}, \mathrm{res}_{+S}, \mathrm{ind}_{+S})$. For each $H \leq G$, the k-module $\widetilde{A}_{S+}(H)$ consists of all k-linear combinations of

$$[K,\sigma]_s := ((\delta_{(s,K)(t,U)}[K,\sigma])_{t\in S})_{U\leq H} \in \widetilde{A}_{S+}(H)$$

for $K \leq H$, $\sigma \in A(K)$, and $s \in C_S(K)$, where $\delta_{(s,K)(t,U)}[K,\sigma] = 0 \in A_+(U)$ if $s \neq t$ or if $K \neq U$, and $\delta_{(s,K)(s,K)}[K,\sigma] = [K,\sigma] \in A_+(K)$. By definition, the induction morphism $\Theta^{A_{+S},\widetilde{A}_S} : \widetilde{A}_{S+} \to A_{+S}$ is a family of k-module homomorphisms $\Theta_H^{A_{+S},\widetilde{A}_S} : \widetilde{A}_{S+}(H) \to A_{+S}(H)$, $H \leq G$, such that

$$\Theta_H^{A_{+S},\widetilde{A}_S}([K,\sigma]_s) = \mathrm{ind}_{+S}{}_K^H((\delta_{(s,K)(t,K)}[K,\sigma])_{t\in S}) = (x_{K,s}(t))_{t\in S}$$

for all $K \leq H$, $\sigma \in A(K)$, and $s \in C_S(K)$, where $x_{K,s}(t) = \mathrm{ind}_{+}{}_{{}^hK}^{H_t} \circ \mathrm{con}_{+}{}_K^h([K,\sigma])$ if $t = {}^hs$ for some $h \in H$, and $x_{K,s}(t) = 0$ otherwise. For each $H \leq G$, it is obvious that $\Theta_H^{A_{+S},\widetilde{A}_S}$ is a bijection. This, combined with Lemma 2.4, shows that $\Theta^{A_{+S},\widetilde{A}_S}$ is an isomorphism of Mackey functors. We have thus proved the proposition. □

By an analogous argument to the proof of Proposition 4.1, the next proposition follows from Lemma 2.4.

Proposition 4.2 *Let $S \in G$-**mon**, and let $A \in \mathbf{Res}_{\mathrm{alg}}(G)_k$. Then the Green functor A_{S+} is isomorphic to A_{+S}.*

We show a generalization of [28, Lemma 3.5] or part of [31, Theorem 3.4].

Corollary 4.3 *Let $S \in G$-**mon**. The Green functor* $\mathrm{C}\Omega(-, S)$ *is isomorphic to* Ω_S.

Proof. Define $\underline{\mathbb{Z}} = (\underline{\mathbb{Z}}, \mathrm{con}, \mathrm{res}) \in \mathbf{Res}_{\mathrm{alg}}(G)_{\mathbb{Z}}$ by $\underline{\mathbb{Z}}(H) = \mathbb{Z}$ and $\mathrm{con}^g_H = \mathrm{res}^H_K = \mathrm{id}_{\mathbb{Z}}$ for all $K \leq H \leq G$ and $g \in G$. Then the Green functor Ω is isomorphic to $\underline{\mathbb{Z}}_+$. Hence the Green functor Ω_S is isomorphic to $\underline{\mathbb{Z}}_{+S}$. By Proposition 3.1, the Green functor $\underline{\mathbb{Z}}_{S+}$ is isomorphic to $\mathrm{C}\Omega(-, S)$. Hence it follows from Proposition 4.2 that the Green functor Ω_S is isomorphic to $\mathrm{C}\Omega(-, S)$. This completes the proof. □

Remark 4.4 Keep the notation of Proposition 3.1 and the proofs of Proposition 4.1 and Corollary 4.3. Given $K \leq H \leq G$ and $s \in C_S(K)$, there exists an H-map $\pi_s : H/K \to \prod_{U \leq H} \mathbb{Z}C_S(U)$ given by

$$\pi_s(hK) = (\delta_{\,{}^hK U}\, {}^h s)_{U \leq H}$$

for all $h \in H$, where δ is the Kronecker delta. The family of $\mathbb{Z}$-lattice homomorphisms

$$\Theta_H : \mathrm{C}\Omega(H, S) \to \Omega_S(H), \quad [H/K, \pi_s] \mapsto (x_{K,s}(t))_{t \in S}$$

for $H \leq G$, where $x_{K,s}(t) = [H_t/{}^hK](= [h \otimes H_s/K])$ if $t = {}^hs$ for some $h \in H$, and $x_{K,s}(t) = 0$ otherwise, defines an isomorphism $\Theta : \mathrm{C}\Omega(-, S) \to \Omega_S$ of Green functors such that the diagram

$$\begin{tikzcd}
\underline{\mathbb{Z}}_{S+} \arrow[r, "q_1"] \arrow[d, "q_2"'] & \widetilde{\underline{\mathbb{Z}}}_{S+} \arrow[r, "\Theta^{\underline{\mathbb{Z}}_{+S}, \widetilde{\underline{\mathbb{Z}}}_S}"] & \underline{\mathbb{Z}}_{+S} \arrow[d, "q_3"] \\
\underline{\mathbb{Z}}_{\otimes S+} \arrow[r, "\upsilon^{\underline{\mathbb{Z}}_{\otimes S}\,-1}"'] & \mathrm{C}\Omega(-, S) \arrow[r, "\Theta"'] & \Omega_S
\end{tikzcd}$$

is commutative, where q_i, $i = 1, 2, 3$, are obvious isomorphisms of Green functors, because $\upsilon^{\underline{\mathbb{Z}}_{\otimes S}} : \mathrm{C}\Omega(-, S) \to \underline{\mathbb{Z}}_{\otimes S+}$ and $\Theta^{\underline{\mathbb{Z}}_{+S}, \widetilde{\underline{\mathbb{Z}}}_S} : \widetilde{\underline{\mathbb{Z}}}_{S+} \to \underline{\mathbb{Z}}_{+S}$ are isomorphisms of Green functors (see also Lemma 2.4).

There exists a bijective correspondence between G-functors introduced by Green [19], for which we mean Mackey functors for G in this paper, and Mackey functors on G-**set** introduced by Dress [16] (cf. [5, Remarks 2.2 and 2.3], [30, Lemma 3.7]). The rest of this section is devoted to the description of the restriction and Mackey bundles, together with the crossed Mackey functors, from Dress point of view on Mackey functors, using finite left G-sets instead of evaluations on subgroups.

Let $S \in G$-**set**. The category G-**set**$\downarrow_{\boldsymbol{S}}$ of G-sets over S is defined as follows:

(i) Objects are pairs (J, w) consisting of $J \in G$-**set** and $w \in \mathrm{Map}_G(J, S)$.

(ii) Morphisms $f : (J_1, w_1) \to (J_2, w_2)$ are defined to be G-maps $f : J_1 \to J_2$ such that $w_1 = w_2 \circ f$.

A contravariant functor $\mathcal{A} : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$, $(J, w) \mapsto \mathcal{A}(J, w)$ is said to be additive if the two canonical embeddings $\iota_1 : (J_1, w_1) \to (J_1 \dot{\cup} J_2, w_1 \dot{+} w_2)$ and $\iota_2 : (J_2, w_2) \to (J_1 \dot{\cup} J_2, w_1 \dot{+} w_2)$ with $(J_1, w_1), (J_2, w_2) \in G\text{-}\mathbf{set}{\downarrow_S}$, where the G-map $w_1 \dot{+} w_2 : J_1 \dot{\cup} J_2 \to S$ is defined by $x \mapsto w_i(x)$ for all $x \in J_i$ with $i = 1, 2$, induce an isomorphism

$$\mathcal{A}(\iota_1) \oplus \mathcal{A}(\iota_2) : \mathcal{A}(J_1 \dot{\cup} J_2, w_1 \dot{+} w_2) \xrightarrow{\sim} \mathcal{A}(J_1, w_1) \oplus \mathcal{A}(J_2, w_2).$$

We denote by $k\text{-}\mathbf{Fun}(G; S)$ the functor category with objects the additive contravariant functors $\mathcal{A} : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$ and morphisms the natural transformations between two such functors. For each $\mathcal{A} \in k\text{-}\mathbf{Fun}(G; S)$, there exists a restriction bundle $A = \mathcal{A}_{\mathrm{Res}}$ composed of

$$A_s = (A_s, \mathrm{con}, \mathrm{res}) \in \mathbf{Res}(G_s)_k, \quad s \in S,$$

given by

$$\begin{aligned}
&\text{(B.0)}\ \ A_s(H) = \mathcal{A}(G/H, {}^{\natural}s),\\
&\text{(B.1)}\ \ \mathrm{con}_s{}^{g}_{H} = \mathcal{A}(G/{}^{g}H \to G/H,\ r\,{}^{g}H \mapsto rgH),\\
&\text{(B.2)}\ \ \mathrm{res}^H_K = \mathcal{A}(G/K \to G/H,\ rK \mapsto rH)
\end{aligned}$$

for all $s \in S$, $K \le H \le G_s$, and $g \in G$, where ${}^{\natural}s : G/H \to S$ is defined by ${}^{\natural}s(rH) = {}^{r}s$ for all $r \in G$. Conversely, for each restriction bundle A composed of

$$A_s = (A_s, \mathrm{con}, \mathrm{res}) \in \mathbf{Res}(G_s)_k, \quad s \in S,$$

together with the crossed conjugation maps $\mathrm{con}_s{}^{g}_{H}$ for $s \in S$, $H \le G_s$, and $g \in G$, there exists a contravariant functor $\mathcal{A} = A^{\mathrm{Fun}} : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$ given by

$$\begin{aligned}
\text{(F.0)}\ \ \mathcal{A}(J, w) &= \left(\prod_{x \in J} A_{w(x)}(G_x)\right)^{G}\\
&= \left\{ (\sigma_x)_{x \in J} \in \prod_{x \in J} A_{w(x)}(G_x) \;\middle|\; \begin{array}{l} \mathrm{con}_{w(x)}{}^{g}_{G_x}(\sigma_x) = \sigma_{{}^{g}x} \\ \text{for all } x \in J \text{ and } g \in G \end{array} \right\},
\end{aligned}$$

$$\text{(F.1)}\ \ \mathcal{A}(f) : \mathcal{A}(J, w) \to \mathcal{A}(J', w'),\ (\sigma_x)_{x \in J} \mapsto (\mathrm{res}^{G_{f(x')}}_{G_{x'}}(\sigma_{f(x')}))_{x' \in J'}$$

for all objects (J, w) and morphisms $f : (J', w') \to (J, w)$ of $G\text{-}\mathbf{set}{\downarrow_S}$, which is additive. Moreover, the categories $k\text{-}\mathbf{Fun}(G; S)$ and $\mathbf{Res}(G; S)_k$ are equivalent.

A bifunctor $\mathcal{X} = (\mathcal{X}^*, \mathcal{X}_*) : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$, $(J, w) \mapsto \mathcal{X}(J, w)$, which consists of a contravariant functor $\mathcal{X}^* : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$, $(J, w) \mapsto \mathcal{X}^*(J, w)$ and a covariant functor $\mathcal{X}_* : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$, $(J, w) \mapsto \mathcal{X}_*(J, w)$ such that $\mathcal{X}(J, w) = \mathcal{X}^*(J, w) = \mathcal{X}_*(J, w)$ for all $(J, w) \in G\text{-}\mathbf{set}{\downarrow_S}$, is called a Mackey functor on $G\text{-}\mathbf{set}{\downarrow_S}$ if the following two conditions are fulfilled by $\mathcal{X}$:

(i) For each pull back diagram in G-$\mathbf{set}{\downarrow_S}$

$$\begin{tikzcd} (J,w) \arrow[r, "f_1"] \arrow[d, "f_2"'] & (J_1,w_1) \arrow[d, "f_{13}"] \\ (J_2,w_2) \arrow[r, "f_{23}"'] & (J_3,w_3) \end{tikzcd}$$

the diagram

$$\begin{tikzcd} \mathcal{X}(J,w) \arrow[r, "\mathcal{X}_*(f_1)"] & \mathcal{X}(J_1,w_1) \\ \mathcal{X}(J_2,w_2) \arrow[u, "\mathcal{X}^*(f_2)"] \arrow[r, "\mathcal{X}_*(f_{23})"'] & \mathcal{X}(J_3,w_3) \arrow[u, "\mathcal{X}^*(f_{13})"'] \end{tikzcd}$$

is commutative.

(ii) The contravariant functor $\mathcal{X}^* : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$ is additive.

Given Mackey functors $\mathcal{X}_1 = ({\mathcal{X}_1}^*, \mathcal{X}_{1*})$ and $\mathcal{X}_2 = ({\mathcal{X}_2}^*, \mathcal{X}_{2*})$ on G-$\mathbf{set}{\downarrow_S}$, a family of k-module homomorphisms $f_{(J,w)} : \mathcal{X}_1(J,w) \to \mathcal{X}_2(J,w)$, $(J,w) \in G$-$\mathbf{set}{\downarrow_S}$, is called a natural transformation of Mackey functors on G-$\mathbf{set}{\downarrow_S}$ if this family is a natural transformation ${\mathcal{X}_1}^* \to {\mathcal{X}_2}^*$ and $\mathcal{X}_{1*} \to \mathcal{X}_{2*}$.

Let k-$\mathbf{Fun}_*(G;S)$ be the functor category with objects the Mackey functors on G-$\mathbf{set}{\downarrow_S}$ and morphisms the natural transformations of Mackey functors on G-$\mathbf{set}{\downarrow_S}$. For each $\mathcal{X} = (\mathcal{X}^*, \mathcal{X}_*) \in k\text{-}\mathbf{Fun}_*(G;S)$, there exists a Mackey bundle $X = \mathcal{X}_{\mathrm{Mack}}$ composed of

$$X_s = (X_s, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Mack}(G_s)_k, \quad s \in S,$$

such that the collection of restriction functors $X_s = (X_s, \mathrm{con}, \mathrm{res}) \in \mathbf{Res}(G_s)_k$, $s \in S$, is the restriction bundle defined to be ${\mathcal{X}^*}_{\mathrm{Res}}$ and the induction maps are given by

$$\text{(B.3)} \quad \mathrm{ind}_K^H = \mathcal{X}_*(G/K \to G/H,\ rK \mapsto rH)$$

for all $s \in S$ and $K \leq H \leq G_s$. Conversely, for each Mackey bundle X composed of

$$X_s = (X_s, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Mack}(G_s)_k, \quad s \in S,$$

there exists a Mackey functor $\mathcal{X} = X^{\mathrm{Fun}_*} = (\mathcal{X}^*, \mathcal{X}_*)$ on G-$\mathbf{set}{\downarrow_S}$ such that the contravariant functor $\mathcal{X}^* : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$ is defined to be X^{Fun} for the restriction bundle X composed of $X_s = (X_s, \mathrm{con}, \mathrm{res}) \in \mathbf{Res}(G_s)_k$, $s \in S$, arising from X by forgetting induction maps and the covariant functor $\mathcal{X}_* : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod}$ is given by

$$\text{(F.2)} \quad \mathcal{X}_*(f) : \mathcal{X}(J',w') \to \mathcal{X}(J,w),\ (\sigma_{x'})_{x' \in J'} \mapsto \left(\sum_{x' \in \overline{G_x \backslash f^{-1}(x)}} \mathrm{ind}_{G_{x'}}^{G_x}(\sigma_{x'}) \right)_{x \in J}$$

for all morphisms $f : (J', w') \to (J, w)$ of G-$\mathbf{set}{\downarrow_S}$, where $\overline{G_x\backslash f^{-1}(x)}$ is a complete set of representatives of G_x-orbits in the inverse image $f^{-1}(x)$ of x under f. Moreover, the categories k-$\mathbf{Fun}_*(G;S)$ and $\mathbf{Mack}(G;S)_k$ are equivalent.

Let $\bullet$ be the one-point G-set. When $S = \bullet$, we write k-$\mathbf{Fun}(G) = k$-$\mathbf{Fun}(G;\bullet)$ and k-$\mathbf{Fun}_*(G) = k$-$\mathbf{Fun}_*(G;\bullet)$ for shortness' sake. Obviously, k-$\mathbf{Fun}(G)$ and k-$\mathbf{Fun}_*(G)$ are regarded as the categories of the contravariant and Mackey functors on G-$\mathbf{set}$, respectively. Moreover, the categories k-$\mathbf{Fun}(G)$ and $\mathbf{Res}(G)_k$ are equivalent, and so are the categories k-$\mathbf{Fun}_*(G)$ and $\mathbf{Mack}(G)$. There exists a unique G-map $S \to \bullet$. We define a functor $\mathbf{Fun}_*(S \to \bullet) : k$-$\mathbf{Fun}_*(G) \to k$-$\mathbf{Fun}_*(G;S)$ by

$$\mathbf{Fun}_*(S \to \bullet)(\mathcal{X}) : G\text{-}\mathbf{set}{\downarrow_S} \to k\text{-}\mathbf{mod},\ (J, w) \mapsto \mathcal{X}(J, (S \to \bullet) \circ w)$$

for all $\mathcal{X} \in k$-$\mathbf{Fun}_*(G)$. Given $\mathcal{X} \in k$-$\mathbf{Fun}_*(G)$, we write $\mathcal{X}^S_\downarrow = (\mathbf{Fun}_*(S \to \bullet))(\mathcal{X})$.

We turn to the Dress construction from Mackey functors on G-$\mathbf{set}{\downarrow_S}$. For each $J \in G$-$\mathbf{set}$, let Pr_S be the projection $J \times S \to S$. Given a G-map $f : J' \to J$ with J, $J' \in G$-$\mathbf{set}$, we denote by $f_S : (J' \times S, \mathrm{Pr}_S) \to (J \times S, \mathrm{Pr}_S)$ the morphism of G-$\mathbf{set}{\downarrow_S}$ induced from $f \times \mathrm{id}_S : J' \times S \to J \times S$. Let $\mathcal{X} = (\mathcal{X}^*, \mathcal{X}_*) \in k$-$\mathbf{Fun}_*(G;S)$. We define $\mathcal{X}_S = ({\mathcal{X}_S}^*, \mathcal{X}_{S*}) \in k$-$\mathbf{Fun}_*(G)$ by

$$\begin{aligned}
&\mathcal{X}_S(J) = \mathcal{X}(J \times S, \mathrm{Pr}_S),\\
&{\mathcal{X}_S}^*(f) = \mathcal{X}^*(f_S) : \mathcal{X}(J \times S, \mathrm{Pr}_S) \to \mathcal{X}(J' \times S, \mathrm{Pr}_S),\\
&\mathcal{X}_{S*}(f) = \mathcal{X}_*(f_S) : \mathcal{X}(J' \times S, \mathrm{Pr}_S) \to \mathcal{X}(J \times S, \mathrm{Pr}_S)
\end{aligned}$$

for all $J \in G$-$\mathbf{set}$ and $f \in \mathrm{Map}_G(J', J)$ with J, $J' \in G$-$\mathbf{set}$. If $X \in \mathbf{Mack}(G;S)_k$ and if $\mathcal{X} = X^{\mathrm{Fun}_*}$, then $X_S \cong (\mathcal{X}_S)_{\mathrm{Mack}}$. Simultaneously, if $\mathcal{X} \in k$-$\mathbf{Fun}_*(G;S)$ and if $X = \mathcal{X}_{\mathrm{Mack}}$, then $\mathcal{X}_S \cong (X_S)^{\mathrm{Fun}_*}$. Given $\mathcal{X} \in k$-$\mathbf{Fun}_*(G)$, the construction $\mathcal{X} \mapsto (\mathcal{X}^S_\downarrow)_S$ is called the Dress construction associated to S (see [5, 30]).

Let $\mathcal{A} \in k$-$\mathbf{Fun}(G)$. Set $\mathcal{A}_{+S} = (((\mathcal{A}_{\mathrm{Res}+})^{\mathrm{Fun}_*})^S_\downarrow)_S$ and $\mathcal{A}_{S+} = (\mathcal{A}_{\mathrm{Res}S+})^{\mathrm{Fun}_*}$. Then for each $J \in G$-$\mathbf{set}$,

$$\mathcal{A}_{+S}(J) = \left(\prod_{(x,s) \in J \times S} \mathcal{A}_{\mathrm{Res}+}(G_{(x,s)}) \right)^G,$$

where the superscript G denotes the set of G-invariants with respect to the action induced by the conjugation maps $G/{}^gG_{(x,s)} \to G/G_{(x,s)}$, $r\,{}^gG_{(x,s)} \mapsto rgG_{(x,s)}$ for $g \in G$ and $(x,s) \in J \times S$. Likewise,

$$\mathcal{A}_{S+}(J) = \left(\prod_{x \in J} \mathcal{A}_{\mathrm{Res}S+}(G_x) \right)^G$$

for each $J \in G$-$\mathbf{set}$. By Proposition 4.1, we know that the family of k-module homomorphisms $\mathcal{A}_{+S}(J) \to \mathcal{A}_{S+}(J)$, $J \in G$-$\mathbf{set}$, given by

$$([U_{(x,s)}, \sigma_{(x,s)}])_{(x,s) \in J \times S} \mapsto \left(\sum_{s \in \overline{G_x\backslash S}} [U_{(x,s)}, (\delta_{st}\sigma_{(x,s)})_{t \in S}] \right)_{x \in J},$$

where $U_{(x,s)} \leq G_{(x,s)}$, $\sigma_{(x,s)} \in \mathcal{A}(G_{(x,s)}/U_{(x,s)})$, and $\overline{G_x \backslash S}$ is a complete set of representatives of G_x-orbits in S, defines an isomorphism $\mathcal{A}_{+S} \to \mathcal{A}_{S+}$ of Mackey functors on G-**set**.

5 Induction formulae for Mackey functors

Let X, Y, $Z \in \mathbf{Mack}(G)_k$. A pairing $X \otimes_k Y \to Z$ is defined to be a family of k-module homomorphisms

$$X(H) \otimes_k Y(H) \to Z(H), \quad x \otimes y \mapsto x \cdot y$$

for $H \leq G$, satisfying the axioms

(P.1) $\mathrm{con}^g_H(x \cdot y) = \mathrm{con}^g_H(x) \cdot \mathrm{con}^g_H(y)$,

(P.2) $\mathrm{res}^H_K(x \cdot y) = \mathrm{res}^H_K(x) \cdot \mathrm{res}^H_K(y)$,

(P.3) (Frobenius axioms)

$$x \cdot \mathrm{ind}^H_K(y') = \mathrm{ind}^H_K(\mathrm{res}^H_K(x) \cdot y'), \quad \mathrm{ind}^H_K(x') \cdot y = \mathrm{ind}^H_K(x' \cdot \mathrm{res}^H_K(y))$$

for all $K \leq H \leq G$, $g \in G$, $x \in X(H)$, $y \in Y(H)$, $x' \in X(K)$ and $y' \in Y(K)$ (cf. [4, 16, 35, 40]).

We need to quote [4, Proposition 1.5(i)] (see also [16, Proposition 4.2] and [35, Proposition 6.1]).

Proposition 5.1 *For any $X \in \mathbf{Mack}(G)_k$, the family of k-module homomorphisms*

$$\Omega_k(H) \otimes_k X(H) \to X(H), \quad [H/K] \otimes x \mapsto \mathrm{ind}^H_K \circ \mathrm{res}^H_K(x)$$

for $H \leq G$ is a pairing, and makes k-modules $X(H)$ for $H \leq G$ into $\Omega_k(H)$-modules.

Suppose that $X \in \mathbf{Mack}(G)_k$. Let $H \leq G$. We can consider $X(H)$ to be a left $\Omega_k(H)$-module with the action given by

$$\left(\sum_{K \leq H} \ell_K [H/K]\right) \cdot x = \sum_{K \leq H} \ell_K \mathrm{ind}^H_K \circ \mathrm{res}^H_K(x)$$

for all $\ell_K \in k$ with $K \leq H$ and $x \in X(H)$. If $|G|$ is invertible in k, then the primitive idempotent $e^{(H)}_K$ of $\Omega_k(H)$ with $K \in \mathrm{Cl}(H)$ (cf. Remark 3.2) acts on $X(H)$ by

$$e^{(H)}_K \cdot x = \frac{1}{|N_H(K)|} \sum_{U \leq K} |U| \mu(U, K) \mathrm{ind}^H_U \circ \mathrm{res}^H_U(x) \tag{III}$$

for all $x \in X(H)$. Moreover, since the identity of $\Omega_k(H)$ is expressed as a sum of orthogonal idempotents $\sum_{K \in \mathrm{Cl}(H)} e^{(H)}_K$, it follows that

$$x = \sum_{K \in \mathrm{Cl}(H)} \frac{1}{|N_H(K)|} \sum_{U \leq K} |U| \mu(U, K) \mathrm{ind}^H_U \circ \mathrm{res}^H_U(x)$$

for all $x \in X(H)$, which is reduced to the formula in Corollary 5.4.

Lemma 5.2 *Let $X \in \mathbf{Mack}(G)_k$. If $|G|$ is invertible in k, then the following statements hold.*

(a) *For any $K < H \le G$ and $x \in X(H)$, $\mathrm{res}^H_K(e^{(H)}_H \cdot x) = 0$.*

(b) *For any $K < H \le G$ and $y \in X(K)$, $e^{(H)}_H \cdot \mathrm{ind}^H_K(y) = 0$.*

(c) *Suppose that $f \in \mathbf{Res}(G)(X, X)_k$. If $e^{(K)}_K \cdot (f_K(x) - x) = 0$ for all $K \le G$ and $x \in X(K)$, then $f = \mathrm{id}_X$, that is, $f_H = \mathrm{id}_{X(H)}$ for all $H \le G$.*

Proof. Let $H \le G$. For any $K < H$, it follows from Proposition 2.2 that

$$\begin{aligned}\rho^{\underline{k}\otimes\{\epsilon\}}_K(\mathrm{res}_{+}{}^H_K(e^{(H)}_H)) &= \mathrm{res}^{+}{}^H_K(\rho^{\underline{k}\otimes\{\epsilon\}}_H(e^{(H)}_H)) \\ &= \frac{1}{|H|}\mathrm{res}^{+}{}^H_K(\rho^{\underline{k}\otimes\{\epsilon\}}_H \circ \eta^{\underline{k}\otimes\{\epsilon\}}_H((x_H(L))_{L\le H})) = 0,\end{aligned}$$

which, together with Proposition 2.2, shows that $\mathrm{res}_{+}{}^H_K(e^{(H)}_H) = 0$. Hence (a) follows from Proposition 5.1 and the axiom (P.2) of a pairing. Moreover, (P.3) yields

$$e^{(H)}_H \cdot \mathrm{ind}^H_K(y) = \mathrm{ind}^H_K(\mathrm{res}_{+}{}^H_K(e^{(H)}_H) \cdot y) = 0$$

for all $K < H$ and $y \in X(K)$. Thus (b) holds. (The statements (a) and (b) are proved in the proof of [4, Proposition 6.2].) To prove (c), we argue by induction on $|H|$. Suppose that $|H| > 1$, and let $x \in X(H)$. By the inductive assumption, $f_U(\mathrm{res}^H_U(x)) = \mathrm{res}^H_U(x)$ for all $U < H$. This, combined with (III), shows that

$$\begin{aligned}e^{(H)}_H \cdot (f_H(x) - x) &= \frac{1}{|H|}\sum_{U\le H}|U|\mu(U,H)\mathrm{ind}^H_U \circ \mathrm{res}^H_U(f_H(x) - x) \\ &= \frac{1}{|H|}\sum_{U\le H}|U|\mu(U,H)\mathrm{ind}^H_U(f_U(\mathrm{res}^H_U(x)) - \mathrm{res}^H_U(x)) \\ &= f_H(x) - x.\end{aligned}$$

Since $e^{(H)}_H \cdot (f_H(x) - x) = 0$, it follows that $f_H(x) = x$. This completes the proof. □

We define a restriction subfunctor $\mathcal{K}^X = (\mathcal{K}^X, \mathrm{con}, \mathrm{res})$ of X by

$$\mathcal{K}^X(H) = \bigcap_{K<H}\{x \in X(H) \mid \mathrm{res}^H_K(x) = 0\}$$

for all $H \le G$. A subgroup H of G is said to be coprimordial for X if $\mathcal{K}^X(H) \ne \{0\}$ (cf. [4]). We denote by $\mathcal{C}(X)$ the set of coprimordial subgroups for X.

Suppose now that A is a restriction subfunctor of X. A canonical induction formula for X from A is defined to be a morphism $\Psi : X \to A_+$ of restriction functors with $\Theta^{X,A} \circ \Psi = \mathrm{id}_X$, where $\Theta^{X,A} : A_+ \to X$ is the induction morphism defined in Section 2 (cf. [4, Definition 3.3]).

Let $\lambda \in \mathbf{Con}(G)(X, A)_k$. Then $(\lambda_K \circ \mathrm{res}^H_K(x))_{K\leq H} \in A^+(H)$ for all $H \leq G$ and $x \in X(H)$. We define $\Psi^{X,A,\lambda} : X \to A_+$ to be a family of k-module homomorphisms $\Psi^{X,A,\lambda}_H : X(H) \to A_+(H)$, $H \leq G$, such that

$$\Psi^{X,A,\lambda}_H(x) = \frac{1}{|H|}\eta^A_H((\lambda_K \circ \mathrm{res}^H_K(x))_{K\leq H})$$

for all $x \in X(H)$, provided $|G|$ is invertible in k. For any $H \leq G$ and $x \in X(H)$,

$$\Psi^{X,A,\lambda}_H(x) = \frac{1}{|H|}\sum_{K\leq H}\sum_{U\leq K}|U|\mu(U,K)[U, \mathrm{res}^K_U \circ \lambda_K \circ \mathrm{res}^H_K(x)]. \tag{IV}$$

The following result is due to Boltje [4, Proposition 6.4].

Proposition 5.3 *Let $X \in \mathbf{Mack}(G)_k$, and let A be a restriction subfunctor of X. Suppose that $|G|$ is invertible in k. Let $\lambda \in \mathbf{Con}(G)(X, A)_k$. Then $\Psi^{X,A,\lambda}$ is a morphism of restriction functors, and the following conditions are equivalent* :

(1) $\Psi^{X,A,\lambda}$ *is a canonical induction formula for X from A*;

(2) $e^{(H)}_H \cdot (\lambda_H(x) - x) = 0$ *for all $H \in \mathcal{C}(X)$ and $x \in X(H)$.*

Proof. Obviously, $\Psi^{X,A,\lambda}$ is a morphism of conjugation functors. Since ρ^A is a morphism of restriction functors, it follows that

$$\eta^A_U \circ \rho^A_U \circ \mathrm{res}_+{}^H_U \circ \eta^A_H = \eta^A_U \circ \mathrm{res}^+{}^H_U \circ \rho^A_H \circ \eta^A_H$$

for all $U \leq H \leq G$. This, combined with Proposition 2.2, shows that

$$\begin{aligned}
\mathrm{res}_+{}^H_U \circ \Psi^{X,A,\lambda}_H(x) &= \frac{1}{|H|}\mathrm{res}_+{}^H_U \circ \eta^A_H\left((\lambda_K \circ \mathrm{res}^H_K(x))_{K\leq H}\right)\\
&= \frac{1}{|U|}\eta^A_U((\lambda_K \circ \mathrm{res}^U_K \circ \mathrm{res}^H_U(x))_{K\leq U})\\
&= \Psi^{X,A,\lambda}_U \circ \mathrm{res}^H_U(x)
\end{aligned}$$

for all $U \leq H \leq G$ and $x \in X(H)$, and thereby, $\Psi^{X,A,\lambda}$ is a morphism of restriction functors. We next prove the equivalence between the conditions (1) and (2). By using Lemma 5.2(b) and (IV), we have

$$e^{(H)}_H \cdot (\Theta^{X,A}_H \circ \Psi^{X,A,\lambda}_H(x) - \lambda_H(x)) = 0$$

for all $H \leq G$ and $x \in X(H)$. Hence (1) implies (2). Suppose that the condition of (2) holds. By Lemma 5.2(a) and hypothesis, $e^{(H)}_H \cdot (\lambda_H(x) - x) = 0$, and hence

$$e^{(H)}_H \cdot (\Theta^{X,A}_H \circ \Psi^{X,A,\lambda}_H(x) - x) = 0$$

for all $H \leq G$ and $x \in X(H)$. This, combined with Lemma 5.2(c), shows that $\Psi^{X,A,\lambda}$ is a canonical induction formula for X from A. We have thus proved the proposition. □

We next define $\lambda^X : X \to \mathcal{K}^X$ to be a family of k-module homomorphisms $\lambda^X_H : X(H) \to \mathcal{K}^X(H)$, $H \leq G$, such that

$$\lambda^X_H(x) = e^{(H)}_H \cdot x \tag{V}$$

for all $x \in X(H)$, provided $|G|$ is invertible in k. By Lemma 5.2(a), this definition makes sense. Clearly, $\lambda^X \in \mathbf{Con}(G)(X, \mathcal{K}^X)_k$. We write $\Psi^{X,\mathcal{K}^X} = \Psi^{X,\mathcal{K}^X,\lambda^X}$ for the sake of simplicity. By Proposition 5.3, $\Psi^{X,\mathcal{K}^X}$ is a canonical induction formula for X from $\mathcal{K}^X$, which is said to be minimal (cf. [4, Example 6.9]).

The following corollary, which is part of [4, Example 6.9], generalizes Brauer's explicit version of Artin's induction theorem for virtual $\mathbb{C}$-characters of G (cf. [3, Corollary 3.3], [8, Satz 2], [41, Corollary 4.5]) and Witherspoon's explicit version of Conlon's induction theorem (cf. [36, Proposition 3.7]).

Corollary 5.4 *Let $X \in \mathbf{Mack}(G)_k$, and suppose that $|G|$ is invertible in k. Then*

$$x = \sum_{K \in \mathrm{Cl}(H) \cap \mathcal{C}(X)} \frac{1}{|N_H(K)|} \sum_{U \leq K} |U| \mu(U,K) \mathrm{ind}^H_U \circ \mathrm{res}^H_U(x)$$

for all $H \leq G$ and $x \in X(H)$.

Proof. Let $H \leq G$ and $x \in X(H)$. Then by (III), Lemma 5.2(a), and (IV), we have

$$\begin{aligned}
\Psi^{X,\mathcal{K}^X}_H(x) &= \frac{1}{|H|} \sum_{K \leq H} \sum_{U \leq K} |U| \mu(U,K) [U, \mathrm{res}^K_U(e^{(K)}_K \cdot \mathrm{res}^H_K(x))] \\
&= \sum_{K \in \mathrm{Cl}(H) \cap \mathcal{C}(X)} \frac{|K|}{|N_H(K)|} [K, e^{(K)}_K \cdot \mathrm{res}^H_K(x)] \\
&= \sum_{K \in \mathrm{Cl}(H) \cap \mathcal{C}(X)} \frac{1}{|N_H(K)|} \sum_{U \leq K} |U| \mu(U,K) [K, \mathrm{ind}^K_U \circ \mathrm{res}^H_U(x)].
\end{aligned}$$

Hence the corollary follows from the fact that $\Psi^{X,\mathcal{K}^X}$ is a canonical induction formula for X from $\mathcal{K}^X$. This completes the proof. □

For each $H \leq G$, we set

$$\mathcal{T}^X(H) = \sum_{K < H} \{\mathrm{ind}^H_K(y) \mid y \in X(K)\}.$$

A subgroup H of G is said to be primordial for X if $\mathcal{T}^X(H) \neq X(H)$ (cf. [35]). We denote by $\mathcal{P}(X)$ the set of primordial subgroups for X.

The following proposition is part of [4, Proposition 6.2]. (This is a special case of a much more general result of Dress [16, Theorems 2 and 3].)

Proposition 5.5 *Let $X \in \mathbf{Mack}(G)_k$, and suppose that $|G|$ is invertible in k. Then*

$$\begin{aligned} &\mathcal{K}^X(H) = \{e_H^{(H)} \cdot x \mid x \in X(H)\}, \\ &\mathcal{T}^X(H) = \{x - e_H^{(H)} \cdot x \mid x \in X(H)\}, \\ &X(H) = \mathcal{K}^X(H) \oplus \mathcal{T}^X(H) \end{aligned}$$

for all $H \leq G$. Moreover $\mathcal{C}(X) = \mathcal{P}(X)$.

Proof. The first two assertions follow from (III) and Lemma 5.2(a), (b). The remaining assertions are straightforward. This completes the proof. □

We define

$$\overline{X} = (\overline{X}, \overline{\mathrm{con}}) \in \mathbf{Con}(G)_k$$

by

$$\overline{X}(H) = \overline{X(H)} := X(H)/\mathcal{T}^X(H) \quad \text{and} \quad \overline{\mathrm{con}}_H^g(\overline{x}) = \overline{\mathrm{con}_H^g(x)}$$

for all $H \leq G$, $g \in G$, and $x \in X(H)$, where $\overline{x} = x + \mathcal{T}^X(H)$ for all $x \in X(H)$. If X is a Green functor, then $\overline{X}$ is an algebra conjugation functor.

Following [35], we define a morphism $\beta : X \to \overline{X}^+$ of Mackey functors by

$$\beta_H(x) = (\overline{\mathrm{res}_K^H(x)})_{K \leq H}$$

for all $H \leq G$ and $x \in X(H)$. If X is a Green functor, then β is a morphism of Green functors. By virtue of Lemma 5.2(b) and Proposition 5.5, there exists an isomorphism $\Delta : \overline{X}^+ \to (\mathcal{K}^X)^+$ of Mackey functors defined to be a family of k-module isomorphisms $\Delta_H : \overline{X}^+(H) \xrightarrow{\sim} (\mathcal{K}^X)^+(H)$, $H \leq G$, such that

$$\Delta_H((\overline{x_K})_{K \leq H}) = (e_K^{(K)} \cdot x_K)_{K \leq H}$$

for all $(x_K)_{K \leq H} \in \prod_{K \leq H} X(K)$. From Proposition 2.2, we know that the diagram

$$\begin{array}{ccc} X(H) & \xrightarrow{\beta_H} & \overline{X}^+(H) \\ {\scriptstyle \Psi_H^{X,\mathcal{K}^X}} \big\downarrow & & \big\downarrow {\scriptstyle \Delta_H} \\ (\mathcal{K}^X)_+(H) & \xrightarrow[\rho_H^{\mathcal{K}^X}]{} & (\mathcal{K}^X)^+(H) \end{array}$$

with $H \leq G$ is commutative, where $\Psi^{X,\mathcal{K}^X} = \Psi^{X,\mathcal{K}^X,\lambda^X}$ (see (IV) and (V)).

The next proposition is due to Thévenaz [35, Corollary 4.4, Theorem 12.3], which is explored on the basis of [32, Proposition 3.4(iii)].

Proposition 5.6 *Let $X \in \mathbf{Mack}(G)_k$, and suppose that $|G|$ is invertible in k. Then β is an isomorphism of Mackey functors. If X is a Green functor, then β is an isomorphism of Green functors.*

Proof. By Proposition 2.2, it suffices to verify that $\Psi^{X,\mathcal{K}^X}$ is an isomorphism of restriction functors. Recall that $\Psi^{X,\mathcal{K}^X}$ is a canonical induction formula for X from $\mathcal{K}^X$. Using the Mackey axiom, Lemma 5.2(a), (b), (IV), (V), and Proposition 5.5, we have

$$\begin{aligned}
&\Psi_H^{X,\mathcal{K}^X} \circ \Theta_H^{X,\mathcal{K}^X}([L,x]) \\
&\quad = \frac{1}{|H|}\sum_{K\leq H}\sum_{U\leq K}|U|\mu(U,K)[U, \mathrm{res}_U^K(e_K^{(K)}\cdot \mathrm{res}_K^H \circ \mathrm{ind}_L^H(x))] \\
&\quad = \sum_{K\in \mathrm{Cl}(H)\cap\mathcal{C}(H)} \frac{|K|}{|N_H(K)|}[K, e_K^{(K)}\cdot \mathrm{res}_K^H\circ \mathrm{ind}_L^H(x)] \\
&\quad = \sum_{K\in \mathrm{Cl}(H)\cap\mathcal{C}(H)} \frac{|K|}{|N_H(K)|}\sum_{KhL\in K\backslash H/L}[K, e_K^{(K)}\cdot \mathrm{ind}_{K\cap {}^hL}^K\circ \mathrm{res}_{K\cap {}^hL}^{{}^hL}\circ \mathrm{con}_L^h(x)] \\
&\quad = \frac{|L|}{|N_H(L)|}\sum_{hL\in N_H(L)/L}[{}^hL, e_{{}^hL}^{({}^hL)}\cdot \mathrm{con}_L^h(x)] \\
&\quad = [L,x]
\end{aligned}$$

for all $H\leq G$ and $[L,x]\in(\mathcal{K}^X)_+(H)$ with $L\in \mathrm{Cl}(H)\cap\mathcal{C}(H)$. Consequently, $\Psi^{X,\mathcal{K}^X}$ is the inverse of $\Theta^{X,\mathcal{K}^X}$. This completes the proof. □

Remark 5.7 By Proposition 2.2, Lemma 5.2(a), and the proof of Proposition 5.6,

$$\begin{aligned}
\beta_H^{-1}((\overline{x_K})_{K\leq H}) &= \Theta_H^{X,\mathcal{K}^X}\circ \frac{1}{|H|}\eta_H^{\mathcal{K}^X}\circ\Delta_H((\overline{x_K})_{K\leq H}) \\
&= \frac{1}{|H|}\sum_{K\in\mathcal{P}(X)}\sum_{U\leq K}|U|\mu(U,K)\mathrm{ind}_U^H\circ \mathrm{res}_U^K(e_K^{(K)}\cdot x_K) \\
&= \sum_{K\in \mathrm{Cl}(H)\cap\mathcal{P}(X)}\frac{|K|}{|N_H(K)|}\mathrm{ind}_K^H(e_K^{(K)}\cdot x_K)
\end{aligned}$$

for all $(\overline{x_K})_{K\leq H}\in \overline{X}^+(H)$ (cf. [35, Proposition 12.5]). Hence

$$x = \sum_{K\in \mathrm{Cl}(H)\cap\mathcal{P}(X)}\frac{|K|}{|N_H(K)|}\mathrm{ind}_K^H(e_K^{(K)}\cdot \mathrm{res}_K^H(x))$$

for all $x\in X(H)$ (see also the final statement of [35, Section 7]). This, combined with (III), yields the induction formula given in Corollary 5.4.

6 Induction formulae for crossed Mackey functors

Let $S\in G$-**set**, and let $X\in \mathbf{Mack}(G;S)_k$. A subgroup H of G is said to be primordial for X if $\mathcal{T}^{X_s}(H)\neq X_s(H)$ for some $s\in C_S(H)$, and is said to be coprimordial for X if $\mathcal{K}^{X_s}(H) \neq \{0\}$ for some $s \in C_S(H)$. Let $\mathcal{P}(X)$ be the set of primordial subgroups of G, and let $\mathcal{C}(X)$ be the set of coprimordial subgroups of G.

We denote by $\mathcal{K}^X$ the restriction bundle for $\mathrm{Stab}(G;S)$ over k composed of $\mathcal{K}^{X_s} \in \mathbf{Res}(G_s)_k$, $s \in S$, such that the crossed conjugation maps are the restriction of those of X. Recall that $(\mathcal{K}^X)_S$ denotes the crossed restriction functor on $\mathcal{K}^X$.

We now define

$$\overline{X}_S = (\overline{X}_S, \overline{\mathrm{con}}_S) \in \mathbf{Con}(G)_k$$

by

$$\overline{X}_S(H) = \prod_{s \in C_S(H)} \overline{X_s(H)} \quad \text{and} \quad \overline{\mathrm{con}}_S{}^g_H((\overline{x(s)})_{s \in C_S(H)}) = (\overline{\mathrm{con}_s{}^g_H(x(s))})_{{}^g s \in C_S({}^g H)}$$

for all $H \leq G$ and $g \in G$. If X is a Green functor and if $S \in G$-**mon**, then $\overline{X}_S$ denotes the algebra conjugation functor with multiplication on $\overline{X}_S(H)$ given by

$$(\overline{x(s)})_{s \in C_S(H)} (\overline{y(t)})_{t \in C_S(H)} = \left(\sum_{(s,t) \in C_S(H) \times C_S(H),\, st = r} \overline{x(s)y(t)} \right)_{r \in C_S(H)}.$$

Moreover, if X is a Green functor and if $S \in G$-**mon**, then we also define

$$\overline{X}_{\otimes S} = (\overline{X}_{\otimes S}, \overline{\mathrm{con}}_{\otimes S}) \in \mathbf{Con}_{\mathrm{alg}}(G)_k$$

by

$$\overline{X}_{\otimes S}(H) = \overline{X(H)} \otimes_k kC_S(H) \quad \text{and} \quad \overline{\mathrm{con}}_{\otimes S}{}^g_H(\overline{x} \otimes s) = \overline{\mathrm{con}^g_H(x)} \otimes {}^g s$$

for all $H \leq G$, $x \in X(H)$, $s \in C_S(H)$, and $g \in G$. In this case, each k-module $\overline{X}_{\otimes S}(H)$ with $H \leq G$ is considered to have an obvious k-algebra structure, so that the algebra conjugation functor $\overline{X}_{\otimes S}$ is isomorphic to $\overline{X}_S$.

Proposition 6.1 *Let $S \in G$-**set**, and let $X \in \mathbf{Mack}(G;S)_k$. If $|G|$ is invertible in k, then for any $H \leq G$, $\mathcal{K}^{X_S}(H) = (\mathcal{K}^X)_S(H)$, and the map*

$$\overline{X_S(H)} \to \overline{X}_S(H), \quad \overline{(x(s))_{s \in S}} \mapsto (\overline{x(s)})_{s \in C_S(H)}$$

is a k-module isomorphism. In particular, $\mathcal{C}(X_S) = \mathcal{C}(X)$ and $\mathcal{P}(X_S) = \mathcal{P}(X)$.

Proof. Let $H \leq G$. If $(x(s))_{s \in S} \in \mathcal{K}^{X_S}(H)$ and if $H_t \neq H$ with $t \in S$, then clearly $\mathrm{res}_S{}^H_{H_t}((x(s))_{s \in S}) = 0$, whence $x(t) = \mathrm{res}^{H_t}_{H_t}(x(t)) = 0$. This, combined with (III) and Proposition 5.5, shows that

$$\begin{aligned}
\mathcal{K}^{X_S}(H) &= \{e_H^{(H)} \cdot (x(s))_{s \in S} \mid (x(s))_{s \in S} \in X_S(H)\} \\
&= \left\{ (e_H^{(H)} \cdot x(s))_{s \in S} \in X_S(H) \;\middle|\; \begin{array}{l} x(s) \in X_s(H) \text{ if } s \in C_S(H), \text{ and} \\ x(s) = 0 \text{ if } s \notin C_S(H) \end{array} \right\} \\
&= \left\{ (x(s))_{s \in S} \in X_S(H) \;\middle|\; \begin{array}{l} x(s) \in \mathcal{K}^{X_s}(H) \text{ if } s \in C_S(H), \text{ and} \\ x(s) = 0 \text{ if } s \notin C_S(H) \end{array} \right\} \\
&= (\mathcal{K}^X)_S(H).
\end{aligned}$$

Thus the first assertion holds. Moreover, by Proposition 5.5,

$$X_S(H) = \mathcal{K}^{X_S}(H) \oplus \mathcal{T}^{X_S}(H) = (\mathcal{K}^X)_S(H) \oplus \mathcal{T}^{X_S}(H)$$

for all $H \leq G$, which, together with Proposition 5.5, yields the second assertion. This completes the proof. □

Given $A = (A, \mathrm{con}) \in \mathbf{Con}(G)_k$ and $K \leq H \leq G$, we set

$$A(K)^{N_H(K)} = \{x \in A(K) \mid \mathrm{con}_K^h(x) = x \text{ for all } h \in N_H(K)\}.$$

The following corollary is concerned with (I) (see Section 1 and Corollary 8.8).

Corollary 6.2 *Let $S \in G$-**set**, and let $X \in \mathbf{Mack}(G; S)_k$. Suppose that $|G|$ is invertible in k. Then the Mackey functor X_S is isomorphic to $(\overline{X}_S)^+$, and the map*

$$\begin{aligned} X_S(H) &\to \prod_{K \in \mathrm{Cl}(H) \cap \mathcal{P}(X)} \overline{X}_S(K)^{N_H(K)}, \\ (x(s))_{s \in S} &\mapsto \left(\left(\overline{\mathrm{res}_K^{H_s}(x(s))} \right)_{s \in C_S(K)} \right)_{K \in \mathrm{Cl}(H) \cap \mathcal{P}(X)} \end{aligned}$$

*with $H \leq G$ is a k-module isomorphism. Moreover, if X is a Green functor and if $S \in G$-**mon**, then the Green functor X_S is isomorphic to $(\overline{X}_{\otimes S})^+$, and the map*

$$\begin{aligned} X_S(H) &\to \prod_{K \in \mathrm{Cl}(H) \cap \mathcal{P}(X)} \overline{X}_{\otimes S}(K)^{N_H(K)}, \\ (x(s))_{s \in S} &\mapsto \left(\sum_{s \in C_S(K)} \overline{\mathrm{res}_K^{H_s}(x(s))} \otimes s \right)_{K \in \mathrm{Cl}(H) \cap \mathcal{P}(X)} \end{aligned}$$

with $H \leq G$ is a k-algebra isomorphism.

Proof. The corollary follows from Propositions 5.6 and 6.1. □

We next state an induction formula for X_S.

Corollary 6.3 *Let $S \in G$-**set**, and let $X \in \mathbf{Mack}(G; S)_k$. If $|G|$ is invertible in k, then*

$$(x(s))_{s \in S} = \sum_{K \in \mathrm{Cl}(H) \cap \mathcal{C}(X)} \frac{1}{|N_H(K)|} \sum_{U \leq K} |U| \mu(U, K) \mathrm{ind}_S{}_U^H \circ \mathrm{res}_S{}_U^H((x(s))_{s \in S})$$

for all $H \leq G$ and $(x(s))_{s \in S} \in X_S(H)$.

Proof. The assertion follows from Corollary 5.4 and Proposition 6.1. □

7 The twisted group algebra $\mathbb{C}^\alpha G$

From now on, we assume that $k = \mathbb{Z}$ and F is an algebraically closed field.

Let $E(G)$ be a finite dimensional F-algebra, and suppose that there exists a collection $\{E_g\}_{g\in G}$ of subspaces of $E(G)$ which satisfy $E_gE_r = E_{gr}$ for all g, $r \in G$ and $E(G) = \oplus_{g\in G}E_g$. Such an F-algebra $E(G)$ is called a fully G-graded F-algebra (see [2, Definition 1.1]). We call $\{E_g\}_{g\in G}$ a fully G-graded system on $E(G)$. Note that the identity of $E(G)$ is contained in E_ϵ (cf. [12]).

Let $H \le G$, and set $E(H) = \oplus_{h\in H}E_h$. Then $E(H)$ is a subalgebra of $E(G)$ with a fully H-graded system $\{E_h\}_{h\in H}$. Let $K \le H$. For each $M \in E(H)$-**mod**, $\mathrm{Eres}^H_K(M)$ denotes the restriction $M|_{E(K)}$ of M to $E(K)$. For each $N \in E(K)$-**mod**, $\mathrm{Eind}^H_K(N)$ denotes the induced $E(H)$-module $E(H) \otimes_{E(K)} N$. Given $N \in E(K)$-**mod** and $h \in H$, we define a conjugate $E({}^hK)$-module $\mathrm{Econ}^h_K(N)$ to be the component

$$E_h \otimes_{E(K)} N = \{u \otimes v \mid u \in E_h \quad \text{and} \quad v \in N\}$$

of $\mathrm{Eind}^H_K(N)$ with the action given by left multiplication in the first factor.

For each $H \le G$, let $R(E(H))$ be the additive group consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of finitely generated left $E(H)$-modules with direct sum for addition. There exist conjugation, restriction, and induction maps

$$\begin{aligned}
&\mathrm{Econ}^g_H : R(E(H)) \to R(E({}^gH)), \quad [M] \mapsto [\mathrm{Econ}^g_H(M)],\\
&\mathrm{Eres}^H_K : R(E(H)) \to R(E(K)), \quad [M] \mapsto [\mathrm{Eres}^H_K(M)],\\
&\mathrm{Eind}^H_K : R(E(K)) \to R(E(H)), \quad [N] \mapsto [\mathrm{Eind}^H_K(N)]
\end{aligned}$$

for $K \le H \le G$ and $g \in G$, where $M \in E(H)$-**mod** and $N \in E(K)$-**mod**. These maps are simply denoted by Econ, Eres, and Eind.

We are now ready to quote Mackey's theorem (cf. [2, Theorem 2.2]).

Theorem 7.1 *Let $E(G)$ be a fully G-graded F-algebra with a fully G-graded system $\{E_g\}_{g\in G}$, and let K, $U \le H \le G$. Then for any $x \in R(E(U))$,*

$$\mathrm{Eres}^H_K \circ \mathrm{Eind}^H_U(x) = \bigoplus_{KhU\in K\backslash H/U} \mathrm{Eind}^K_{K\cap {}^hU} \circ \mathrm{Eres}^{{}^hU}_{K\cap {}^hU} \circ \mathrm{Econ}^h_U(x).$$

By Theorem 7.1, the family of $\mathbb{Z}$-modules $RE(H) := R(E(H))$, $H \le G$, together with Econ, Eres, and Eind, defines $RE = (RE, \mathrm{Econ}, \mathrm{Eres}, \mathrm{Eind}) \in \mathbf{Mack}(G)_{\mathbb{Z}}$. We call this Mackey functor the $E(G)$-representation functor.

Let $\alpha : G \times G \to F^\times$ be a normalized 2-cocycle, that is,

$$\alpha(rs, t)\alpha(r, s) = \alpha(r, st)\alpha(s, t)$$

for all r, s, $t \in G$, and $\alpha(s, t) = 1$ whenever either s or t is equal to ϵ. Given $H \le G$, we denote by $F^\alpha H$ the F-algebra with a basis $\{\overline{s}\}_{s\in H}$ and multiplication given by

$$\overline{s}\,\overline{t} = \alpha(s, t)\overline{st}$$

for all $s, t \in H$, and call it the twisted group algebra. Observe that $F^\alpha G$ is a fully G-graded F-algebra with a fully G-graded system $\{F\overline{s}\}_{s\in G}$. We now write $R_\alpha(H) = R(F^\alpha H)$ for all $H \leq G$, and denote by

$$R_\alpha = (R_\alpha, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \ (\in \mathbf{Mack}(G)_{\mathbb{Z}})$$

the $F^\alpha G$-representation functor.

Given $H \leq G$ and $M \in \mathbb{C}^\alpha H$-**mod**, we define a map $\chi_M : H \to \mathbb{C}$ by

$$\chi_M(h) = \mathrm{Tr}(\overline{h}, M)$$

for all $h \in H$, and call it the α-character of H afforded by M, where $\mathrm{Tr}(\overline{h}, M)$ is the trace of the action of $\overline{h}$ on M (cf. [22, p. 351]).

If the characteristic of F does not divide $|G|$, then $F^\alpha H$ with $H \leq G$ is semisimple (see, *e.g.*, [22, Theorem 3.2.10]).

We prove directly, via the representation theory of $\mathbb{C}^\alpha G$, the following generalization of a well-known fact for the $\mathbb{C}G$-representation functor.

Lemma 7.2 *Suppose that $F = \mathbb{C}$. Then $\mathcal{C}(R_\alpha)$ is the set of cyclic subgroups of G.*

Proof. Let $H \leq G$. Suppose that $M \cong N$ with $M, N \in \mathbb{C}^\alpha H$-**mod**. Then it follows from [22, Proposition 7.1.9] that $\chi_M = \chi_N$. By [22, Theorem 7.1.10], the α-characters of H afforded by all nonisomorphic irreducible $\mathbb{C}^\alpha H$-modules are linearly independent. This means that, if H is not cyclic, then $\mathcal{K}^{R_\alpha}(H) = \{0\}$. Thus every coprimordial subgroup for R_α is cyclic. Suppose now that $H = \langle r \rangle$. We prove $H \in \mathcal{C}(R_\alpha)$. By the proof of [22, Lemma 5.8.13], there exists a map $\delta : H \to \mathbb{C}$ such that the map

$$\mathbb{C}^\alpha H \to \mathbb{C}H, \quad \overline{h} \mapsto \delta(h)h$$

is a $\mathbb{C}$-algebra isomorphism. Hence $\chi_M(r) \neq 0$ for some $M \in \mathbb{C}^\alpha H$-**mod**. Suppose now that R_α is extended to $\mathbb{Q}R_\alpha \in \mathbf{Mack}(G)_{\mathbb{Q}}$ by $\mathbb{Q}$-linearly. Then $[M] \notin \mathcal{T}^{\mathbb{Q}R_\alpha}(H)$, and thereby, $H \in \mathcal{P}(\mathbb{Q}R_\alpha)$. Obviously, $\mathcal{C}(\mathbb{Q}R_\alpha) = \mathcal{C}(R_\alpha)$. Moreover, it follows from Proposition 5.5 that $\mathcal{P}(\mathbb{Q}R_\alpha) = \mathcal{C}(\mathbb{Q}R_\alpha)$. Thus $H \in \mathcal{C}(R_\alpha)$, completing the proof. □

We provide another lemma (cf. [4, Example 9.7], [34, Lemma 8.2]).

Lemma 7.3 *Suppose that $F = \mathbb{C}$. Let $U \trianglelefteq K \leq G$, and suppose that K/U is cyclic. Let $N \in \mathbb{C}^\alpha U$-**mod** with $\dim_{\mathbb{C}}(N) = 1$, and suppose that for each $r \in K$, N is isomorphic to $\mathrm{con}_U^r(N)$. Let $M \in \mathbb{C}^\alpha K$-**mod**, and suppose that M is irreducible. If N is an irreducible constituent of $\mathrm{res}_U^K(M)$, then N is isomorphic to $\mathrm{res}_U^K(M)$.*

Proof. By [22, Theorem 6.2.4], N is extensible to a left $\mathbb{C}^\alpha K$-module. This, combined with [22, Corollary 6.4.4], shows that there exist precisely $e = |K : U|$ nonisomorphic left $\mathbb{C}^\alpha K$-modules M_i, $i = 1, \ldots, e$, extending N. Thus it follows from

[22, Theorem 5.6.2] that $\mathrm{ind}_U^K(N) = \oplus_{i=1}^e M_i$. Moreover, if N is an irreducible constituent of $\mathrm{res}_U^K(M)$, then M is an irreducible constituent $\mathrm{ind}_U^K(N)$, and thereby, $M \cong M_i$ for some i. This completes the proof. $\square$

8 The twisted quantum double $D^\omega(G)$ of a finite group

Let $(FG)^*$ be the F-algebra consisting of all F-linear maps from the group algebra FG to F with pointwise addition and multiplication. For each $s \in G$, we define an element ϕ_s of $(FG)^*$ by

$$\phi_s(g) = \begin{cases} 1 & \text{if } s = g \in G, \\ 0 & \text{if } s \neq g \in G. \end{cases}$$

The elements ϕ_s, $s \in G$, form an F-basis of $(FG)^*$.

Let $\omega : G \times G \times G \to F^\times$ be a normalized 3-cocycle, that is,

$$\omega(g, r, s)\omega(g, rs, t)\omega(r, s, t) = \omega(gr, s, t)\omega(g, r, st)$$

for all g, r, s, $t \in G$, and $\omega(g, r, s) = 1$ whenever one of g, r or s is equal to ϵ. Given g, r, $s \in G$, we define

$$\theta_s(g, r) = \frac{\omega(s, g, r)\omega(g, r, {}^{(gr)^{-1}}s)}{\omega(g, {}^{g^{-1}}s, r)}$$

and

$$\gamma_s(g, r) = \frac{\omega(g, r, s)\omega(s, {}^{s^{-1}}g, {}^{s^{-1}}r)}{\omega(g, s, {}^{s^{-1}}r)}.$$

The twisted quantum double $D^\omega(G)$ of G with respect to ω (cf. [14, 23, 26, 38]) is the quasi-triangular quasi-Hopf algebra with underlying vector space $(FG)^* \otimes_F FG$,

multiplication	$(\phi_s \otimes g)(\phi_t \otimes r) = \theta_s(g, r)\phi_s\phi_{{}^g t} \otimes gr,$
unit	$1_{D^\omega(G)} = \sum_{s \in G} \phi_s \otimes \epsilon,$
comultiplication	$\Delta(\phi_r \otimes g) = \sum_{s,\, t \in G,\, st = r} \gamma_g(s, t)(\phi_s \otimes g) \otimes (\phi_t \otimes g),$
counit	$\varepsilon(\phi_s \otimes g) = \delta_{s\epsilon},$
Drinfel'd associator	$\Phi = \sum_{r,\, s,\, t \in G} \omega(r, s, t)^{-1}(\phi_r \otimes \epsilon) \otimes (\phi_s \otimes \epsilon) \otimes (\phi_t \otimes \epsilon),$
universal R-matrix	$\mathcal{R} = \sum_{s,\, t \in G} (\phi_s \otimes \epsilon) \otimes (\phi_t \otimes s),$
antipode	$S(\phi_s \otimes g) = \theta_{s^{-1}}(g, g^{-1})^{-1}\gamma_g(s, s^{-1})^{-1}\phi_{{}^{g^{-1}}s^{-1}} \otimes g^{-1}.$

For verification, we need to apply the identities

$$\begin{aligned}
\theta_s(g,r)\theta_s(gr,t) &= \theta_{g^{-1}s}(r,t)\theta_s(g,rt), \\
\theta_{st}(g,r)\gamma_{gr}(s,t) &= \gamma_g(s,t)\gamma_r({}^{g^{-1}}s, {}^{g^{-1}}t)\theta_s(g,r)\theta_t(g,r), \\
\gamma_g(rs,t)\gamma_g(r,s)\omega({}^{g^{-1}}r, {}^{g^{-1}}s, {}^{g^{-1}}t) &= \gamma_g(r,st)\gamma_g(s,t)\omega(r,s,t)
\end{aligned}$$

for all g, r, s, $t \in G$. We denote by $R(D^\omega(G))$ the representation ring of $D^\omega(G)$, which is the commutative ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of finitely generated left $D^\omega(G)$-modules with direct sum for addition and tensor product for multiplication.

Let $H \leq G$. We define a subalgebra $D^\omega_G(H)$ of $D^\omega(G)$ to be

$$D^\omega_G(H) = \sum_{s\in G,\, h\in H} F\phi_s \otimes h.$$

We view each $h \in H$ as $\sum_{s\in G} \phi_s \otimes h \in D^\omega_G(H)$. Each $\phi_s \in (FG)^*$ where $s \in G$ is identified with $\phi_s \otimes \epsilon \in D^\omega_G(H)$.

We consider $D^\omega(G)$ to be a fully G-graded F-algebra with a fully G-graded system $\{\sum_{s\in G} F\phi_s \otimes g\}_{g\in G}$, and denote by

$$RD^\omega_G = (RD^\omega_G, \mathrm{Dcon}, \mathrm{Dres}, \mathrm{Dind}) \ (\in \mathbf{Mack}(G)_{\mathbb{Z}})$$

the $D^\omega(G)$-representation functor.

Let $H \leq G$ and $s \in G$. If g, r, $t \in H_s$, then

$$\theta_s(g,r) = \gamma_s(g,r) = \frac{\omega(s,g,r)\omega(g,r,s)}{\omega(g,s,r)}$$

and

$$\theta_s(tg,r)\theta_s(t,g) = \theta_s(t,gr)\theta_s(g,r).$$

Thus we obtain a normalized 2-cocycle

$$\theta_s : H_s \times H_s \to F^\times, \quad (g,r) \mapsto \theta_s(g,r).$$

We denote by G^c the G-monoid G on which G acts by conjugation ${}^r s$ with r, $s \in G$, and denote by $\overline{H\backslash G^c}$ a complete set of representatives of H-orbits in G^c.

For each $s \in G^c$, there exists a two-sided ideal $D^\omega_s(H)$ of $D^\omega_G(H)$ defined by

$$D^\omega_s(H) = \sum_{rH_s\in H/H_s} \sum_{h\in H} F\phi_{{}^r s} \otimes h.$$

Obviously, $D^\omega_G(H)$ is expressed as a direct sum of $D^\omega_s(H)$, $s \in \overline{H\backslash G^c}$, and thereby, every left $D^\omega_G(H)$-module M is decomposed into a direct sum of the submodules $D^\omega_s(H)M$, $s \in \overline{H\backslash G^c}$. Moreover, every left $D^\omega_s(H)$-module with $s \in G^c$ is naturally viewed as a left $D^\omega_G(H)$-modules.

Let $s \in G^c$, and define a left ideal $E_s^\omega(H)$ of $D_s^\omega(H)$ by

$$E_s^\omega(H) = \sum_{rH_s \in H/H_s} \sum_{h \in rH_s} F\phi_{{}^r s} \otimes h = \sum_{h \in H} F\phi_{{}^h s} \otimes h.$$

We identify the twisted group algebra $F^{\theta_s}H_s$ with $\sum_{h \in H_s} F\phi_s \otimes h$ which is a subspace of the F-space $E_s^\omega(H)$, and identify $\overline{h} \in F^{\theta_s}H_s$ for $h \in H_s$ with $\phi_s \otimes h \in E_s^\omega(H)$. In this context, $E_s^\omega(H)$ is considered as a right $F^{\theta_s}H_s$-module with the action given by right multiplication.

Given $M \in D_G^\omega(H)$-**mod** and $s \in G^c$, we set $\phi_s M = \{\phi_s x \mid x \in M\}$ and view it as a left $F^{\theta_s}H_s$-module with the action given by left multiplication.

We state a fundamental lemma about representations of $D_G^\omega(H)$ with $H \leq G$, which is similar to [38, Lemma 1.1].

Lemma 8.1 *Let $H \leq G$, and let $s \in G^c$. Then there exists an equivalence between the categories $F^{\theta_s}H_s$-**mod** and $D_s^\omega(H)$-**mod** given by the functors*

$$\zeta_{H,s}^1 : F^{\theta_s}H_s\text{-}\mathbf{mod} \to D_s^\omega(H)\text{-}\mathbf{mod}, \quad N \mapsto E_s^\omega(H) \otimes_{F^{\theta_s}H_s} N$$

and

$$\zeta_{H,s}^2 : D_s^\omega(H)\text{-}\mathbf{mod} \to F^{\theta_s}H_s\text{-}\mathbf{mod}, \quad M \mapsto \phi_s M,$$

where $D_s^\omega(H)$ acts on $E_s^\omega(H) \otimes_{F^{\theta_s}H_s} N$ by left multiplication in the first factor.

Proof. Let $M \in D_s^\omega(H)$-**mod**, and let $N \in F^{\theta_s}H_s$-**mod**. The map

$$N \to \phi_s E_s^\omega(H) \otimes_{F^{\theta_s}H_s} N, \quad x \mapsto \phi_s \otimes x$$

is an $F^{\theta_s}H_s$-module isomorphism. We define a map $f : M \to E_s^\omega(H) \otimes_{F^{\theta_s}H_s} \phi_s M$ by

$$f(x) = \sum_{rH_s \in H/H_s} \frac{1}{\theta_{{}^r s}(r, r^{-1})} (\phi_{{}^r s} \otimes r) \otimes (\phi_s \otimes r^{-1})x$$

for all $x \in M$. This map is independent of the choice of representatives r of H/H_s, because

$$\theta_{{}^r s}(rt, (rt)^{-1})\theta_s(t^{-1}, r^{-1}) = \theta_{{}^r s}(r, r^{-1})\theta_{{}^r s}(rt, t^{-1})$$

for all $r \in H$ and $t \in H_s$. Let h, h', $r \in H$, and suppose that ${}^h s = {}^{h'r} s$. Then

$$\phi_{{}^h s} \otimes h' = \frac{1}{\theta_{{}^h s}(h, tr^{-1})} (\phi_{{}^h s} \otimes h)(\phi_s \otimes tr^{-1})$$

for some $t \in H_s$. We have

$$\theta_{{}^h s}(h, h^{-1}) = \theta_s(h^{-1}, h) \quad \text{and} \quad \theta_s(tr^{-1}, r)\theta_s(t, r^{-1}) = \theta_{{}^r s}(r, r^{-1}).$$

Hence

$$\begin{aligned}(\phi_{{}^h s} \otimes h)f(x) &= (\phi_{{}^h s} \otimes h) \otimes \phi_s x \\ &= \frac{1}{\theta_s(h^{-1}, h)}(\phi_{{}^h s} \otimes h) \otimes (\phi_s \otimes h^{-1})(\phi_{{}^h s} \otimes h)x \\ &= f((\phi_{{}^h s} \otimes h)x)\end{aligned}$$

and

$$\begin{aligned}(\phi_s \otimes tr^{-1})f(x) &= \frac{\theta_s(tr^{-1}, r)}{\theta_{{}^r s}(r, r^{-1})}(\phi_s \otimes t) \otimes (\phi_s \otimes r^{-1})x \\ &= \frac{\theta_s(tr^{-1}, r)\theta_s(t, r^{-1})}{\theta_{{}^r s}(r, r^{-1})}\phi_s \otimes \phi_s(\phi_s \otimes tr^{-1})x \\ &= f((\phi_s \otimes tr^{-1})x)\end{aligned}$$

for all $x \in M$. This implies that f is a $D^\omega_s(H)$-module homomorphism. Moreover, the inverse $f^{-1} : E^\omega_s(H) \otimes_{F^{\theta_s}H_s} \phi_s M \to M$ of f is given by

$$f^{-1}((\phi_{{}^h s} \otimes h) \otimes \phi_s x) = (\phi_{{}^h s} \otimes h)x$$

for all $h \in H$ and $x \in M$. Thus the lemma holds. □

Keep the notation of Lemma 8.1. Let $s \in G^c$ and $g \in G$. Given $H \leq G_s$ and $N \in F^{\theta_s}H$-**mod**, we define an $F^{\theta_{{}^g s}}{}^gH$-module $\mathrm{con}_s{}^g_H(N)$ to be

$$\mathrm{con}_s{}^g_H(N) = \zeta^2_{{}^gH,\,{}^g s} \circ \mathrm{Dcon}^g_H \circ \zeta^1_{H,s}(N) = (\phi_{{}^g s} \otimes g) \otimes_{D^\omega_G(H)} (E^\omega_s(H) \otimes_{F^{\theta_s}H} N),$$

where $\mathrm{Dcon}^g_H \circ \zeta^1_{H,s}(N)$ is viewed as a left $D^\omega_{{}^g s}({}^gH)$-module. Given $H \leq G$ and $M \in D^\omega_G(H)$-**mod**, the map

$$\begin{gathered}\phi_{{}^g s}\mathrm{Dcon}^g_H(M)(= (\phi_{{}^g s} \otimes g) \otimes_{D^\omega_G(H)} M) \to \mathrm{con}_s{}^g_{H_s}(\phi_s M), \\ (\phi_{{}^g s} \otimes g) \otimes x \mapsto (\phi_{{}^g s} \otimes g) \otimes (\phi_s \otimes \phi_s x)\end{gathered}$$

is an $F^{\theta_{{}^g s}}{}^gH_{{}^g s}$-module isomorphism.

To study $D^\omega(G)$-representation functor, we also require the next lemma.

Lemma 8.2 *Let $H \leq G$, $s \in G^c$, and $h \in H$. The following statements hold.*

(a) *For any $N \in F^{\theta_s}H_s$-**mod**,*

$$\zeta^1_{H,s}(N) \cong \zeta^1_{H,\,{}^h s} \circ \mathrm{con}_s{}^h_{H_s}(N)$$

as $D^\omega_G(H)$-modules.

(b) *For any $M \in D^\omega_G(H)$-**mod**,*

$$\phi_{{}^h s}M \cong \mathrm{con}_s{}^h_{H_s}(\phi_s M)$$

as $F^{\theta_{{}^h s}}H_{{}^h s}$-modules.

Proof. (a) Observe that $\zeta^1_{H,s}(N) = E^\omega_s(H) \otimes_{F^{\theta_s}H_s} N$ and

$$\begin{aligned}\zeta^1_{H,{}^hs} \circ \mathrm{con}_s{}^h_{H_s}(N) &= \zeta^1_{H,{}^hs} \circ \zeta^2_{H_{{}^hs},{}^hs} \circ \mathrm{Dcon}^h_{H_s} \circ \zeta^1_{H_s,s}(N) \\ &= E^\omega_{{}^hs}(H) \otimes_{F^{\theta_{{}^hs}}H_{{}^hs}} ((\phi_{{}^hs} \otimes h) \otimes_{D^\omega_G(H_s)} (E^\omega_s(H_s) \otimes_{F^{\theta_s}H_s} N)).\end{aligned}$$

We define a map $f_1 : \zeta^1_{H,s}(N) \to \zeta^1_{H,{}^hs} \circ \mathrm{con}_s{}^h_{H_s}(N)$ by

$$f_1((\phi_{{}^rs} \otimes r) \otimes x) = \frac{1}{\theta_{{}^rs}(rh^{-1}, h)}(\phi_{{}^rs} \otimes rh^{-1}) \otimes ((\phi_{{}^hs} \otimes h) \otimes (\phi_s \otimes x))$$

for all $r \in H$ and $x \in N$. Let $r \in H$ and $x \in N$. For any $t \in H_s$,

$$\theta_{{}^hs}(h,t)\theta_s(h^{-1}, hth^{-1})\theta_s(th^{-1}, h) = \theta_s(h^{-1}, h)\theta_{{}^hs}(hth^{-1}, h)$$

and

$$\theta_{{}^rs}(r, th^{-1})\theta_{{}^rs}(rth^{-1}, h) = \theta_{{}^rs}(r, t)\theta_s(th^{-1}, h),$$

whence

$$\begin{aligned}f_1((\phi_s \otimes \bar{t}x)) &= \frac{1}{\theta_s(h^{-1}, h)}(\phi_s \otimes h^{-1}) \otimes ((\phi_{{}^hs} \otimes h) \otimes (\phi_s \otimes \bar{t}x)) \\ &= \frac{\theta_{{}^hs}(h,t)}{\theta_s(h^{-1}, h)}(\phi_s \otimes h^{-1}) \otimes ((\phi_{{}^hs} \otimes ht) \otimes (\phi_s \otimes x)) \\ &= \frac{\theta_{{}^hs}(h,t)\theta_s(h^{-1}, hth^{-1})}{\theta_s(h^{-1}, h)\theta_{{}^hs}(hth^{-1}, h)}(\phi_s \otimes th^{-1}) \otimes ((\phi_{{}^hs} \otimes h) \otimes (\phi_s \otimes x)) \\ &= f_1((\phi_s \otimes t) \otimes x)\end{aligned}$$

and

$$\begin{aligned}(\phi_{{}^rs} \otimes r)f_1((\phi_s \otimes t) \otimes x) &= \frac{\theta_{{}^rs}(r, th^{-1})}{\theta_s(th^{-1}, h)}(\phi_{{}^rs} \otimes rth^{-1}) \otimes ((\phi_{{}^hs} \otimes h) \otimes (\phi_s \otimes x)) \\ &= \frac{\theta_{{}^rs}(r,t)}{\theta_{{}^rs}(rth^{-1}, h)}(\phi_{{}^rs} \otimes rth^{-1}) \otimes ((\phi_{{}^hs} \otimes h) \otimes (\phi_s \otimes x)) \\ &= f_1(\theta_{{}^rs}(r,t)(\phi_{{}^rs} \otimes rt) \otimes x) \\ &= f_1((\phi_{{}^rs} \otimes r)(\phi_s \otimes t) \otimes x).\end{aligned}$$

Thus

$$\begin{aligned}f_1((\phi_{{}^rs} \otimes r)(\phi_s \otimes t) \otimes x) &= (\phi_{{}^rs} \otimes r)f_1((\phi_s \otimes t) \otimes x) \\ &= (\phi_{{}^rs} \otimes r)f_1((\phi_s \otimes \bar{t}x)) \\ &= f_1((\phi_{{}^rs} \otimes r) \otimes \bar{t}x)\end{aligned}$$

for all $t \in H_s$, and thereby, f_1 is well-defined. Obviously, f_1 is a bijection. Let h', h'', $r \in H$, and suppose that ${}^{h'}s = {}^{h''r}s$. Then

$$\phi_{{}^{h'}s} \otimes h'' = \frac{1}{\theta_{{}^{h'}s}(h', tr^{-1})}(\phi_{{}^{h'}s} \otimes h')(\phi_s \otimes tr^{-1})$$

for some $t \in H_s$. By the preceding argument,

$$(\phi_{\,{}^{h'}s} \otimes h')f_1((\phi_s \otimes t) \otimes x) = f_1((\phi_{\,{}^{h'}s} \otimes h')(\phi_s \otimes t) \otimes x).$$

Moreover, since

$$\theta_s(tr^{-1}, rh^{-1})\theta_s(th^{-1}, h) = \theta_s(tr^{-1}, r)\theta_{\,{}^{r}s}(rh^{-1}, h),$$

it follows that

$$\begin{aligned}
&(\phi_s \otimes tr^{-1})f_1((\phi_{\,{}^{r}s} \otimes r) \otimes x) \\
&\qquad = \frac{\theta_s(tr^{-1}, rh^{-1})}{\theta_{\,{}^{r}s}(rh^{-1}, h)}(\phi_s \otimes th^{-1}) \otimes ((\phi_{\,{}^{h}s} \otimes h) \otimes (\phi_s \otimes x)) \\
&\qquad = \frac{\theta_s(tr^{-1}, r)}{\theta_s(th^{-1}, h)}(\phi_s \otimes th^{-1}) \otimes ((\phi_{\,{}^{h}s} \otimes h) \otimes (\phi_s \otimes x)) \\
&\qquad = f_1(\theta_s(tr^{-1}, r)(\phi_s \otimes t) \otimes x) \\
&\qquad = f_1((\phi_s \otimes tr^{-1})(\phi_{\,{}^{r}s} \otimes r) \otimes x).
\end{aligned}$$

This means that f_1 is a $D^\omega_G(H)$-module isomorphism. Consequently, (a) holds.
(b) Since $\mathrm{Dcon}^h_H(M) \cong M$ as $D^\omega_G(H)$-modules, it follows that

$$\phi_{\,{}^{h}s}M \cong \phi_{\,{}^{h}s}\mathrm{Dcon}^h_H(M) \cong \mathrm{con}_s{}^h_{H_s}(\phi_s M)$$

as $F^{\theta_{\,{}^{h}s}}H_{\,{}^{h}s}$-modules. Thus (b) holds.

We give an alternative proof of (b). Observe that

$$\begin{aligned}
\mathrm{con}_s{}^h_{H_s}(\phi_s M) &= \phi_{\,{}^{h}s}\mathrm{Dcon}^h_{H_s} \circ \zeta^1_{H_s,s}(\phi_s M) \\
&= (\phi_{\,{}^{h}s} \otimes h) \otimes_{D^\omega_G(H_s)} (E^\omega_s(H_s) \otimes_{F^{\theta_s}H_s} \phi_s M).
\end{aligned}$$

We define a map $f_2 : \phi_{\,{}^{h}s}M \to \mathrm{con}_s{}^h_{H_s}(\phi_s M)$ by

$$f_2(\phi_{\,{}^{h}s}x) = \frac{1}{\theta_{\,{}^{h}s}(h, h^{-1})}(\phi_{\,{}^{h}s} \otimes h) \otimes (\phi_s \otimes (\phi_s \otimes h^{-1})x)$$

for all $x \in M$. Since $\theta_{\,{}^{h}s}(h, h^{-1}) = \theta_s(h^{-1}, h)$, it follows that f_2 is a bijection. Let $r \in H_{\,{}^{h}s}$. Then

$$\theta_{\,{}^{h}s}(r, h)\theta_s(h^{-1}rh, h^{-1}) = \theta_{\,{}^{h}s}(h, h^{-1}rh)\theta_s(h^{-1}, r),$$

and thereby,

$$\begin{aligned}
\overline{r}f_2(\phi_{\,{}^{h}s}x) &= \frac{\theta_{\,{}^{h}s}(r, h)}{\theta_{\,{}^{h}s}(h, h^{-1})}(\phi_{\,{}^{h}s} \otimes rh) \otimes (\phi_s \otimes (\phi_s \otimes h^{-1})x) \\
&= \frac{\theta_{\,{}^{h}s}(r, h)\theta_s(h^{-1}rh, h^{-1})}{\theta_{\,{}^{h}s}(h, h^{-1})\theta_{\,{}^{h}s}(h, h^{-1}rh)}(\phi_{\,{}^{h}s} \otimes h) \otimes (\phi_s \otimes (\phi_s \otimes h^{-1}r)x) \\
&= \frac{1}{\theta_{\,{}^{h}s}(h, h^{-1})}(\phi_{\,{}^{h}s} \otimes h) \otimes (\phi_s \otimes (\phi_s \otimes h^{-1})\overline{r}\phi_{\,{}^{h}s}x) \\
&= f_2(\overline{r}\phi_{\,{}^{h}s}x)
\end{aligned}$$

for all $x \in M$. Hence f_2 is an $F^{\theta_{h_s}}H_{h_s}$-module isomorphism, completing the proof. $\square$

There exists a family of $\mathbb{Z}$-lattice homomorphisms

$$\mathrm{con}_{s\,H}^{\;g} : R(F^{\theta_s}H) \to R(F^{\theta_{g_s}}{}^gH)$$

for $s \in G^c$, $H \leq G_s$, and $g \in G$ such that

$$\mathrm{con}_{s\,H}^{\;g}([N]) = [\mathrm{con}_{s\,H}^{\;g}(N)] = [\zeta^2_{{}^gH,\,{}^gs} \circ \mathrm{Dcon}_H^g \circ \zeta^1_{H,s}(N)]$$

for all $N \in F^{\theta_s}H$-**mod**, which is called the crossed conjugation maps. The following lemma asserts that this family satisfies the axioms of crossed conjugation maps.

Lemma 8.3 *Let $s \in G^c$, and suppose that $R_{\theta_s} = (R_{\theta_s}, \mathrm{con}, \mathrm{res}, \mathrm{ind})$ is the $F^{\theta_s}G_s$-representation functor. Then*

$$\begin{aligned}
&(\mathrm{C.0})\ \ \mathrm{con}_{s\,H}^{\;t} = \mathrm{con}_H^t,\\
&(\mathrm{C.1})\ \ \mathrm{con}_{{}^rs\,{}^rH}^{\;g} \circ \mathrm{con}_{s\,H}^{\;r} = \mathrm{con}_{s\,H}^{\;gr},\\
&(\mathrm{C.2})\ \ \mathrm{con}_{s\,K}^{\;g} \circ \mathrm{res}_K^H = \mathrm{res}_{{}^gK}^{{}^gH} \circ \mathrm{con}_{s\,H}^{\;g},\\
&(\mathrm{C.3})\ \ \mathrm{con}_{s\,H}^{\;g} \circ \mathrm{ind}_K^H = \mathrm{ind}_{{}^gK}^{{}^gH} \circ \mathrm{con}_{s\,K}^{\;g}
\end{aligned}$$

for all $K \leq H \leq G_s$, $g, r \in G$, and $t \in G_s$.

Proof. Let $H \leq G_s$. Observe that $D_s^\omega(H) = E_s^\omega(H) = \sum_{h\in H} F\phi_s \otimes h$. Then

$$\begin{aligned}
&\mathrm{con}_H^t([M]) = [\zeta^2_{{}^tH,s} \circ \mathrm{Dcon}_H^t \circ \zeta^1_{H,s}(M)],\\
&\mathrm{res}_K^H([M]) = [\zeta^2_{K,s} \circ \mathrm{Dres}_K^H \circ \zeta^1_{H,s}(M)],\\
&\mathrm{ind}_K^H([N]) = [F^{\theta_s}H \otimes_{F^{\theta_s}K} N] = [\zeta^2_{H,s} \circ \mathrm{Dind}_K^H \circ \zeta^1_{K,s}(N)]
\end{aligned}$$

for all $t \in G_s$, $K \leq H$, $M \in F^{\theta_s}H$-**mod**, and $N \in F^{\theta_s}K$-**mod**, where $\mathrm{Dres}_K^H \circ \zeta^1_{H,s}(M)$ is viewed as a left $D_s^\omega(K)$-module, and $\mathrm{Dind}_K^H \circ \zeta^1_{K,s}(N)$ is viewed as a left $D_s^\omega(H)$-module. Hence (C.0)–(C.3) follow from Lemma 8.1. This completes the proof. $\square$

By Lemma 8.3, the Mackey functors

$$R_s^\theta := R_{\theta_s} = (R_{\theta_s}, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Mack}(G_s)_{\mathbb{Z}}, \quad s \in G^c,$$

together with the crossed conjugation maps $\mathrm{con}_{s\,H}^{\;g} : R(F^{\theta_s}H) \to R(F^{\theta_{g_s}}{}^gH)$ for $s \in G^c$, $H \leq G_s$, and $g \in G$, defines a Mackey bundle for $\mathrm{Stab}(G; G^c)$ over $\mathbb{Z}$, where R_{θ_s} is the $F^{\theta_s}G_s$-representation functor. We denote this Mackey bundle by R^θ.

Recall that $R^\theta_{G^c}$ denotes the crossed Mackey functor on R^θ. Let $H \leq G$. We now define $\mathbb{Z}$-lattice homomorphisms

$$\Gamma_H : RD_G^\omega(H) \to R^\theta_{G^c}(H), \quad [M] \mapsto ([\phi_s M])_{s\in G^c} = ([\zeta^2_{H,s}(D_s^\omega(H)M)])_{s\in G^c}$$

and

$$\Gamma'_H : R^{\theta}_{G^c}(H) \to RD^{\omega}_G(H), \quad ([N(s)])_{s\in G^c} \mapsto \sum_{s\in \overline{H\backslash G^c}} [\zeta^1_{H,s}(N(s))].$$

By virtue of Lemma 8.2, this definition makes sense. From Lemma 8.1, we know that $\Gamma_H \circ \Gamma'_H = \mathrm{id}_{R^{\theta}_{G^c}(H)}$ and $\Gamma'_H \circ \Gamma_H = \mathrm{id}_{RD^{\omega}_G(H)}$. Thus $\Gamma_H^{-1} = \Gamma'_H$.

The following theorem is a key to induction formulae for RD^{ω}_G.

Theorem 8.4 *The Mackey functor RD^{ω}_G is isomorphic to $R^{\theta}_{G^c}$. Really, the family of $\mathbb{Z}$-lattice isomorphisms $\Gamma_H : RD^{\omega}_G(H) \to R^{\theta}_{G^c}(H)$, $H \le G$, defines an isomorphism $\Gamma : RD^{\omega}_G \to R^{\theta}_{G^c}$ of Mackey functors.*

Proof. Let $K \le H \le G$, and let $g \in G$. Obviously, the diagrams

$$\begin{array}{ccc} RD^{\omega}_G(H) & \xrightarrow{\Gamma_H} & R^{\theta}_{G^c}(H) \\ {\scriptstyle \mathrm{Dcon}^g_H}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{con}_{G^c}{}^g_H} \\ RD^{\omega}_G({}^gH) & \xrightarrow[\Gamma_{{}^gH}]{} & R^{\theta}_{G^c}({}^gH) \end{array} \quad \text{and} \quad \begin{array}{ccc} RD^{\omega}_G(H) & \xrightarrow{\Gamma_H} & R^{\theta}_{G^c}(H) \\ {\scriptstyle \mathrm{Dres}^H_K}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{res}_{G^c}{}^H_K} \\ RD^{\omega}_G(K) & \xrightarrow[\Gamma_K]{} & R^{\theta}_{G^c}(K) \end{array}$$

are commutative. Let $N \in D^{\omega}_G(K)$-**mod**. Then $\mathrm{Dind}^H_K(N) = D^{\omega}_G(H) \otimes_{D^{\omega}_G(K)} N$. Let $s \in G^c$, and let $\{h_1, \ldots, h_\ell\}$ be a complete set of representatives of $H_s\backslash H/K$. For each integer i with $1 \le i \le \ell$, let $\{r_{i1}, \cdots, r_{in_i}\}$ be a left transversal of $H_s \cap {}^{h_i}K$ in H_s. Obviously, $\{r_{i1}h_i, \cdots, r_{in_i}h_i \mid i = 1, \ldots, \ell\}$ is a left transversal of K in H. We now obtain

$$\mathrm{Dind}^H_K(N) = D^{\omega}_G(H) \otimes_{D^{\omega}_G(K)} N = \sum_{i=1}^{\ell}\sum_{j=1}^{n_i} F r_{ij}h_i \otimes_{D^{\omega}_G(K)} N.$$

Set $t_i = {}^{h_i^{-1}}s$, $i = 1, \ldots, \ell$. Then

$$\mathrm{con}_{t_i}{}^{h_i}_{K_{t_i}}(\phi_{t_i}N) = (\phi_s \otimes h_i) \otimes_{D^{\omega}_G(K_{t_i})} (E^{\omega}_{t_i}(K_{t_i}) \otimes_{F^{\theta_{t_i}}K_{t_i}} \phi_{t_i}N)$$

for all i, and the map

$$\phi_s \mathrm{Dind}^H_K(N) \to \sum_{i=1}^{\ell} F^{\theta_s}H_s \otimes_{F^{\theta_s}({}^{h_i}K)_s} \mathrm{con}_{t_i}{}^{h_i}_{K_{t_i}}(\phi_{t_i}N),$$
$$(\phi_s \otimes r_{ij}h_i) \otimes x \mapsto \overline{r_{ij}} \otimes ((\phi_s \otimes h_i) \otimes (\phi_{t_i} \otimes \phi_{t_i}x))$$

is an $F^{\theta_s}H_s$-module isomorphism. We now conclude that the diagram

$$\begin{array}{ccc} RD^{\omega}_G(H) & \xrightarrow{\Gamma_H} & R^{\theta}_{G^c}(H) \\ {\scriptstyle \mathrm{Dind}^H_K}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{ind}_{G^c}{}^H_K} \\ RD^{\omega}_G(K) & \xrightarrow[\Gamma_K]{} & R^{\theta}_{G^c}(K) \end{array}$$

is commutative. This completes the proof. □

Remark 8.5 Suppose that $F = \mathbb{C}$. Let $\mathrm{conj}(G)$ be a full set of nonconjugate elements in G, which is regarded as $\overline{G\backslash G^c}$. By the proof of [38, Theorem 2.2], the map

$$\mathbb{C} \otimes_{\mathbb{Z}} R^{\theta}_{G^c}(G) \to \prod_{s\in\mathrm{conj}(G)} Z(\mathbb{C}^{\theta_s}G_s), \quad ([M_s])_{s\in G^c} \mapsto \left(\sum_{g\in G_s} \mathrm{Tr}\,(\overline{s}, M_g)\overline{g}\right)_{s\in\mathrm{conj}(G)},$$

where $Z(\mathbb{C}^{\theta_s}G_s)$ is the center of $\mathbb{C}^{\theta_s}G_s$, is a $\mathbb{C}$-space isomorphism. Moreover, from Theorem 8.4 and the proof of [38, Lemma 2.1], we know that the map

$$\mathbb{C} \otimes_{\mathbb{Z}} R(D^{\omega}(G)) \to \prod_{s\in\mathrm{conj}(G)} Z(\mathbb{C}^{\theta_s}G_s), \quad [M] \mapsto \left(\sum_{g\in G_s} \mathrm{Tr}\,(\overline{s}, \phi_g M)\overline{g}\right)_{s\in\mathrm{conj}(G)}$$

is a $\mathbb{C}$-algebra isomorphism, which was proved by Witherspoon [38, Theorem 2.2] (see also [24, 2.2(g)] and [37, p. 316]).

If ω is trivial, that is, $\omega(g, r, s) = 1$ for all g, r, $s \in G$, then we simply write $D(G) = D^{\omega}(G)$, $D_G(H) = D^{\omega}_G(H)$ with $H \leq G$, and $RD_G = RD^{\omega}_G$. The $\mathbb{C}$-algebra $D(G)$ is called the quantum double of G (cf. [14, 25, 37]).

For each $H \leq G$, $R(D_G(H))$ denotes the ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of finitely generated left $D_G(H)$-modules with direct sum for addition and tensor product for multiplication. Given $K \leq H \leq G$, $M \in D_G(H)$-**mod**, and $N \in D_G(K)$-**mod**, the maps

$$M \otimes (D_G(H) \otimes_{D_G(K)} N) \to D_G(H) \otimes_{D_G(K)} (M|_{D_G(K)} \otimes N),$$
$$u \otimes (h \otimes v) \mapsto h \otimes (h^{-1}u \otimes v)$$

and

$$(D_G(H) \otimes_{D_G(K)} N) \otimes M \to D_G(H) \otimes_{D_G(K)} (N \otimes M|_{D_G(K)}),$$
$$(h \otimes v) \otimes u \mapsto h \otimes (v \otimes h^{-1}u),$$

where $h \in H$, are $D_G(H)$-module isomorphisms. These facts mean that Frobenius axioms hold for RD_G. Thus RD_G is a Green functor (cf. [37, Section 5]).

Let $a(G)$ be the representation ring of FG, that is, the commutative ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes of finitely generated left FG-modules with direct sum for addition and tensor product for multiplication (see, *e.g.*, [11, §80D]). We define

$$a = (a, \mathrm{con}, \mathrm{res}, \mathrm{ind}) \in \mathbf{Green}(G)_{\mathbb{Z}}$$

to be the family of $\mathbb{Z}$-algebras $a(H)$, $H \leq G$, with usual conjugation, restriction, and induction maps, and call it the FG-representation functor. If ω is trivial, then R^{θ} is the FG-representation functor. Recall that a_{G^c} denotes the crossed Mackey functor on a, which is obtained by the crossing of a by G^c.

There is an important consequence of Theorem 8.4 (cf. [30, Theorem 5.5]).

Corollary 8.6 *The Green functor RD_G is isomorphic to a_{G^c}. Really, the family of $\mathbb{Z}$-algebra isomorphisms $\Gamma_H : RD_G(H) \to a_{G^c}(H)$, $H \le G$, defines an isomorphism $\Gamma : RD_G \to a_{G^c}$ of Green functors.*

Proof. Let $H \le G$, and let $r \in G^c$. Given M_1, $M_2 \in D_G(H)$-**mod**, the map

$$\begin{aligned}
&\sum_{(s,t)\in\overline{H_r\backslash G^c\times G^c},\, st=r} \mathrm{ind}_{H_{s,t}}^{H_r}(\mathrm{res}_{H_{s,t}}^{H_s}(\phi_s M_1)\otimes_F \mathrm{res}_{H_{s,t}}^{H_t}(\phi_t M_2))\\
&\quad= \sum_{(s,t)\in\overline{H_r\backslash G^c\times G^c},\, st=r} FH_r \otimes_{FH_{s,t}} (\mathrm{res}_{H_{s,t}}^{H_s}(\phi_s M_1)\otimes_F \mathrm{res}_{H_{s,t}}^{H_t}(\phi_t M_2))\\
&\qquad\qquad \to \sum_{(s,t)\in G^c\times G^c,\, st=r} \phi_s M_1 \otimes_F \phi_t M_2 \cong \phi_r(M_1\otimes M_2),\\
&\qquad h\otimes(\phi_s x_1\otimes \phi_t x_2) \mapsto (\phi_{{}^h s}\otimes h)x_1 \otimes (\phi_{{}^h t}\otimes h)x_2
\end{aligned}$$

is an FH_r-modules isomorphism. Thus Γ_H is a $\mathbb{Z}$-algebra isomorphism. Consequently, the corollary follows from Theorem 8.4. This completes the proof. □

Remark 8.7 Keep the notation of Section 3, and assume further that $S = G^c$. We view each $(J,\pi) \in \mathbf{El}(G\text{-}\mathbf{set}, T_G^{\mathbb{Z}\otimes G^c})$ as the set of all pairs (x,π) for $x \in J$, and call (J,π) a crossed G-set (cf. [6, Definition 2.1], [17, Definition 4.2.1], [29, (1.2)]). Let $H \le G$, and let $s \in C_G(H)$. The G-map $\pi_s : G/H \to \prod_{U\le G} \mathbb{Z}C_G(U)$ is defined by

$$\pi_s(rH) = (\delta_{{}^r HU}\, {}^r s)_{U\le G}$$

for all $r \in G$ (see Remark 4.4). The F-span $\langle (G/H,\pi_s)\rangle_F$ of the crossed G-set $(G/H,\pi_s)$ is viewed as a left $D(G)$-module with the action given by

$$(\phi_t\otimes g)(rH,\pi_s) = \delta_{t\, {}^{gr}s}(grH,\pi_s)$$

for all g, r, $t \in G$ (cf. [39, p. 18]), and the F-span $\langle G_s/H\rangle_F$ of the G_s-set G_s/H is naturally is viewed as a left FG_s-module. Assume now that ω is trivial. Then

$$\zeta^1(\langle G_s/H\rangle_F) = \left(\sum_{r\in G} F\phi_{{}^r s}\otimes r\right)\otimes_{FG_s}\langle G_s/H\rangle_F,$$

and the map

$$\zeta^1(\langle G_s/H\rangle_F) \to \langle (G/H,\pi_s)\rangle_F, \quad (\phi_{{}^r s}\otimes r)\otimes H \mapsto (rH,\pi_s)$$

is a $D(G)$-module isomorphism. The isomorphism $\Theta : \mathrm{C}\Omega(-,G^c) \to \Omega_{G^c}$ of Green functors is defined in Remark 4.4, and the isomorphism $\Gamma : RD_G \to a_{G^c}$ of Green functors is defined in Corollary 8.6. We define $\Xi : \Omega_{G^c} \to a_{G^c}$ to be a family of $\mathbb{Z}$-algebra homomorphisms $\Xi_H : \Omega_{G^c}(H) \to a_{G^c}(H)$, $H \le G$, such that

$$\Xi_H(([J(t)])_{t\in G^c}) = ([\langle J(t)\rangle_F])_{t\in G^c},$$

where $J(t) \in H_t$-**set** and $\langle J(t)\rangle_F$ is the F-span of $J(t)$ viewed as a left FH_t-module. Clearly, $\Xi \in \mathbf{Green}(G)(\Omega_{G^c}, a_{G^c})_{\mathbb{Z}}$. We now conclude that

$$[\langle (G/H, \pi_s)\rangle_F] = [\zeta^1(\langle G_s/H\rangle_F)] = \Gamma_G^{-1} \circ \Xi_G \circ \Theta_G([G/H, \pi_s]).$$

We obtain another important consequence of Theorem 8.4, which includes (I) stated in Section 1 (see also [37, Theorem 5.5]).

Corollary 8.8 *Suppose that $F = \mathbb{C}$. Then the map*

$$\mathbb{Q} \otimes_{\mathbb{Z}} R(D^\omega(G)) \to \prod_{H \in \mathrm{Cl}(G,Cyc)} \mathbb{Q} \otimes_{\mathbb{Z}} \left(\prod_{s \in C_G(H)} \overline{R(\mathbb{C}^{\theta_s} H)} \right)^{N_G(H)},$$
$$[M] \mapsto \left(\left(\overline{\mathrm{res}_H^{G_s}(\phi_s M)} \right)_{s \in C_G(H)} \right)_{H \in \mathrm{Cl}(G,Cyc)}$$

is a $\mathbb{Q}$-space isomorphism. Moreover, the map

$$\mathbb{Q} \otimes_{\mathbb{Z}} R(D(G)) \to \prod_{H \in \mathrm{Cl}(G,Cyc)} \mathbb{Q} \otimes_{\mathbb{Z}} \left(\overline{a(H)} \otimes_{\mathbb{Z}} \mathbb{Z}C_G(H) \right)^{N_G(H)},$$
$$[M] \mapsto \left(\sum_{s \in C_G(H)} \overline{\mathrm{res}_H^{G_s}(\phi_s M)} \otimes s \right)_{H \in \mathrm{Cl}(G,Cyc)}$$

is a $\mathbb{Q}$-algebra isomorphism.

Proof. Suppose that R^θ is extended to $\mathbb{Q}R^\theta \in \mathbf{Mack}(G)_{\mathbb{Q}}$ by $\mathbb{Q}$-linearly, and suppose that a is extended to $\mathbb{Q}a \in \mathbf{Mack}(G)_{\mathbb{Q}}$ by $\mathbb{Q}$-linearly. Then it follows from Proposition 5.5 and Lemma 7.2 that both $\mathcal{P}(\mathbb{Q}R^\theta)$ and $\mathcal{P}(\mathbb{Q}a)$ are the set of cyclic subgroups of G. Hence the first assertion is a consequence of Corollary 6.2 with $X = \mathbb{Q}R^\theta$ and Theorem 8.4, and the second one is a consequence of Corollary 6.2 with $X = \mathbb{Q}a$ and Corollary 8.6. This completes the proof. □

We end this section with a canonical version of [28, Theorem 4.1], which states a generalization of Artin's induction theorem.

Corollary 8.9 *Suppose that $F = \mathbb{C}$. Then for any $M \in D^\omega(G)$-**mod**,*

$$[M] = \sum_{H \in \mathrm{Cl}(G,Cyc)} \frac{1}{|N_G(H)|} \sum_{K \leq H} |K|\mu(K,H)[D^\omega(G) \otimes_{D_G^\omega(K)} (M|_{D_G^\omega(K)})].$$

Proof. By an analogous argument to the proof of Corollary 8.8, the assertion follows from Corollary 6.3 with $X = \mathbb{Q}R^\theta$ and Theorem 8.4. This completes the proof. □

9 Fundamental theorems for the plus constructions

We continue to assume that $k = \mathbb{Z}$. Throughout this section, A denotes a restriction functor for G over $\mathbb{Z}$ and $\mathcal{B}$ a stable $\mathbb{Z}$-basis of A. Let $H \leq G$. We set

$$\mathcal{G}_A(H) = \coprod_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \mathbb{Z}.$$

For each $(K,\sigma) \in \mathfrak{R}(H,\mathcal{B})$, $N_H(K,\sigma)$ denotes the stabilizer of (K,σ) in H, that is,

$$N_H(K,\sigma) = \{h \in N_H(K) \mid \mathrm{con}_K^h(\sigma) = \sigma\}.$$

There exists a $\mathbb{Z}$-module isomorphism $\kappa_H^A : \mathcal{G}_A(H) \xrightarrow{\sim} A^+(H)$ given by

$$\kappa_H^A((\delta_{(K,\sigma)\,(U,\tau)})_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})}) = (y_L^{(K,\sigma)})_{L\leq H},$$

where

$$y_L^{(K,\sigma)} = \begin{cases} \displaystyle\sum_{hN_H(K,\sigma)\in N_H(K)/N_H(K,\sigma)} \mathrm{con}_K^{rh}(\sigma) & \text{if } L = {}^rK \text{ for some } r \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Given $K \leq H$ and $\chi \in A(K)$, there exist integers $\langle \chi, \sigma \rangle$, $\sigma \in \mathcal{B}(K)$, such that

$$\chi = \sum_{\sigma\in\mathcal{B}(K)} \langle \chi, \sigma \rangle \sigma.$$

We now define a $\mathbb{Z}$-module homomorphism $\varphi_{A,H} : A_+(H) \to \mathcal{G}_A(H)$ by

$$\varphi_{A,H}([K,\sigma]) = \left(\sum_{hK\in H/K,\, U\leq {}^hK} \langle \mathrm{res}_U^{^hK} \circ \mathrm{con}_K^h(\sigma), \tau \rangle \right)_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})}$$

for all $(K,\sigma) \in \mathfrak{R}(H,\mathcal{B})$, and call it the Burnside homomorphism. Obviously, the diagram

$$\begin{tikzcd} A_+(H) \arrow[r, "\varphi_{A,H}"] \arrow[dr, "\rho_H^A"'] & \mathcal{G}_A(H) \arrow[d, "\kappa_H^A"] \\ & A^+(H) \end{tikzcd}$$

is commutative, and thereby, $\varphi_{A,H}$ is a monomorphism (see Proposition 2.2).

For each $(K,\sigma) \in \mathfrak{R}(H,\mathcal{B})$, we set

$$W_H(K,\sigma) = N_H(K,\sigma)/K.$$

Remark 9.1 For each $(K,\sigma) \in \mathfrak{R}(H,\mathcal{B})$, $|W_H(K,\sigma)|$ divides each component of $\varphi_{A,H}([K,\sigma])$. By an argument analogous to the proof of [11, Proposition 80.15], we can show that the elements $(1/|W_H(K,\sigma)|)\varphi_{A,H}([K,\sigma])$ for $(K,\sigma) \in \mathfrak{R}(H,\mathcal{B})$ form a $\mathbb{Z}$-basis of $\mathcal{G}_A(H)$, that is,

$$\mathcal{G}_A(H) = \bigoplus_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} \frac{1}{|W_H(U,\tau)|}\varphi_{A,H}([U,\tau])\mathbb{Z}.$$

The following lemma, which is similar to [43, Lemma 2.7 (Cauchy-Frobenius)] (see also [34, Lemma 4.1]), plays a crucial role in the proof of Theorem 9.4.

Lemma 9.2 *Let $H \leq G$, and suppose that (K,σ), $(U,\tau) \in \mathfrak{R}(H,\mathcal{B})$. Then for any $Q \leq W_H(U,\tau)$,*

$$\sum_{rU\in Q} \sum_{hK\in H/K,\,\langle r\rangle U\leq {}^hK} \langle \mathrm{res}^{{}^hK}_U \circ \mathrm{con}^h_K(\sigma),\tau\rangle \equiv 0 \pmod{|Q|}.$$

Proof. We set

$$I_U = \{hK \in H/K \mid U \leq {}^hK \quad \text{and} \quad \langle \mathrm{res}^{{}^hK}_U \circ \mathrm{con}^h_K(\sigma),\tau\rangle \neq 0\}$$

and set

$$I_{rU} = \{hK \in I_U \mid \langle r\rangle U \leq {}^hK\}$$

for each $rU \in Q$. View I_U as a left Q-set with the action given by

$$rUhK = rhK$$

for all $rU \in Q$ and $hK \in I_U$. Then

$$I_{rU} = \{hK \in I_U \mid rUhK = hK\}$$

for each $rU \in Q$. Hence

$$\begin{aligned}\sum_{rU\in Q}\sum_{hK\in I_{rU}} \langle \mathrm{res}^{{}^hK}_U \circ \mathrm{con}^h_K(\sigma),\tau\rangle &= \sum_{hK\in I_U}\sum_{rU\in Q_{hK}} \langle \mathrm{res}^{{}^hK}_U \circ \mathrm{con}^h_K(\sigma),\tau\rangle \\ &= \sum_{hK\in I_U} |Q_{hK}|\cdot\langle \mathrm{res}^{{}^hK}_U \circ \mathrm{con}^h_K(\sigma),\tau\rangle,\end{aligned}$$

where Q_{hK} is the stabilizer of hK in Q. Observe now that

$$\begin{aligned}\langle \mathrm{res}^{{}^hK}_U \circ \mathrm{con}^h_K(\sigma),\tau\rangle &= \langle \mathrm{con}^r_U \circ \mathrm{res}^{{}^hK}_U \circ \mathrm{con}^h_K(\sigma), \mathrm{con}^r_U(\tau)\rangle \\ &= \langle \mathrm{res}^{{}^{rh}K}_U \circ \mathrm{con}^{rh}_K(\sigma),\tau\rangle\end{aligned}$$

for all $rU \in Q$ and $hK \in I_U$. Then

$$\sum_{rU \in Q} \sum_{hK \in I_{rU}} \langle \mathrm{res}_U^{hK} \circ \mathrm{con}_K^h(\sigma), \tau\rangle = \sum_{hK \in \overline{Q\backslash I_U}} |O(hK)| \cdot |Q_{hK}| \cdot \langle \mathrm{res}_U^{hK} \circ \mathrm{con}_K^h(\sigma), \tau\rangle$$
$$\equiv 0 \pmod{|Q|},$$

where $\overline{Q\backslash I_U}$ is a complete set of representatives of Q-orbits in I_U and $O(hK)$ is the Q-orbit containing hK. This completes the proof. □

We define an obstruction group of $A_+(H)$ by

$$\mathrm{Obs}_A(H) = \coprod_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} \mathbb{Z}/|W_H(U,\tau)|\mathbb{Z}.$$

By Lemma 2.3,

$$\mathrm{Im}\varphi_{A,H} = \bigoplus_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} \varphi_{A,H}([U,\tau])\mathbb{Z}.$$

Hence it follows from Remark 9.1 that

$$\mathcal{G}_A(H)/\mathrm{Im}\varphi_{A,H} \cong \mathrm{Obs}_A(H).$$

Let p be a prime. By Lemma 2.3, $[K,\sigma]$, $(K,\sigma) \in \mathfrak{R}(H,\mathcal{B})$, form a $\mathbb{Z}_{(p)}$-basis of $A_+(H)_{(p)}$, that is,

$$A_+(H)_{(p)} = \bigoplus_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \mathbb{Z}_{(p)}[K,\sigma].$$

We identify $\mathcal{G}_A(H)_{(p)}$ with

$$\coprod_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \mathbb{Z}_{(p)},$$

and identify $\mathrm{Obs}_A(H)_{(p)}$ with

$$\coprod_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \mathbb{Z}_{(p)}/|W_H(K,\sigma)|_p\mathbb{Z}_{(p)} \left(\cong \coprod_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \mathbb{Z}/|W_H(K,\sigma)|_p\mathbb{Z}\right).$$

Let $\varphi_{A,H}^{(p)}$ be the monomorphism from $A_+(H)_{(p)}$ to $\mathcal{G}_A(H)_{(p)}$ determined by $\varphi_{A,H}$. Then by the preceding argument,

$$\mathcal{G}_A(H)_{(p)}/\mathrm{Im}\varphi_{A,H}^{(p)} \cong \mathrm{Obs}_A(H)_{(p)}.$$

We write $\varphi_{A,H}^{(\infty)} = \varphi_{A,H}$.

For each $(K,\sigma) \in \mathfrak{R}(H,\mathcal{B})$, $W_H(K,\sigma)_p$ denotes a Sylow p-subgroup of $W_H(K,\sigma)$, and $W_H(K,\sigma)_\infty$ denotes $W_H(K,\sigma)$.

We denote by Λ the set consisting of all primes and the symbol ∞. Assume that $p \in \Lambda$. If $(U,\tau) \in \mathfrak{R}(H,\mathcal{B})$ and if $(x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \in \mathcal{G}_A(H)_{(p)}$, then we set $x_{h.(U,\tau)} = x_{(U,\tau)}$ for all $h \in H$. There exists a $\mathbb{Z}_{(p)}$-module homomorphism $\psi^{(p)}_{(U,\tau)} : \mathcal{G}_A(H)_{(p)} \to \mathbb{Z}_{(p)}/|W_H(U,\tau)|_p\mathbb{Z}_{(p)}$ with $(U,\tau) \in \mathfrak{R}(H,\mathcal{B})$ given by

$$\psi^{(p)}_{(U,\tau)}\left((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}\right) \equiv \sum_{\substack{rU\in W_H(U,\tau)_p,\\ \nu\in\mathcal{B}(\langle r\rangle U)}} x_{(\langle r\rangle U,\nu)} \cdot \langle \mathrm{res}^{\langle r\rangle U}_U(\nu), \tau\rangle \pmod{|W_H(U,\tau)|_p}$$

for all $(x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \in \mathcal{G}_A(H)_{(p)}$. When p is a prime, $\psi^{(p)}_{(U,\tau)}$ is independent of the choice of a Sylow p-subgroup $W_H(U,\tau)_p$ of $W_H(U,\tau)$, because

$$\begin{aligned}\sum_{\substack{rU\in W_H(U,\tau)_p,\\ \nu\in\mathcal{B}(\langle r\rangle U)}} x_{(\langle r\rangle U,\nu)} \cdot \langle \mathrm{res}^{\langle r\rangle U}_U(\nu), \tau\rangle &= \sum_{\substack{rU\in W_H(U,\tau)_p,\\ \nu\in\mathcal{B}(\langle r\rangle U)}} x_{({}^h\langle r\rangle U,\, {}^h\nu)} \cdot \langle \mathrm{res}^{{}^h\langle r\rangle U}_U({}^h\nu), \tau\rangle \\ &= \sum_{\substack{rU\in {}^{hU}W_H(U,\tau)_p,\\ \nu\in\mathcal{B}(\langle r\rangle U)}} x_{(\langle r\rangle U,\nu)} \cdot \langle \mathrm{res}^{\langle r\rangle U}_U(\nu), \tau\rangle\end{aligned}$$

for all $h \in N_H(U,\tau)$, where ${}^h\nu = \mathrm{con}^h_{\langle r\rangle U}(\nu)$.

We define a $\mathbb{Z}_{(p)}$-module homomorphism $\psi^{(p)}_{A,H} : \mathcal{G}_A(H)_{(p)} \to \mathrm{Obs}_A(H)_{(p)}$ by

$$\psi^{(p)}_{A,H}((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}) = (\psi^{(p)}_{(U,\tau)}((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}))_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})}$$

for all $(x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \in \mathcal{G}_A(H)_{(p)}$, and call it the Cauchy-Frobenius homomorphism.

Lemma 9.3 *Assume that $p \in \Lambda$. For each $H \leq G$, $\psi^{(p)}_{A,H}$ is an epimorphism.*

Proof. The proof is straightforward. See also the proof of [34, Lemma 4.3]. □

The following theorem is a generalization of [43, Proposition 2.9] (see also [13, Proposition 1.3.5], [29, Theorem 4.4], [34, Theorem 4.5], and [42, Lemma 2.1]).

Theorem 9.4 (Fundamental theorem) *Assume that $p \in \Lambda$. For each $H \leq G$, the sequence*

$$0 \longrightarrow A_+(H)_{(p)} \xrightarrow{\varphi^{(p)}_{A,H}} \mathcal{G}_A(H)_{(p)} \xrightarrow{\psi^{(p)}_{A,H}} \mathrm{Obs}_A(H)_{(p)} \longrightarrow 0$$

of $\mathbb{Z}_{(p)}$-modules is exact.

Proof. By Proposition 2.2, $\varphi_{A,H}^{(p)}$ is a monomorphism. Moreover, Lemma 9.3 states that $\psi_{A,H}^{(p)}$ is an epimorphism. Hence it remains to verify that $\mathrm{Im}\,\varphi_{A,H}^{(p)} = \mathrm{Ker}\,\psi_{A,H}^{(p)}$. By definition and Lemma 9.2,

$$\begin{aligned}
&\psi_{(U,\tau)}^{(p)}\left(\varphi_{A,H}^{(p)}([K,\sigma])\right)\\
&\qquad = \psi_{(U,\tau)}^{(p)}\left(\left(\sum_{hK\in H/K,\, L\leq {}^hK} \langle \mathrm{res}_L^{{}^hK}\circ \mathrm{con}_K^h(\sigma),\nu\rangle\right)_{(L,\nu)\in\mathfrak{R}(H,\mathcal{B})}\right)\\
&\qquad \equiv \sum_{\substack{rU\in W_H(U,\tau)_p,\\ \nu\in\mathcal{B}(\langle r\rangle U)}}\ \sum_{hK\in H/K,\,\langle r\rangle U\leq {}^hK} \langle \mathrm{res}_{\langle r\rangle U}^{{}^hK}\circ\mathrm{con}_K^h(\sigma),\nu\rangle\cdot\langle \mathrm{res}_U^{\langle r\rangle U}(\nu),\tau\rangle\\
&\qquad \equiv \sum_{rU\in W_H(U,\tau)_p}\ \sum_{hK\in H/K,\,\langle r\rangle U\leq {}^hK} \langle \mathrm{res}_U^{{}^hK}\circ\mathrm{con}_K^h(\sigma),\tau\rangle\\
&\qquad \equiv 0 \pmod{|W_H(U,\tau)|_p}
\end{aligned}$$

for all (K,σ), $(U,\tau)\in\mathfrak{R}(H,\mathcal{B})$. Hence we have $\mathrm{Im}\,\varphi_{A,H}^{(p)}\subseteq \mathrm{Ker}\,\psi_{A,H}^{(p)}$. Suppose next that $x = (x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}\in \mathrm{Ker}\,\psi_{A,H}^{(p)}$, and set

$$\mathfrak{R}(x) = \{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B}) \mid x_{(K,\sigma)}\neq 0\}.$$

We define a partially order $\leq_H$ of $\mathfrak{R}(H,\mathcal{B})$ by

$$(U,\tau)\leq_H (K,\sigma) :\Longleftrightarrow U\leq {}^hK \text{ and } \langle \mathrm{res}_U^{{}^hK}\circ\mathrm{con}_K^h(\sigma),\tau\rangle\neq 0 \text{ for some } h\in H,$$

and define $\mathfrak{R}_0(x)$ to be the set of maximal elements of $\mathfrak{R}(x)$ with respect to $\leq_H$. If $x\neq 0$, let ℓ_x be the smallest integer such that $|K|\leq \ell_x$ for all $(K,\sigma)\in\mathfrak{R}_0(x)$. Set $\ell_0 = 0$ for convenience' sake. Using induction on ℓ_x, we show that $x\in\mathrm{Im}\,\varphi_{A,H}^{(p)}$. If $\ell_x = 0$, then clearly, $x = 0\in\mathrm{Im}\,\varphi_{A,H}^{(p)}$. Assume that $x\neq 0$. For each $(U,\tau)\in\mathfrak{R}_0(x)$,

$$\psi_{(U,\tau)}^{(p)}(x)\equiv x_{(U,\tau)} \pmod{|W_H(U,\tau)|_p},$$

whence $x_{(U,\tau)} = y_{(U,\tau)}\cdot|W_H(U,\tau)|_p$ for some $y_{(U,\tau)}\in\mathbb{Z}_{(p)}$. Now set

$$y = x - \sum_{(U,\tau)\in\mathfrak{R}_0(x)} y_{(U,\tau)}\cdot\frac{|W_H(U,\tau)|_p}{|W_H(U,\tau)|}\,\varphi_{A,H}^{(p)}([U,\tau]).$$

Then by the definition of $\varphi_{A,H}^{(p)}$, we have $\ell_y<\ell_x$. Since $y\in\mathrm{Ker}\,\psi_{A,H}^{(p)}$, it follows from the inductive assumption that $y\in\mathrm{Im}\,\varphi_{A,H}^{(p)}$. This means that $x\in\mathrm{Im}\,\varphi_{A,H}^{(p)}$. Thus we have $\mathrm{Im}\,\varphi_{A,H}^{(p)}\supseteq\mathrm{Ker}\,\psi_{A,H}^{(p)}$. This completes the proof. □

For each $(U,\tau)\in\mathfrak{R}(H,\mathcal{B})$, we set

$$\mathfrak{S}(H,\mathcal{B})_{\geq(U,\tau)} = \{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B}) \mid U\leq K \quad\text{and}\quad \langle\mathrm{res}_U^K(\sigma),\tau\rangle\neq 0\}.$$

Lemma 9.5 *Let $H \leq G$. For any $(x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \in \mathcal{G}_A(H)$,*

$$\begin{aligned}\eta_H^A &\circ \kappa_H^A \left((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}\right) \\ &= \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} \frac{|H|}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B})_{\geq(U,\tau)}} \mu(U,K)x_{(K,\sigma)} \cdot \langle \mathrm{res}_U^K(\sigma), \tau\rangle [U,\tau].\end{aligned}$$

Proof. By definition,

$$\begin{aligned}\eta_H^A &\circ \kappa_H^A \left((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}\right) \\ &= \sum_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \sum_{rN_H(K)\in H/N_H(K)} \sum_{U\leq {}^rK} |U|\mu(U, {}^rK)x_{(K,\sigma)} \\ &\qquad\qquad \times \sum_{hN_H(K,\sigma)\in N_H(K)/N_H(K,\sigma)} [U, \mathrm{res}_U^{{}^rK} \circ \mathrm{con}_K^{rh}(\sigma)] \\ &= \sum_{(U,\tau)\in\mathfrak{S}(H,\mathcal{B})} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B})_{\geq(U,\tau)}} |U|\mu(U,K)x_{(K,\sigma)} \cdot \langle \mathrm{res}_U^K(\sigma), \tau\rangle [U,\tau] \\ &= \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} \frac{|H|}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B})_{\geq(U,\tau)}} \mu(U,K)x_{(K,\sigma)} \cdot \langle \mathrm{res}_U^K(\sigma), \tau\rangle [U,\tau],\end{aligned}$$

completing the proof. □

There exists a $\mathbb{Z}$-module homomorphism $\xi_{(U,\tau)} : \mathcal{G}_A(H) \to \mathbb{Z}/|W_H(U,\tau)|\mathbb{Z}$ with $(U,\tau) \in \mathfrak{R}(H,\mathcal{B})$ given by

$$\begin{aligned}\xi_{(U,\tau)} &\left((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}\right) \\ &\equiv \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B})_{\geq(U,\tau)}} \mu(U,K)x_{(K,\sigma)} \cdot \langle \mathrm{res}_U^K(\sigma), \tau\rangle \pmod{|W_H(U,\tau)|}.\end{aligned}$$

We now define a $\mathbb{Z}$-module homomorphism $\xi_{A,H} : \mathcal{G}_A(H) \to \mathrm{Obs}_A(H)$ by

$$\xi_{A,H}((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}) = (\xi_{(U,\tau)}((x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}))_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})}$$

for all $(x_{(K,\sigma)})_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})} \in \mathcal{G}_A(H)$.

The next theorem is similar to [10, Corollary 4.2] (see also [20, Theorem 1.1], [29, Corollary 5.3], and [43, Theorem 8.3]).

Theorem 9.6 (Second fundamental theorem) *For each $H \leq G$, the sequence*

$$0 \longrightarrow A_+(H) \xrightarrow{\varphi_{A,H}} \mathcal{G}_A(H) \xrightarrow{\xi_{A,H}} \mathrm{Obs}_A(H) \longrightarrow 0$$

of $\mathbb{Z}$-modules is exact.

Proof. By Proposition 2.2, $\varphi_{A,H}$ is a monomorphism. Moreover, it is easily verified that $\xi_{A,H}$ is an epimorphism. Combining Proposition 2.2 with Lemma 9.5, we have $\mathrm{Im}\,\varphi_{A,H} = \mathrm{Ker}\,\xi_{A,H}$. This completes the proof. □

10 Integral canonical induction formulae

Let $X \in \mathbf{Mack}(G)_{\mathbb{Z}}$, and let A be a restriction subfunctor of X. If $E = \mathbb{Q}$ or $E = \mathbb{Z}_{(p)}$ with $p \in \Lambda$, then X is extended to $EX \in \mathbf{Mack}(G)_E$ by E-linearly, and A is also extended to $EA \in \mathbf{Res}(G)_E$ by E-linearly.

We assume that $\lambda \in \mathbf{Con}(G)(X, A)_{\mathbb{Z}}$ and $\mathcal{B}$ is a stable $\mathbb{Z}$-basis of A. By Proposition 5.3, there exists a morphism $\Psi^{X,A,\lambda} : \mathbb{Q}X \to \mathbb{Q}A_+$ of restriction functors defined to be a family of $\mathbb{Q}$-space homomorphisms $\Psi_H^{X,A,\lambda} : \mathbb{Q}X(H) \to \mathbb{Q}A_+(H)$, $H \leq G$, such that

$$\Psi_H^{X,A,\lambda}(x) = \frac{1}{|H|}\eta_H^A((\lambda_K \circ \mathrm{res}_K^H(x))_{K \leq H})$$

for all $x \in X(H)$. Given $H \leq G$, $x \in X(H)$ and $(U, \tau) \in \mathfrak{R}(H, \mathcal{B})$, we set

$$m_\tau(x) = \frac{1}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B})_{\geq(U,\tau)}} \mu(U,K)\langle \lambda_K \circ \mathrm{res}_K^H(x), \sigma\rangle \cdot \langle \mathrm{res}_U^K(\sigma), \tau\rangle.$$

By (IV),

$$\begin{aligned}
\Psi_H^{X,A,\lambda}(x) &= \frac{1}{|H|}\sum_{K\leq H}\sum_{U\leq K}|U|\mu(U,K)[U, \mathrm{res}_U^K \circ \lambda_K \circ \mathrm{res}_K^H(x)] \\
&= \frac{1}{|H|}\sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B})}\sum_{U\leq K}|U|\mu(U,K)\langle \lambda_K \circ \mathrm{res}_K^H(x), \sigma\rangle[U, \mathrm{res}_U^K(\sigma)] \\
&= \frac{1}{|H|}\sum_{(U,\tau)\in\mathfrak{S}(H,\mathcal{B})}|N_H(U,\tau)| \cdot m_\tau(x)[U,\tau] \\
&= \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} m_\tau(x)[U,\tau]
\end{aligned}$$

for all $H \leq G$ and $x \in X(H)$.

If $\Psi_H^{X,A,\lambda}(x) \in \mathbb{Z}_{(p)}A_+(H)$ with $p \in \Lambda$ for all $H \leq G$ and $x \in X(H)$, then we view $\Psi^{X,A,\lambda}$ as a morphism $\Psi^{X,A,\lambda} : \mathbb{Z}_{(p)}X \to \mathbb{Z}_{(p)}A_+$ of restriction functors defined to be a family of $\mathbb{Z}_{(p)}$-module homomorphisms $\Psi_H^{X,A,\lambda} : \mathbb{Z}_{(p)}X(H) \to \mathbb{Z}_{(p)}A_+(H)$, $H \leq G$, such that

$$\Psi_H^{X,A,\lambda}(x) = \frac{1}{|H|}\eta_H^A((\lambda_K \circ \mathrm{res}_K^H(x))_{K \leq H})$$

for all $x \in X(H)$.

The following theorem is part of [4, Corollary 9.4].

Theorem 10.1 *Let $X \in \mathbf{Mack}(G)_{\mathbb{Z}}$, and let A be a restriction subfunctor of X. Assume that $\lambda \in \mathbf{Con}(G)(X, A)_{\mathbb{Z}}$, $\mathcal{B}$ is a stable $\mathbb{Z}$-basis of A, $p \in \Lambda$, and*

$$\langle \lambda_U \circ \mathrm{res}_U^H(x), \tau\rangle = \sum_{\sigma\in\mathcal{B}(K)} \langle \lambda_K \circ \mathrm{res}_K^H(x), \sigma\rangle \cdot \langle \mathrm{res}_U^K(\sigma), \tau\rangle \qquad (*_p)$$

for all $U \trianglelefteq K \leq H \leq G$, $x \in X(H)$, *and* $\tau \in \mathcal{B}(U)$ *such that* K/U *is a cyclic* p*-group and* $\mathrm{con}_U^r(\tau) = \tau$ *for all* $r \in K$. *Then*

$$\sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} m_\tau(x)[U,\tau] \in \mathbb{Z}_{(p)}A_+(H)$$

for all $H \leq G$ *and* $x \in X(H)$.

The condition $(*_p)$ is the condition $(*_\pi)$ in [4, Theorem 9.3, Corollary 9.4] with $\pi = \{p\}$. We apply Theorem 9.4 to the proof of this theorem.

Proof of Theorem 10.1. By Proposition 2.2 and Lemma 9.5,

$$\begin{aligned}
&\varphi_{A,H}^{(p)}\left(|H| \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} m_\tau(x)[U,\tau]\right) \\
&\qquad\qquad = \varphi_{A,H}^{(p)} \circ \eta_H^A \circ \kappa_H^A((\langle \lambda_K \circ \mathrm{res}_K^H(x), \sigma\rangle)_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}) \\
&\qquad\qquad = |H|(\langle \lambda_K \circ \mathrm{res}_K^H(x), \sigma\rangle)_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\psi_{(U,\tau)}^{(p)}((\langle \lambda_K \circ \mathrm{res}_K^H(x), \sigma\rangle)_{(K,\sigma)\in\mathfrak{R}(H,\mathcal{B})}) \\
&\qquad\qquad \equiv \sum_{\substack{rU\in W_H(U,\tau)_p, \\ \nu\in\mathcal{B}(\langle r\rangle U)}} \langle \lambda_{\langle r\rangle U} \circ \mathrm{res}_{\langle r\rangle U}^H(x), \nu\rangle \cdot \langle \mathrm{res}_U^{\langle r\rangle U}(\nu), \tau\rangle \\
&\qquad\qquad \equiv \sum_{rU\in W_H(U,\tau)_p} \langle \lambda_U \circ \mathrm{res}_U^H(x), \tau\rangle \\
&\qquad\qquad \equiv 0 \pmod{|W_H(U,\tau)|_p}
\end{aligned}$$

for each $(U,\tau) \in \mathfrak{R}(H,\mathcal{B})$. Hence the assertion follows from Theorem 9.4. This completes the proof. □

The following corollary is crucial to a canonical choice of Brauer's induction theorem on X (cf. [4, Corollary 9.5]).

Corollary 10.2 *Keep the hypothesis of Theorem* 10.1, *and assume further that*

$$e_H^{(H)} \cdot (\lambda_H(x) - x) = 0$$

for all $H \in \mathcal{C}(\mathbb{Q}X)$ *and* $x \in X(H)$. *Then* $\Psi^{X,A,\lambda}$ *is a canonical induction formula for* $\mathbb{Z}_{(p)}X$ *from* $\mathbb{Z}_{(p)}A$, *and*

$$\Psi_H^{X,A,\lambda}(x) = \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B})} m_\tau(x)[U,\tau]$$

for all $H \leq G$ *and* $x \in X(H)$.

Proof. By Proposition 5.3, $\Psi^{X,A,\lambda}$ is a canonical induction formula for $\mathbb{Q}X$ from $\mathbb{Q}A$. Hence the corollary follows from Theorem 10.1. This completes the proof. $\square$

In the remaining part of this section, we assume the following situation.

Hypothesis 10.3

(i) $S \in G$-**set**.

(ii) $X \in \mathbf{Mack}(G;S)_{\mathbb{Z}}$.

(iii) $A \in \mathbf{Res}(G;S)_{\mathbb{Z}}$ such that for each $s \in S$, A_s is a restriction subfunctor of X_s and the crossed conjugation maps $\mathrm{con}_{s\,H}^{\,g}$ for $H \leq G_s$ and $g \in G$ are the restriction of those of X.

(iv) $\lambda_s \in \mathbf{Con}(G_s)(X_s, A_s)_{\mathbb{Z}}$, $s \in S$, which satisfy

$$\mathrm{con}_{s\,H}^{\,g} \circ \lambda_{s\,H} = \lambda_{{}^g s\ {}^g H} \circ \mathrm{con}_{s\,H}^{\,g}$$

for all $s \in S$, $H \leq G_s$, and $g \in G$.

(v) For each $s \in S$, $\mathcal{B}_s$ is a stable $\mathbb{Z}$-basis of A_s such that

$$\mathcal{B}_{{}^g s}({}^g H) = \{\mathrm{con}_{s\,H}^{\,g}(\sigma_s) \mid \sigma_s \in \mathcal{B}_s(H)\}$$

for all $H \leq G_s$ and $g \in G$.

Obviously, the crossed restriction functor A_S on A is a restriction subfunctor of the crossed Mackey functor X_S on X. We define $\lambda_S : X_S \to A_S$ to be a family of $\mathbb{Z}$-module homomorphisms $\lambda_{S\,H} : X_S(H) \to A_S(H)$, $H \leq G$, such that

$$\lambda_{S\,H}((x(s))_{s\in S}) = (y_H(s))_{s\in S}$$

for all $(x(s))_{s\in S} \in X_S(H)$, where $y_H(s) = \lambda_{s\,H}(x(s))$ if $s \in C_S(H)$, and $y_H(s) = 0$ otherwise. Clearly, $\lambda_S \in \mathbf{Con}(G)(X_S, A_S)_{\mathbb{Z}}$.

We define a stable $\mathbb{Z}$-basis $\mathcal{B}_S$ of A_S to be a family of $\mathbb{Z}$-bases $\mathcal{B}_S(H)$ of $A_S(H)$, $H \leq G$, such that

$$\mathcal{B}_S(H) = \{(\delta_{st}\sigma_s)_{t\in S} \in A_S(H) \mid s \in C_S(H) \text{ and } \sigma_s \in \mathcal{B}_s(H)\}$$

for all $H \leq G$, where $\delta_{st}\sigma_s = 0$ if $s \neq t$ and $\delta_{ss}\sigma_s = \sigma_s$.

Lemma 10.4 *Let $U \leq K \leq H \leq G$. Assume that $\tau_s \in \mathcal{B}_s(U)$ with $s \in C_S(H)$ and*

$$\langle \lambda_{s\,U} \circ \mathrm{res}_U^H(x), \tau_s\rangle = \sum_{\sigma_s \in \mathcal{B}_s(K)} \langle \lambda_{s\,K} \circ \mathrm{res}_K^H(x), \sigma_s\rangle \cdot \langle \mathrm{res}_U^K(\sigma_s), \tau_s\rangle$$

for all $x \in X_s(H)$*. Set* $\tau = (\delta_{st}\tau_s)_{t\in S} \in \mathcal{B}_S(U)$*. Then*

$$\langle \lambda_{S\,U} \circ {\mathrm{res}_S}_U^H((x(t))_{t\in S}), \tau\rangle = \sum_{\sigma\in\mathcal{B}_S(K)} f_{\sigma,\tau}((x(t))_{t\in S})$$

for all $(x(t))_{t\in S} \in X_S(H)$*, where*

$$f_{\sigma,\tau}((x(t))_{t\in S}) = \langle \lambda_{S\,K} \circ {\mathrm{res}_S}_K^H((x(t))_{t\in S}), \sigma\rangle \cdot \langle {\mathrm{res}_S}_U^K(\sigma), \tau\rangle.$$

Proof. Let $(x(t))_{t\in S} \in X_S(H)$. If $\sigma_s \in \mathcal{B}_s(K)$ and if $\sigma = (\delta_{st}\sigma_s)_{t\in S} \in \mathcal{B}_S(K)$, then

$$f_{\sigma,\tau}((x(t))_{t\in S}) = \langle \lambda_{s\,K} \circ \mathrm{res}_K^H(x(s)), \sigma_s\rangle \cdot \langle \mathrm{res}_U^K(\sigma_s), \tau_s\rangle.$$

Hence

$$\begin{aligned}\sum_{\sigma\in\mathcal{B}_S(K)} f_{\sigma,\tau}((x(t))_{t\in S}) &= \sum_{\sigma_s\in\mathcal{B}_s(K)} \langle \lambda_{s\,K} \circ \mathrm{res}_K^H(x(s)), \sigma_s\rangle \cdot \langle \mathrm{res}_U^K(\sigma_s), \tau_s\rangle \\ &= \langle \lambda_{s\,U} \circ \mathrm{res}_U^H(x(s)), \tau_s\rangle \\ &= \langle \lambda_{S\,U} \circ {\mathrm{res}_S}_U^H((x(t))_{t\in S}), \tau\rangle,\end{aligned}$$

completing the proof. □

We are now in position to show a result about an integral canonical induction formula for X_S from A_S.

Proposition 10.5 *Assume that* $p \in \Lambda$ *and*

$$\langle \lambda_{s\,U} \circ \mathrm{res}_U^H(x), \tau_s\rangle = \sum_{\sigma_s\in\mathcal{B}_s(K)} \langle \lambda_{s\,K} \circ \mathrm{res}_K^H(x), \sigma_s\rangle \cdot \langle \mathrm{res}_U^K(\sigma_s), \tau_s\rangle$$

for all $U \trianglelefteq K \leq H \leq G$*,* $s \in C_S(H)$*,* $x \in X_s(H)$*, and* $\tau_s \in \mathcal{B}_s(U)$ *such that* K/U *is a cyclic* p*-group and* ${\mathrm{con}_s}_U^r(\tau_s) = \tau_s$ *for all* $r \in K$*. Assume further that*

$$e_H^{(H)} \cdot (\lambda_{s\,H}(x) - x) = 0$$

for all $H \in \mathcal{C}(\mathbb{Q}X)$*,* $s \in C_S(H)$*, and* $x \in X_s(H)$*. Then* Ψ^{X_S,A_S,λ_S} *is a canonical induction formula for* $\mathbb{Z}_{(p)}X_S$ *from* $\mathbb{Z}_{(p)}A_S$*, and*

$$\Psi_H^{X_S,A_S,\lambda_S}((x(s))_{s\in S}) = \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B}_S)} m_\tau((x(s))_{s\in S})[U,\tau]$$

for all $H \leq G$ *and* $(x(s))_{s\in S} \in X_S(H)$*, where*

$$\begin{aligned}&m_\tau((x(s))_{s\in S}) \\ &\quad= \frac{1}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B}_S)_{\geq(U,\tau)}} \mu(U,K) \\ &\qquad\qquad \times \langle \lambda_{S\,K} \circ {\mathrm{res}_S}_K^H((x(s))_{s\in S}), \sigma\rangle \cdot \langle {\mathrm{res}_S}_U^K(\sigma), \tau\rangle.\end{aligned}$$

Proof. By Lemma 10.4, the condition $(*_p)$ of Theorem 10.1 holds for $X = X_S$, $A = A_S$, $\lambda = \lambda_S$, and $\mathcal{B} = \mathcal{B}_S$. Suppose that $H \in \mathcal{C}(\mathbb{Q}X_S)$. Let $(x(s))_{s\in S} \in X_S(H)$, and set

$$(y(s))_{s\in S} = e_H^{(H)} \cdot (\lambda_{S\,H}((x(s))_{s\in S}) - (x(s))_{s\in S}).$$

Then Proposition 5.5 yields $(y(s))_{s\in S} \in \mathcal{K}^{X_S}(H)$. Using an argument analogous to the proof of Proposition 6.1, we have

$$y(s) = \begin{cases} e_H^{(H)} \cdot (\lambda_{s\,H}(x(s)) - x(s)) & \text{if } s \in C_S(H), \\ 0 & \text{otherwise.} \end{cases}$$

This implies that $(y(s))_{s\in S} = 0$, because $\mathcal{C}(\mathbb{Q}X_S) = \mathcal{C}(\mathbb{Q}X)$ by Proposition 6.1. Thus the proposition is a consequence of Corollary 10.2. This completes the proof. $\square$

11 Induction formulae for representations of $\mathbb{C}^\alpha G$

Let $\alpha : G \times G \to \mathbb{C}^\times$ be a normalized 2-cocycle, and keep the notation of Section 7. For each $H \leq G$, let $\mathrm{Irr}_\alpha(H)$ be the set of isomorphism classes of irreducible left $\mathbb{C}^\alpha H$-modules, and let $\mathrm{Lin}_\alpha(H)$ be the set of isomorphism classes of one-dimensional left $\mathbb{C}^\alpha H$-modules. We denote by R_α^{ab} the restriction subfunctor of the $\mathbb{C}^\alpha G$-representation functor R_α such that $R_\alpha^{\mathrm{ab}}(H)$ with $H \leq G$ is the $\mathbb{Z}$-span of $\mathrm{Lin}_\alpha(H)$, and define a morphism $\lambda^\alpha : R_\alpha \to R_\alpha^{\mathrm{ab}}$ of conjugation functors by

$$\lambda_H^\alpha(\chi) = \begin{cases} \chi & \text{if } \chi \in \mathrm{Lin}_\alpha(H), \\ 0 & \text{otherwise} \end{cases}$$

for all $H \leq G$ and $\chi \in \mathrm{Irr}_\alpha(H)$. Obviously, there exists a stable $\mathbb{Z}$-basis $\mathcal{B}^\alpha$ of R_α^{ab} such that $\mathcal{B}^\alpha(H) = \mathrm{Lin}_\alpha(H)$ for all $H \leq G$. From Lemma 7.3, we know that the condition $(*_p)$ of Theorem 10.1 holds for $X = R_\alpha$, $A = R_\alpha^{\mathrm{ab}}$, $\lambda = \lambda^\alpha$, $\mathcal{B} = \mathcal{B}^\alpha$, and $p = \infty$. Observe that by Lemma 7.2, $\mathcal{C}(\mathbb{Q}R_\alpha)$ is the set of cyclic subgroups of G. Then for any $H \in \mathcal{C}(\mathbb{Q}R_\alpha)$, $R_\alpha^{\mathrm{ab}}(H) = R_\alpha(H)$ and $\lambda_H^\alpha = \mathrm{id}_{R_\alpha(H)}$ (see also the proof of Lemma 7.2). Hence it follows from Corollary 10.2 that $\Psi^{R_\alpha, R_\alpha^{\mathrm{ab}}, \lambda^\alpha}$ is a canonical induction formula for R_α from R_α^{ab} and

$$\Psi_H^{R_\alpha, R_\alpha^{\mathrm{ab}}, \lambda^\alpha}(\chi) = \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B}^\alpha)} m_\tau^\alpha(\chi)[U,\tau]$$

for all $\chi \in R_\alpha(H)$, where

$$m_\tau^\alpha(\chi) = \frac{1}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B}^\alpha)_{\geq(U,\tau)}} \mu(U,K)\langle \lambda_K^\alpha \circ \mathrm{res}_K^H(\chi), \sigma\rangle.$$

Note that $\langle \mathrm{res}_U^K(\sigma), \tau\rangle = 1$ for any $(K,\sigma) \in \mathfrak{S}(H,\mathcal{B}^\alpha)_{\geq(U,\tau)}$ with $(U,\tau) \in \mathfrak{R}(H,\mathcal{B}^\alpha)$. Consequently, we have the following.

Proposition 11.1 *Under the above notation,*

$$\chi = \sum_{(U,\tau)\in\mathfrak{R}(G,\mathcal{B}^\alpha)} m_\tau^\alpha(\chi)\mathrm{ind}_U^G(\tau)$$

for all $\chi \in R_\alpha(G)$*.*

If α is trivial, that is, $\alpha(s,t) = 1$ for all s, $t \in G$, then Proposition 11.1 yields a canonical choice of Brauer's induction theorem on $\mathbb{C}$-characters of G, which is due to Boltje [3] (cf. [4, Examples 1.8(a), 6.13(a), 9.7]).

A subgroup H of G is said to be hyper-elementary if H has a cyclic normal p-complement, or equivalently $O^p(H)$ is cyclic, for some prime p. Assume now that $p \in \Lambda$, and define a morphism $\lambda^{p,\alpha} : R_\alpha \to R_\alpha^{\mathrm{ab}}$ of conjugation functors by

$$\lambda_H^{p,\alpha}(\chi) = \begin{cases} \chi & \text{if } O^p(H) \text{ is cyclic and if } \chi \in \mathrm{Lin}_\alpha(H), \\ 0 & \text{otherwise} \end{cases}$$

for all $H \leq G$ and $\chi \in \mathrm{Irr}_\alpha(H)$. (Note that $\lambda^{\infty,\alpha} = \lambda^\alpha$.) Then it follows from Lemma 7.3 that the condition $(*_p)$ of Theorem 10.1 holds for $X = R_\alpha$, $A = R_\alpha^{\mathrm{ab}}$, $\lambda = \lambda^{p,\alpha}$, and $\mathcal{B} = \mathcal{B}^\alpha$. Moreover, $\lambda_H^{p,\alpha} = \mathrm{id}_{R_\alpha(H)}$ for any $H \in \mathcal{C}(\mathbb{Q}R_\alpha)$, because $\mathcal{C}(\mathbb{Q}R_\alpha)$ is the set of cyclic subgroups of G. Hence it follows from Corollary 10.2 that $\Psi^{R_\alpha,R_\alpha^{\mathrm{ab}},\lambda^{p,\alpha}}$ is a canonical induction formula for $\mathbb{Z}_{(p)}R_\alpha$ from $\mathbb{Z}_{(p)}R_\alpha^{\mathrm{ab}}$, and

$$\Psi_H^{R_\alpha,R_\alpha^{\mathrm{ab}},\lambda^{p,\alpha}}(\chi) = \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B}^\alpha)} m_\tau^{p,\alpha}(\chi)[U,\tau]$$

for all $\chi \in R_\alpha(H)$, where

$$m_\tau^{p,\alpha}(\chi) = \frac{1}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B}^\alpha)_{\geq(U,\tau)}} \mu(U,H)\langle\lambda_K^{p,\alpha} \circ \mathrm{res}_K^H(\chi), \sigma\rangle.$$

In particular, Proposition 11.1 is reduced to the type of hyper-elementary groups.

Proposition 11.2 *Let* $\Lambda(G)$ *denote the set of all primes dividing* $|G|$*. Under the above notation, if the condition*

$$\sum_{p\in\Lambda(G)} \ell_p \frac{|G|}{|G|_p} = 1$$

holds for integers ℓ_p*,* $p \in \Lambda(G)$*, then*

$$\chi = \sum_{(U,\tau)\in\mathfrak{H}(G,\mathcal{B}^\alpha)} \left(\sum_{p\in\Lambda(G)} \ell_p \frac{|G|}{|G|_p} m_\tau^{p,\alpha}(\chi) \right) \mathrm{ind}_U^G(\tau)$$

for all $\chi \in R_\alpha(G)$*, where*

$$\mathfrak{H}(G,\mathcal{B}^\alpha) = \{(U,\tau) \in \mathfrak{R}(G,\mathcal{B}^\alpha) \mid O^p(U) \textit{ is cyclic}\}.$$

If α is trivial, then Proposition 11.2 is [34, Theorem 8.7].

Remark 11.3 A subgroup H of G is said to be elementary if H is the direct product of a p-group and a cyclic group of order prime to p for some prime p. We denote by $\mathfrak{E}(G)$ the set of elementary subgroups of G. By [22, Theorem 7.5.3], every α-character is expressed as a $\mathbb{Z}$-linear combination of α-characters induced from α-characters of degree 1 of elementary subgroups of G (see also [11, Brauer Induction Theorem 15.9]). Hence it follows from [22, Proposition 7.1.1, Theorem 7.1.11] that each $\chi \in R_\alpha(G)$ is expressed as a $\mathbb{Z}$-linear combination of the elements $\mathrm{ind}_U^G(\tau)$ for $(U,\tau) \in \mathfrak{R}(G,\mathcal{B}^\alpha)$ with $U \in \mathfrak{E}(G)$.

12 Induction formulae for representations of $D^\omega(G)$

Let $\omega : G \times G \times G \to \mathbb{C}^\times$ be a normalized 3-cocycle, and keep the notation of Section 8. Recall that R^θ is the Mackey bundle composed of $R_{\theta_s} \in \mathbf{Mack}(G_s)_\mathbb{Z}$, $s \in G^c$, equipped with the crossed conjugation maps ${\mathrm{con}_s}^g_H$ for $s \in G^c$, $H \leq G_s$, and $g \in G$. Given $s \in G^c$, the restriction subfunctor $R^{\mathrm{ab}}_{\theta_s}$ of R_{θ_s}, the morphism $\lambda^{p,\theta_s} : R_{\theta_s} \to R^{\mathrm{ab}}_{\theta_s}$ of conjugation functors, where $p \in \Lambda$, and the stable $\mathbb{Z}$-basis $\mathcal{B}^{\theta_s}$ such that $\mathcal{B}^{\theta_s}(H) = \mathrm{Lin}_{\theta_s}(H)$ for all $H \leq G_s$ are defined in Section 11. Let $R^{\mathrm{ab}\theta}$ be the restriction bundle composed of $R^{\mathrm{ab}}_{\theta_s} \in \mathbf{Res}(G_s)_\mathbb{Z}$, $s \in G^c$, such that the crossed conjugation maps ${\mathrm{con}_s}^g_H$ for $s \in G^c$, $H \leq G_s$, and $g \in G$ are the restriction of those of R^θ. The crossed Mackey functor

$$R^\theta_{G^c} = (R^\theta_{G^c}, \mathrm{con}_{G^c}, \mathrm{res}_{G^c}, \mathrm{ind}_{G^c}) \in \mathbf{Mack}(G)_\mathbb{Z}$$

on R^θ and the crossed restriction functor

$$R^{\mathrm{ab}\theta}_{G^c} = (R^{\mathrm{ab}\theta}_{G^c}, \mathrm{con}_{G^c}, \mathrm{res}_{G^c}) \in \mathbf{Res}(G)_\mathbb{Z}$$

on $R^{\mathrm{ab}\theta}$ are defined in Section 4. Suppose that the morphism $\lambda^{p,\theta}_{G^c} : R^\theta_{G^c} \to R^{\mathrm{ab}\theta}_{G^c}$ of conjugation functors and the stable $\mathbb{Z}$-basis $\mathcal{B}^\theta_{G^c}$ of $R^{\mathrm{ab}\theta}_{G^c}$ are λ_{G^c} and $\mathcal{B}_{G^c}$ defined in Section 10 with $S = G^c$, $X = R^\theta$, $A = R^{\mathrm{ab}\theta}$, $\lambda_s = \lambda^{p,\theta_s}$, and $\mathcal{B}_s = \mathcal{B}^{\theta_s}$, respectively.

Lemma 12.1 *Assume that $p \in \Lambda$. Then $\Psi^{R^\theta_{G^c}, R^{\mathrm{ab}\theta}_{G^c}, \lambda^{p,\theta}_{G^c}}$ is a canonical induction formula for $\mathbb{Z}_{(p)}R^\theta_{G^c}$ from $\mathbb{Z}_{(p)}R^{\mathrm{ab}\theta}_{G^c}$ such that*

$$\Psi_H^{R^\theta_{G^c}, R^{\mathrm{ab}\theta}_{G^c}, \lambda^{p,\theta}_{G^c}}((x(s))_{s\in G^c}) = \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B}^\theta_{G^c})} m_\tau^{p,\theta}((x(s))_{s\in G^c})[U,\tau]$$

for all $H \leq G$ and $(x(s))_{s\in G^c} \in R^\theta_{G^c}(H)$, where

$$m_\tau^{p,\theta}((x(s))_{s\in G^c}) = \frac{1}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B}^\theta_{G^c})_{\geq(U,\tau)}} \mu(U,K)\langle \lambda^{p,\theta}_{G^cK} \circ {\mathrm{res}_{G^c}}^H_K((x(s))_{s\in G^c}), \sigma\rangle.$$

Proof. The argument before Proposition 11.2 means that the assumptions of Proposition 10.5 hold for $s \in G^c$, $X_s = R_{\theta_s}$, $A_s = R^{\mathrm{ab}}_{\theta_s}$, $\lambda_s = \lambda^{p,\theta_s}$, and $\mathcal{B}_s = \mathcal{B}^{\theta_s}$. Hence the lemma follows from Proposition 10.5. This completes the proof. □

Keep the notation of Section 11. For each $H \leq G$, we set

$$\mathrm{Irr}(D^\omega_G(H)) = \left\{ [M] \in D^\omega_G(H)\text{-}\overline{\mathbf{mod}} \;\middle|\; \begin{array}{l} [\phi_s M] \in \mathrm{Irr}_{\theta_s}(H_s) \text{ for some } s \in \overline{H\backslash G^c}, \\ \text{and } \phi_t M = \{0\} \text{ for any } t \in \overline{H\backslash G^c} \\ \text{with } s \neq t \end{array} \right\}$$

and

$$\mathrm{Lin}(D^\omega_G(H)) = \left\{ [N] \in D^\omega_G(H)\text{-}\overline{\mathbf{mod}} \;\middle|\; \begin{array}{l} \dim_{\mathbb{C}}(\phi_s N) = 1 \text{ for some } s \in C_G(H), \\ \text{and } \phi_t N = \{0\} \text{ for any } t \in G^c \\ \text{with } s \neq t \end{array} \right\}.$$

By Theorem 8.4, $D^\omega_G(H)$ with $H \leq G$ is a semisimple algebra, and $\mathrm{Irr}(D^\omega_G(H))$ is the set of isomorphism classes of irreducible left $D^\omega_G(H)$-modules. Let $R^{\mathrm{ab}}D^\omega_G$ be a restriction subfunctor of RD^ω_G such that $R^{\mathrm{ab}}D^\omega_G(H)$ with $H \leq G$ is the $\mathbb{Z}$-span of $\mathrm{Lin}(D^\omega_G(H))$. For each $p \in \Lambda$, we define a morphism $\lambda^{p,\omega}_G : RD^\omega_G \to R^{\mathrm{ab}}D^\omega_G$ of conjugation functors by

$$\lambda^{p,\omega}_{G\,H}(\chi) = \begin{cases} \chi & \text{if } O^p(H) \text{ is cyclic and if } \chi \in \mathrm{Lin}(D^\omega_G(H)), \\ 0 & \text{otherwise} \end{cases}$$

for all $H \leq G$ and $\chi \in \mathrm{Irr}(D^\omega_G(H))$. Obviously, there exists a stable $\mathbb{Z}$-basis $\mathcal{B}^\omega_G$ of $R^{\mathrm{ab}}D^\omega_G$ such that $\mathcal{B}^\omega_G(H) = \mathrm{Lin}(D^\omega_G(H))$ for all $H \leq G$. Given $H \leq G$ and $(U, \tau) \in \mathfrak{R}(H, \mathcal{B}^\omega_G)$, we set

$$W_H(U, \tau) = \{hU \in N_H(U)/U \mid \mathrm{Dcon}^h_U(\tau) = \tau\}.$$

Theorem 12.2 *Assume that $p \in \Lambda$. Then $\Psi^{RD^\omega_G, R^{\mathrm{ab}}D^\omega_G, \lambda^{p,\omega}_G}$ is a canonical induction formula for $\mathbb{Z}_{(p)}RD^\omega_G$ from $\mathbb{Z}_{(p)}R^{\mathrm{ab}}D^\omega_G$ such that*

$$\Psi_H^{RD^\omega_G, R^{\mathrm{ab}}D^\omega_G, \lambda^{p,\omega}_G}(\chi) = \sum_{(U,\tau)\in\mathfrak{R}(H,\mathcal{B}^\omega_G)} m^{p,\omega}_\tau(\chi)[U, \tau]$$

for all $H \leq G$ and $\chi \in RD^\omega_G(H)$, where

$$m^{p,\omega}_\tau(\chi) = \frac{1}{|W_H(U,\tau)|} \sum_{(K,\sigma)\in\mathfrak{S}(H,\mathcal{B}^\omega_G)_{\geq(U,\tau)}} \mu(U, K)\langle \lambda^{p,\omega}_{G\,K} \circ \mathrm{Dres}^H_K(\chi), \sigma\rangle.$$

Proof. We define a morphism $\Gamma^{\mathrm{ab}} : R^{\mathrm{ab}}D^{\omega}_{G} \to R^{\mathrm{ab}\theta}_{G^c}$ of restriction functors by

$$\Gamma^{\mathrm{ab}}_{H} : R^{\mathrm{ab}}D^{\omega}_{G}(H) \to R^{\mathrm{ab}\theta}_{G^c}(H), \quad x \to \Gamma_H(x)$$

for all $H \le G$, where Γ_H is defined in Section 8. By Theorem 8.4, Γ^{ab} is an isomorphism of restriction functors. For each $H \le G$, the diagram

$$\begin{array}{ccc} RD^{\omega}_{G}(H) & \xrightarrow{\Gamma_H} & R^{\theta}_{G^c}(H) \\ {\scriptstyle \lambda^{p,\omega}_{G\,H}}\Big\downarrow & & \Big\downarrow{\scriptstyle \lambda^{p,\theta}_{G^c H}} \\ R^{\mathrm{ab}}D^{\omega}_{G}(H) & \xrightarrow[\Gamma^{\mathrm{ab}}_{H}]{} & R^{\mathrm{ab}\theta}_{G^c}(H) \end{array}$$

is commutative, and Γ^{ab}_{H} induces a one to one correspondence

$$\mathcal{B}^{\omega}_{G}(H) \ni \sigma \mapsto \Gamma_H(\sigma) \in \mathcal{B}^{\theta}_{G^c}(H).$$

Hence the theorem follows from Theorem 8.4 and Lemma 12.1. $\square$

We are now successful in finding an analogy of Brauer's induction theorem on $\mathbb{C}$-characters of G.

Corollary 12.3 *Keep the notation of Theorem* 12.2, *and let* $M \in D^{\omega}(G)$-**mod**. *Then*

$$[M] = \sum_{(U,[N])\in\mathfrak{R}(G,\mathcal{B}^{\omega}_{G})} m^{\infty,\omega}_{[N]}([M])[D^{\omega}(G) \otimes_{D^{\omega}_{G}(U)} N].$$

If the condition

$$\sum_{p\in\Lambda(G)} \ell_p \frac{|G|}{|G|_p} = 1$$

holds for integers ℓ_p, $p \in \Lambda(G)$, *then*

$$[M] = \sum_{(U,[N])\in\mathfrak{H}(G,\mathcal{B}^{\omega}_{G})} \left(\sum_{p\in\Lambda(G)} \ell_p \frac{|G|}{|G|_p} m^{p,\omega}_{[N]}([M]) \right) [D^{\omega}(G) \otimes_{D^{\omega}_{G}(U)} N],$$

where

$$\mathfrak{H}(G,\mathcal{B}^{\omega}_{G}) = \{(U,[N]) \in \mathfrak{R}(G,\mathcal{B}^{\omega}_{G}) \mid O^p(U) \textit{ is cyclic}\}.$$

Remark 12.4 By Lemma 8.1, there exists an equivalence between the categories $\mathbb{C}H$-**mod** and $D^{\omega}_{\epsilon}(H)$-**mod**. Moreover, if $\alpha : G \times G \to \mathbb{C}^{\times}$ is the trivial 2-cocycle, then the statements of Propositions 11.1 and 11.2 are special cases of Corollary 12.3.

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