Involutes of fronts in the Euclidean plane

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Involutes of fronts in the Euclidean plane

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Abstract For a regular plane curve, an involute of it is the trajectory described by the end of a stretched string unwinding from a point of the curve. Even for a regular curve, the involute always has a singularity. By using a moving frame along the front and the curvature of the Legendre immersion in the unit tangent bundle, we define an involute of the front in the Euclidean plane and give properties of it. We also consider a relationship between evolutes and involutes of fronts without inflection points. As a result, the evolutes and the involutes of fronts without inflection points are corresponding to the differential and the integral of the curvature of the Legendre immersion.

Keywords involute, evolute, front, Legendre immersion, inflection point

Mathematics Subject Classification (2000) 58M05, 68R15, 57R45.

1 Introduction

The notions of involutes (or, evolvents) and evolutes were studied by C. Huygens in his work [15] and investigated in physics, classical analysis, differential geometry and singularity theory of planar curves (cf. [6], [8], [12], [13], [14],

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[20]). For a regular plane curve, an involute of it is the trajectory described by the end of stretched string unwinding from a base point of the curve. As a remarkable property of a regular curve without inflection points, the involute of the regular curve has a $3/2$ cusp at the base point.

On the other hand, the evolute of a regular plane curve is also classical object (cf. [6], [12], [13]). The evolute of a regular curve without inflection points is given by not only the locus of all its centres of curvature, but also an envelope of the normal lines of the regular curve. It is well-known that the relationship between involutes and evolutes of regular plane curves. The evolute of an involute is the original curve, less portions of zero or undefined curvature. The properties of evolutes were also discussed by using squared distance functions, the theories of Lagrangian, Legendrian singularity and further concepts (cf. [2], [3], [4], [7], [10], [18], [19], [21], [23]).

In this paper, we define involutes of curves with singular points which are called fronts. In section 2, we recall the definitions for the involute and the evolute of regular plane curves. Moreover, for a Legendre curve (a Legendre immersion) in the unit tangent bundle, we give a moving frame along the frontal (the front) and the curvature of the Legendre curve (the Legendre immersion) (cf. [9]). By using them, we define an involute of the front. We also recall the definition of the evolute of the front without inflection points (cf. [10]). We discuss properties of involutes without inflection points. For example, the involute of the front without inflection points is also a front without inflection points. We also give relationships between evolutes and involutes of fronts without inflection points by using the curvature of Legendre immersions. In section 3, we analyse singular points of the involute of the front without inflection points. Moreover, we give a relationship between singular points of the involute of the front and vertices. Since the involute of the front without inflection points is also a front without inflection points, we can repeat the involute of the front. We give a formula of the n-th involute of the front in section 4. We introduce a special parametrisation for Legendre immersions without inflection points in section 5. By using this parametrisation, the evolute and the involute of the front without inflection points are corresponding to the differential and the integral of the curvature of the Legendre immersion. We give not only the relationship between the contact of Legendre immersions, evolutes and involutes, but investigate also the case of the same shape of the front and the n-th evolute (or, the n-th involute) of the front under the same parametrisation.

We shall assume throughout the whole paper that all maps and manifolds are C^∞, unless the contrary is explicitly stated.

2 Basic notations and definitions

We recall the definitions of the involute and the evolute of a regular curve (cf. [6], [12], [13]). Let I be an interval of $\mathbb{R}$ and let $\mathbb{R}^2$ be the Euclidean plane with the inner product $\boldsymbol{a}\cdot\boldsymbol{b} = a_1b_1 + a_2b_2$, where $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2) \in \mathbb{R}^2$.

Suppose that $\gamma : I \to \mathbb{R}^2$ is a regular plane curve, that is, $\dot{\gamma}(t) = (d\gamma/dt)(t) \neq 0$ for any $t \in I$. We have the unit tangent vector $\boldsymbol{t}(t) = \dot{\gamma}(t)/|\dot{\gamma}(t)|$ and the unit normal vector $\boldsymbol{n}(t) = J(\boldsymbol{t}(t))$, where $|\dot{\gamma}(t)| = \sqrt{\dot{\gamma}(t) \cdot \dot{\gamma}(t)}$ and J is the anti-clockwise rotation by $\pi/2$ on $\mathbb{R}^2$. Then we have the Frenet formula

$$\begin{pmatrix} \dot{\boldsymbol{t}}(t) \\ \dot{\boldsymbol{n}}(t) \end{pmatrix} = \begin{pmatrix} 0 & |\dot{\gamma}(t)|\kappa(t) \\ -|\dot{\gamma}(t)|\kappa(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t}(t) \\ \boldsymbol{n}(t) \end{pmatrix},$$

where the curvature is given by $\kappa(t) = \dot{\boldsymbol{t}}(t) \cdot \boldsymbol{n}(t)/|\dot{\gamma}(t)| = \det(\dot{\gamma}(t), \ddot{\gamma}(t))/|\dot{\gamma}(t)|^3$. Note that the curvature $\kappa(t)$ is independent of the choice of a parametrisation up to sign.

In this paper, we consider involutes and evolutes of plane curves. For $t_0 \in I$, the *involute* $Inv(\gamma, t_0) : I \to \mathbb{R}^2$ *of a regular plane curve* $\gamma : I \to \mathbb{R}^2$ *at* t_0 is given by $Inv(\gamma, t_0)(t) = \gamma(t) - \int_{t_0}^{t} |\dot{\gamma}(u)| du \boldsymbol{t}(t)$ and the *evolute* $Ev(\gamma) : I \to \mathbb{R}^2$ *of a regular plane curve* $\gamma : I \to \mathbb{R}^2$ is given by $Ev(\gamma)(t) = \gamma(t) + (1/\kappa(t))\boldsymbol{n}(t)$, away from inflection points, that is, $\kappa(t) \neq 0$.

We give examples of an involute and an evolute of a regular curve, for more examples see [6], [12], [13].

Example 1 Let $\gamma : [-\pi, \pi) \to \mathbb{R}^2$ be a circle $\gamma(t) = (r\cos t, r\sin t)$ with radius $r > 0$. Then the involute of the circle at t_0 is

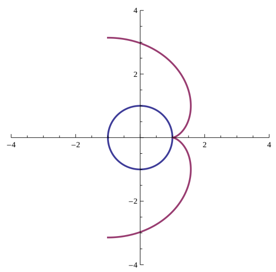
$$Inv(\gamma, t_0)(t) = (r\cos t + (t - t_0)r\sin t, r\sin t - (t - t_0)r\cos t).$$

In Figure 1, the involute of the circle with $r = 1$ at $t_0 = 0$ is depicted.

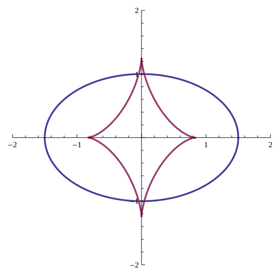
Example 2 Let $\gamma : [0, 2\pi) \to \mathbb{R}^2$ be an ellipse $\gamma(t) = (a\cos t, b\sin t)$ with $a, b > 0$ and $a \neq b$. Then the evolute of the ellipse is

$$Ev(\gamma)(t) = \left(\frac{a^2 - b^2}{a} \cos^3 t, -\frac{a^2 - b^2}{b} \sin^3 t \right).$$

In Figure 2, the evolute of the ellipse with $a = 3/2, b = 1$ is shown.



The involute of the circle at 0.
Figure 1.



The evolute of the ellipse.
Figure 2.

The following properties are well-known in classical differential geometry of curves:

Proposition 1 *Let $\gamma : I \to \mathbb{R}^2$ be a regular curve and $t_0 \in I$.*

(1) *If t is a regular point of $Inv(\gamma, t_0)$, then we have $Ev(Inv(\gamma, t_0))(t) = \gamma(t)$.*

(2) *If t and t_0 are regular points of $Ev(\gamma)$ and not inflection points of γ, then we have $Inv(Ev(\gamma), t_0)(t) = \gamma(t) + (1/\kappa(t_0))\boldsymbol{n}(t)$.*

Even if γ is a regular curve, the base point t_0 is always a singular point of the involute $Inv(\gamma, t_0)$ and also the evolute $Ev(\gamma)$ may have singularities, see Figures 1 and 2. For a singular point of $Inv(\gamma, t_0)$ (respectively, $Ev(\gamma)$), $Ev(Inv(\gamma, t_0))(t)$ (respectively, $Inv(Ev(\gamma), t_0)(t)$) cannot be defined.

In general, if γ is not a regular curve, then we cannot define the involute and the evolute of the curve as above. In [10], we defined the evolute of the front without inflection points in the Euclidean plane, see Definition 2. In this paper, we define an involute of the front in the Euclidean plane, see Definition 3. These are generalisations of evolutes and involutes of regular plane curves. In order to define an evolute and an involute of the front, we review Legendre curves in the unit tangent bundle, the Frenet formula and the curvature of the Legendre curve (cf. [9]).

We say that $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ is *a Legendre curve* if $(\gamma, \nu)^*\theta = 0$, where θ is the canonical contact 1-form on the unit tangent bundle $T_1\mathbb{R}^2 = \mathbb{R}^2 \times S^1$ and S^1 is the unit circle (cf. [2], [3], [4]). This condition is equivalent to $\dot{\gamma}(t) \cdot \nu(t) = 0$ for all $t \in I$. Moreover, if (γ, ν) is an immersion, we call (γ, ν) *a Legendre immersion*. We say that $\gamma : I \to \mathbb{R}^2$ is *a frontal* (respectively, *a front* or *a wave front*) if there exists a smooth mapping $\nu : I \to S^1$ such that (γ, ν) is a Legendre curve (respectively, a Legendre immersion).

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve. Then we have the Frenet formula of the frontal γ as follows. We put $\boldsymbol{\mu}(t) = J(\nu(t))$. We call the pair $\{\nu(t), \boldsymbol{\mu}(t)\}$ a *moving frame along the frontal* $\gamma(t)$ in $\mathbb{R}^2$ and the Frenet formula of the frontal (or, the Legendre curve) which is given by

$$\begin{pmatrix} \dot{\nu}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) \\ -\ell(t) & 0 \end{pmatrix} \begin{pmatrix} \nu(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \dot{\nu}(t) \cdot \boldsymbol{\mu}(t)$. Moreover, there exists a smooth function $\beta(t)$ such that $\dot{\gamma}(t) = \beta(t)\boldsymbol{\mu}(t)$. The pair (ℓ, β) is an important invariant of Legendre curves (or, frontals). We call the pair $(\ell(t), \beta(t))$ *the curvature of the Legendre curve* (with respect to the parameter t).

Definition 1 Let (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves. We say that (γ, ν) and $(\widetilde{\gamma}, \widetilde{\nu})$ are *congruent as Legendre curves* if there exists a congruence C on $\mathbb{R}^2$ such that $\widetilde{\gamma}(t) = C(\gamma(t)) = A(\gamma(t)) + \boldsymbol{b}$ and $\widetilde{\nu}(t) = A(\nu(t))$ for all $t \in I$, where C is given by a rotation A and a translation $\boldsymbol{b}$ on $\mathbb{R}^2$.

We have the existence and the uniqueness for Legendre curves in the unit tangent bundle analogously to the case of regular plane curves, see [9].

Theorem 1 (The Existence Theorem) *Let $(\ell, \beta) : I \to \mathbb{R}^2$ be a smooth mapping. There exists a Legendre curve $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ whose associated curvature of the Legendre curve is (ℓ, β).*

Theorem 2 (The Uniqueness Theorem) *Let (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu}) : I \to \mathbb{R}^2 \times S^1$ be Legendre curves whose curvatures of Legendre curves (ℓ,β) and $(\widetilde{\ell},\widetilde{\beta})$ coincide. Then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ are congruent as Legendre curves.*

In fact, the Legendre curve whose associated curvature of the Legendre curve is (ℓ,β), is given by the form

$$\gamma(t) = \left(-\int \beta(t)\sin\left(\int \ell(t)dt\right)dt,\ \int \beta(t)\cos\left(\int \ell(t)dt\right)dt\right),$$
$$\nu(t) = \left(\cos\int \ell(t)dt,\ \sin\int \ell(t)dt\right).$$

We give examples of Legendre curves.

Example 3 One of the typical examples of a front (and hence a frontal) is a regular plane curve. Let $\gamma : I \to \mathbb{R}^2$ be a regular plane curve. In this case, we may take $\nu : I \to S^1$ by $\nu(t) = \boldsymbol{n}(t)$. Then it is easy to check that $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion (a Legendre curve). By a direct calculation, the relationship between the curvature of the Legendre curve $(\ell(t),\beta(t))$ and the curvature $\kappa(t)$ is given by $\ell(t) = -\beta(t)\kappa(t)$.

Example 4 Let n, m and k be natural numbers with $m = n+k$. Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be $\gamma(t) = (t^n/n, t^m/m)$, $\nu(t) = (1/\sqrt{t^{2k}+1})\left(-t^k, 1\right)$. It is easy to see that (γ,ν) is a Legendre curve, and a Legendre immersion when $n = 1$ or $k = 1$. We call γ of *type* (n, m). For example, the frontal of type $(2,3)$ has the $3/2$ cusp (A_2 singularity) at $t = 0$, that of type $(3,4)$ has the $4/3$ cusp (E_6 singularity) at $t = 0$, see Figure 3. By definition, we have $\boldsymbol{\mu}(t) = (1/\sqrt{t^{2k}+1})(-1, -t^k)$ and $\ell(t) = kt^{k-1}/(t^{2k}+1), \beta(t) = -t^{n-1}\sqrt{t^{2k}+1}$.

Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre curve with the curvature of the Legendre immersion (ℓ,β). We say that $t_0 \in I$ is *an inflection point of the frontal* (or, *of a Legendre curve* (γ,ν)) if $\ell(t_0) = 0$. Note that if t_0 is a regular point of γ, the definition of the inflection point coincides with the usual inflection point for regular curves by Example 3. If a Legendre curve (γ,ν) does not have inflection points, then (γ,ν) is a Legendre immersion.

In [10], we have defined the evolute of the front without inflection points in the Euclidean plane by using parallel curves of the front. Here, we recall an alternative definition of the evolute of the front as follows, see Theorem 3.3 in [10]. Throughout the paper, we assume that $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion without inflection points. We denote the curvature of the Legendre immersion by (ℓ,β).

Definition 2 The evolute $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ of the front γ without inflection points is given by $\mathcal{E}v(\gamma)(t) = \gamma(t) - (\beta(t)/\ell(t))\nu(t)$.

The definition of the evolute $\mathcal{E}v(\gamma)$ of the front is a generalisation of the evolute $Ev(\gamma)$ of a regular curve γ. For properties of the evolute of the front see [10]. We define the involute of the front as follows:

Definition 3 The involute $\mathcal{I}nv(\gamma, t_0) : I \to \mathbb{R}^2$ of the front γ at $t_0 \in I$ is given by $\mathcal{I}nv(\gamma, t_0)(t) = \gamma(t) - \int_{t_0}^t \beta(u)du\boldsymbol{\mu}(t)$.

For a regular plane curve $\gamma : I \to \mathbb{R}^2$, we consider $\boldsymbol{n}(t) = \nu(t)$ in Example 3. It follows that $\boldsymbol{t}(t) = -\boldsymbol{\mu}(t)$ and $|\dot{\gamma}(t)| = -\beta(t)$. Therefore, we have the following result.

Proposition 2 *For a regular curve $\gamma : I \to \mathbb{R}^2$ and any $t_0 \in I$, we have $\mathcal{I}nv(\gamma, t_0) = Inv(\gamma, t_0)$.*

Remark 1 The evolute and the involute of the front are independent of the choice of a parametrisation. Moreover, if the set of regular points of γ is dense, then $\mathcal{E}v(\gamma)$ and $\mathcal{I}nv(\gamma, t_0)$ are uniquely determined by γ, namely, they do not depend on the choice of ν.

Proposition 3 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature of the Legendre immersion (ℓ, β) and without inflection points.*

(1) *The evolute $\mathcal{E}v(\gamma) : I \to \mathbb{R}^2$ is a front. More precisely, the evolute $(\mathcal{E}v(\gamma)(t), J(\nu(t))) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature*

$$\left(\ell(t),\ \frac{d}{dt}\left(\frac{\beta(t)}{\ell(t)}\right)\right).$$

(2) *The involute $\mathcal{I}nv(\gamma, t_0) : I \to \mathbb{R}^2$ is a front for any $t_0 \in I$. More precisely, the involute $(\mathcal{I}nv(\gamma, t_0)(t), J^{-1}(\nu(t))) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature*

$$\left(\ell(t),\ \ell(t)\int_{t_0}^t \beta(u)du\right).$$

Proof (1) By using the Frenet formula of the front, we have $\dot{\mathcal{E}v}(\gamma)(t) = (d/dt)\,(\beta(t)/\ell(t))\,J(\boldsymbol{\mu}(t))$. Therefore, we have $\dot{\mathcal{E}v}(\gamma)(t) \cdot J(\nu(t)) = 0$. Since $(d/dt)(J(\nu(t))) = J(\dot{\nu}(t)) = J(\ell(t)\boldsymbol{\mu}(t)) = \ell(t)J(\boldsymbol{\mu}(t)) \neq 0$, it holds that the evolute $(\mathcal{E}v(\gamma)(t), J(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), (d/dt)(\beta(t)/\ell(t)))$.

(2) By direct calculation, we have $\dot{\mathcal{I}nv}(\gamma, t_0)(t) = \ell(t)\int_{t_0}^t \beta(u)duJ^{-1}(\boldsymbol{\mu}(t))$. Therefore, we have $\dot{\mathcal{I}nv}(\gamma, t_0)(t) \cdot J^{-1}(\nu(t)) = 0$. Since $(d/dt)(J^{-1}(\nu(t))) = J^{-1}(\dot{\nu}(t)) = J^{-1}(\ell(t)\boldsymbol{\mu}(t)) = \ell(t)J^{-1}(\boldsymbol{\mu}(t)) \neq 0$, it holds that the involute $(\mathcal{I}nv(\gamma, t_0)(t), J^{-1}(\nu(t)))$ is a Legendre immersion with the curvature $(\ell(t), \ell(t)\int_{t_0}^t \beta(u)du)$. $\square$

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ, β) and without inflection points. By Proposition 3, $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ and $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ are also a Legendre immersion without inflection points for any $t_0 \in I$. We give a justification of Proposition 1 with singular points.

Proposition 4 *For any $t_0 \in I$, we have the following:*

(1) $\mathcal{E}v(\mathcal{I}nv(\gamma, t_0))(t) = \gamma(t)$.

(2) $\mathcal{I}nv(\mathcal{E}v(\gamma), t_0)(t) = \gamma(t) - (\beta(t_0)/\ell(t_0))\nu(t)$.

Proof (1) By the definition of the evolute and Proposition 3, we have

$$\mathcal{E}v(\mathcal{I}nv(\gamma,t_0))(t)=\mathcal{I}nv(\gamma,t_0)(t)-\frac{\ell(t)\int_{t_0}^t\beta(u)du}{\ell(t)}J^{-1}(\nu(t))=\gamma(t).$$

(2) By the definition of the involute and Proposition 3, we have

$$\mathcal{I}nv(\mathcal{E}v(\gamma),t_0)(t)=\mathcal{E}v(\gamma)(t)-\int_{t_0}^t\frac{d}{du}\left(\frac{\beta(u)}{\ell(u)}\right)duJ(\boldsymbol{\mu}(t))=\gamma(t)-\frac{\beta(t_0)}{\ell(t_0)}\nu(t).$$

□

For a given Legendre immersion $(\gamma,\nu):I\to\mathbb{R}^2\times S^1$, we consider an existence condition of a Legendre immersion $(\widetilde{\gamma},\widetilde{\nu}):I\to\mathbb{R}^2\times S^1$ such that $\mathcal{E}v(\widetilde{\gamma})(t)=\gamma(t)$ or $\mathcal{I}nv(\widetilde{\gamma},t_0)(t)=\gamma(t)$ for some t_0. By using Proposition 4 and Theorem 2, we have the following Corollaries.

Corollary 1 (1) *If* $(\widetilde{\gamma}(t),\widetilde{\nu}(t))=(\mathcal{I}nv(\gamma,t_0)(t)+\lambda J^{-1}(\nu(t)),J^{-1}(\nu(t)))$ *for any* $t_0\in I$ *and any* $\lambda\in\mathbb{R}$*, then we have* $\mathcal{E}v(\widetilde{\gamma})(t)=\gamma(t)$.

(2) *If* $(\widetilde{\gamma}(t),\widetilde{\nu}(t))=(\mathcal{E}v(\gamma)(t),J(\nu(t)))$ *and* t_0 *is a singular point of* γ*, then we have* $\mathcal{I}nv(\widetilde{\gamma},t_0)(t)=\gamma(t)$.

Proof (1) Since $\widetilde{\gamma}(t)$ is a parallel curve of $\mathcal{I}nv(\gamma,t_0)(t)$, it holds that $\mathcal{E}v(\widetilde{\gamma})(t)$ $=\mathcal{E}v(\mathcal{I}nv(\gamma,t_0))(t)$. It follows from Proposition 4 that $\mathcal{E}v(\widetilde{\gamma})(t)=\gamma(t)$.

(2) By Proposition 4 and $\beta(t_0)=0$, we have $\mathcal{I}nv(\widetilde{\gamma},t_0)(t)=\gamma(t)$. □

Conversely, we have the following result.

Corollary 2 *Let* (γ,ν) *and* $(\widetilde{\gamma},\widetilde{\nu}):I\to\mathbb{R}^2\times S^1$ *be Legendre immersions with the curvatures* (ℓ,β) *and* $(\widetilde{\ell},\widetilde{\beta})$ *respectively, and without inflection points.*

(1) *If* $(\mathcal{E}v(\widetilde{\gamma})(t),J(\widetilde{\nu}(t)))$ *and* $(\gamma(t),\nu(t))$ *are congruent as Legendre immersions, then* $(\widetilde{\gamma}(t),\widetilde{\nu}(t))$ *and* $(\mathcal{I}nv(\gamma,t_0)(t)+(\widetilde{\beta}(t_0)/\widetilde{\ell}(t_0))J^{-1}(\nu(t)),J^{-1}(\nu(t)))$ *are congruent as Legendre immersions for any* $t_0\in I$.

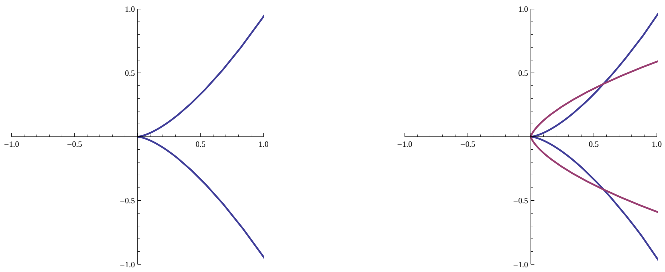
(2) *Let* $t_0\in I$. *If* $(\mathcal{I}nv(\widetilde{\gamma},t_0)(t),J^{-1}(\widetilde{\nu}(t)))$ *and* $(\gamma(t),\nu(t))$ *are congruent as Legendre immersions, then* $(\widetilde{\gamma}(t),\widetilde{\nu}(t))$ *and* $(\mathcal{E}v(\gamma)(t),J(\nu(t)))$ *are congruent as Legendre immersions, and* t_0 *is a singular point of* γ.

We give an example of an involute of a front. Examples of evolutes of fronts are presented in [10].

Example 5 Let $(\gamma,\nu):\mathbb{R}\to\mathbb{R}^2\times S^1$ be of type $(2,3)$ in Example 4, $\gamma(t)=\left(t^2/2,t^3/3\right),\nu(t)=(1/\sqrt{t^2+1})\left(-t,1\right)$. We have $\boldsymbol{\mu}(t)=(-1/\sqrt{t^2+1})(1,t)$, $\ell(t)=1/(t^2+1)$ and $\beta(t)=-t\sqrt{t^2+1}$. It follows that the involute of the 3/2 cusp at $t_0\in\mathbb{R}$ is given by

$$\mathcal{I}nv(\gamma,t_0)(t)=\left(\frac{t^2}{6}-\frac{1}{3}+\frac{1}{3}\frac{(t_0^2+1)^{\frac{3}{2}}}{\sqrt{t^2+1}},\ -\frac{t}{3}+\frac{1}{3}\frac{(t_0^2+1)^{\frac{3}{2}}}{\sqrt{t^2+1}}t\right).$$

Note that the involute of the 3/2 cusp at $t_0=0$ is diffeomorphic to the 4/3 cusp at 0, see Figure 3 and Corollary 3 below.



The 3/2 cusp. The involute of the 3/2 cusp at 0.

Figure 3.

3 Properties of involutes of fronts

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ, β) and without inflection points. We give properties of the involute of the front.

Proposition 5 *For any points $t_0, t_1 \in I$, $\mathcal{I}nv(\gamma, t_1)$ is a parallel curve of $\mathcal{I}nv(\gamma, t_0)$.*

Proof By the definition of the involute, we have

$$\mathcal{I}nv(\gamma, t_1)(t) = \gamma(t) - \int_{t_1}^{t} \beta(u)du\boldsymbol{\mu}(t) = \mathcal{I}nv(\gamma, t_0)(t) + \int_{t_1}^{t_0} \beta(u)du J^{-1}(\nu(t)).$$

Since $J^{-1}(\nu(t))$ is the unit normal of $\mathcal{I}nv(\gamma, t_0)(t)$, $\mathcal{I}nv(\gamma, t_1)$ is a parallel curve of $\mathcal{I}nv(\gamma, t_0)$. □

We analyse singular points of the involute of the front.

Proposition 6 *Let $t_0 \in I$.*

(1) *t_1 is a singular point of $\mathcal{I}nv(\gamma, t_0)$ if and only if $\int_{t_0}^{t_1} \beta(s)ds = 0$.*

(2) *Suppose that t_1 is a singular point of $\mathcal{I}nv(\gamma, t_0)$. Then $\mathcal{I}nv(\gamma, t_0)$ is diffeomorphic to the $3/2$ cusp at t_1 if and only if $\beta(t_1) \neq 0$.*

(3) *Suppose that t_1 is a singular point of $\mathcal{I}nv(\gamma, t_0)$. Then $\mathcal{I}nv(\gamma, t_0)$ is diffeomorphic to the $4/3$ cusp at t_1 if and only if $\beta(t_1) = 0$ and $\dot{\beta}(t_1) \neq 0$.*

Proof (1) By differentiating of the involute of the front, we have $\dot{\mathcal{I}}nv(\gamma, t_0)(t) = \ell(t) \int_{t_0}^{t} \beta(u)du\nu(t)$. By the assumption $\ell(t) \neq 0$ for all $t \in I$, we have the result.

(2) From the Frenet formula of the front, we have

$$\ddot{\mathcal{I}}nv(\gamma, t_0)(t) = \left(\dot{\ell}(t) \int_{t_0}^{t} \beta(u)du + \ell(t)\beta(t) \right) \nu(t) + \ell(t)^2 \int_{t_0}^{t} \beta(u)du \ \boldsymbol{\mu}(t).$$

By (1), we obtain $\ddot{\mathcal{I}}nv(\gamma, t_0)(t_1) = \ell(t_1)\beta(t_1)\nu(t_1)$. Moreover, we have

$$\dddot{\mathcal{I}}nv(\gamma, t_0)(t_1) = \left(2\dot{\ell}(t_1)\beta(t_1) + \ell(t_1)\dot{\beta}(t_1) \right) \nu(t_1) + 2\ell(t_1)^2\beta(t_1)\boldsymbol{\mu}(t_1).$$

Thus, $\det\Big(\ddot{\mathcal{I}nv}(\gamma,t_0)(t_1),\dddot{\mathcal{I}nv}(\gamma,t_0)(t_1)\Big)=2\ell(t_1)^3\beta(t_1)^2\neq 0$ if and only if $\beta(t_1)\neq 0$. Therefore, we obtain (2).

(3) By (2), $\det\Big(\ddot{\mathcal{I}nv}(\gamma,t_0)(t_1),\dddot{\mathcal{I}nv}(\gamma,t_0)(t_1)\Big)=0$ if and only if $\beta(t_1)=0$. Moreover, under the conditions $\int_{t_0}^{t_1}\beta(u)du=0$ and $\beta(t_1)=0$, we have

$$\mathcal{I}nv^{(4)}(\gamma,t_0)(t_1)=\Big(3\dot{\ell}(t_1)\dot{\beta}(t_1)+\ell(t_1)\ddot{\beta}(t_1)\Big)\nu(t_1)+3\ell(t_1)^2\dot{\beta}(t_1)\boldsymbol{\mu}(t_1)$$

and hence $\det\big(\dddot{\mathcal{I}nv}(\gamma,t_0)(t_1),\mathcal{I}nv^{(4)}(\gamma,t_0)(t_1)\big)=3\ell(t_1)^3\dot{\beta}(t_1)^2$. Thus,

$$\det\Big(\ddot{\mathcal{I}nv}(\gamma,t_0)(t_1),\dddot{\mathcal{I}nv}(\gamma,t_0)(t_1)\Big)=0,$$

$$\det\Big(\dddot{\mathcal{I}nv}(\gamma,t_0)(t_1),\mathcal{I}nv^{(4)}(\gamma,t_0)(t_1)\Big)\neq 0$$

if and only if $\beta(t_1)=0,\dot{\beta}(t_1)\neq 0$. It follows that $\mathcal{I}nv(\gamma,t_0)$ is diffeomorphic to the 4/3 cusp at t_1 (cf. [5], [16], [17]). Therefore, we obtain (3). □

By Proposition 6, we have the following Corollary:

Corollary 3 *Under the above notations, we have the following.*

(1) *$\mathcal{I}nv(\gamma,t_0)$ is diffeomorphic to the 3/2 cusp at t_0 if and only if t_0 is a regular point of γ.*

(2) *$\mathcal{I}nv(\gamma,t_0)$ is diffeomorphic to the 4/3 cusp at t_0 if and only if γ is diffeomorphic to the 3/2 cusp at t_0.*

Remark 2 In this paper, we assume that the front does not have inflection points, though we can define the involute of the front with inflection points. In this case, the involute of the front is a frontal (cf. [11]). We can find other kinds of singularities of the involute, see [1], [11], [22].

Lemma 1 *If $t_1\in I\setminus\{t_0\}$ is a singular point of $\mathcal{I}nv(\gamma,t_0)$, then there exists at least one singular point of γ in the open interval (t_0,t_1) (respectively, (t_1,t_0)) when $t_0<t_1$ (respectively, $t_1<t_0$).*

Proof We show the case for $t_0<t_1$. By the mean value theorem for integration, there exists a point $\xi\in(t_0,t_1)$ such that $\int_{t_0}^{t_1}\beta(u)du=\beta(\xi)(t_1-t_0)$. Since t_1 is a singular point of $\mathcal{I}nv(\gamma,t_0)$, we have $\int_{t_0}^{t_1}\beta(u)du=0$. It follows that $\beta(\xi)=0$, that is, ξ is a singular point of γ. □

Next we discuss a relationship between singular points of an involute of the front and vertices. Let (γ,ν) be a Legendre immersion with the curvature of the Legendre immersion (ℓ,β) and without inflection points. We say that a point t_0 is a *vertex of the front* γ (or, *of the Legendre immersion* (γ,ν)) if $(d/dt)(\beta/\ell)(t_0)=0$, equivalently $(d/dt)\mathcal{E}v(t_0)=0$, that is, a singular point of the evolute. Note that if t_0 is a regular point of γ, the definition of the vertex coincides with the usual vertex for regular curves (cf. [10]).

In this paper, we say that a Legendre immersion $(\gamma,\nu):[a,b]\to\mathbb{R}^2\times S^1$ is *closed* if $(\gamma^{(n)}(a),\nu^{(n)}(a))=(\gamma^{(n)}(b),\nu^{(n)}(b))$ for all $n\in\mathbb{N}\cup\{0\}$, where

$\gamma^{(n)}(a)$, $\nu^{(n)}(a)$, $\gamma^{(n)}(b)$ and $\nu^{(n)}(b)$ means one-sided differential. If $(\gamma,\nu): [a,b] \to \mathbb{R}^2 \times S^1$ is a closed Legendre immersion, then either a and b are regular points of γ or singular points of γ. When a and b are singular points of γ, we treat these singular points as one singular point of γ.

Note that if $(\mathcal{I}nv(\gamma,t_0), J^{-1}(\nu)) : [a,b] \to \mathbb{R}^2 \times S^1$ is a closed Legendre immersion, then (γ,ν) is also closed. By Lemma 1 and Proposition 3.11 in [10], we have the following Lemma.

Lemma 2 *Let $(\gamma,\nu) : [a,b] \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points. Suppose that $(\mathcal{I}nv(\gamma,t_0), J^{-1}(\nu))$ is a closed Legendre immersion and singular points of γ and $\mathcal{I}nv(\gamma,t_0)$ are finite. Then*

$$\sharp\Sigma(\mathcal{I}nv(\gamma,t_0)) \leq \sharp\Sigma(\gamma) \leq \sharp V(\gamma), \tag{1}$$

where $\Sigma(\mathcal{I}nv(\gamma,t_0))$ (respectively, $\Sigma(\gamma)$) is the set of singular points of the involute $\mathcal{I}nv(\gamma,t_0)$ (respectively, γ) and $V(\gamma)$ is the set of vertices of the front γ.

Proof We show the first inequality. Suppose that $s_0,\dots,s_n$ are singular points of $\mathcal{I}nv(\gamma,t_0)$ such that $a < s_0 < s_1 < \cdots < s_n < b$. By Lemma 1, there is at least one singular point of γ in the open interval (s_{i-1},s_i) for each $i = 1,\dots,n$. We show that there is at least one singular point of γ in $(s_n,b] \cup [a,s_0)$. Since $(\mathcal{I}nv(\gamma,t_0), J^{-1}(\nu))$ and (γ,ν) are closed Legendre immersions, we have $\int_{t_0}^b \beta(u)du = \int_{t_0}^a \beta(u)du$, that is, $\int_a^b \beta(u)du = 0$. If $\beta(t) > 0$ (respectively, $\beta(t) < 0$) on $(s_n,b] \cup [a,s_0)$, then $\int_{t_0}^t \beta(u)du$ is a monotone increasing function (respectively, monotone decreasing function) on $(s_n,b]$ and $[a,s_0)$. Hence $0 = \int_{t_0}^{s_n} \beta(u)du < \int_{t_0}^b \beta(u)du$ and $\int_{t_0}^a \beta(u)du < \int_{t_0}^{s_0} \beta(u)du = 0$ (respectively, $0 = \int_{t_0}^{s_n} \beta(u)du > \int_{t_0}^b \beta(u)du$ and $\int_{t_0}^a \beta(u)du > \int_{t_0}^{s_0} \beta(u)du = 0$). This implies $\int_a^b \beta(u)du = \int_{t_0}^b \beta(u)du - \int_{t_0}^a \beta(u)du > 0$ (respectively, $\int_a^b \beta(u)du = \int_{t_0}^b \beta(u)du - \int_{t_0}^a \beta(u)du < 0$). This contradicts the fact $\int_a^b \beta(u)du = 0$. Therefore, there exists a point $\xi \in (s_n,b] \cup [a,s_0)$ such that $\beta(\xi) = 0$.

Next, we suppose that $s_0,\dots,s_n$ are singular points of $\mathcal{I}nv(\gamma,t_0)$ such that $a = s_0 < s_1 < \cdots < s_n = b$. In this case, there are n singular points of the involute (note that we treat a and b as one singular point). By Lemma 1, there is at least one singular point of γ in each interval (s_{i-1},s_i), $i = 1,\dots n$. Hence the inequality holds.

The second inequality is a direct consequence of the proof of Proposition 3.11 in [10]. Also see Remark 3 below. □

Remark 3 By definition, the set of vertices of the front γ is the set of singular points of $\mathcal{E}v(\gamma)$. By Proposition 4, we can also prove the second inequality of (1) in Lemma 2 by the same method as the first inequality.

Remark 4 If $(\mathcal{I}nv(\gamma,t_0), J^{-1}(\nu)) : [a,b] \to \mathbb{R}^2 \times S^1$ is a closed Legendre immersion, then $\mathcal{I}nv(\gamma,t_0)(a) = \mathcal{I}nv(\gamma,t_0)(b)$ and hence $\int_a^b \beta(s)ds = 0$. It follows that γ must have a singular point. As a consequence, if γ is a regular curve, then $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$ cannot be a closed Legendre immersion (cf. Figure 1).

We give conditions that the front has at least four vertices. (cf. [10]).

Proposition 7 *Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion without inflection points. Suppose that $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu))$ is a closed Legendre immersion.*

(1) *If $\mathcal{I}nv(\gamma, t_0)$ has at least four singular points, then γ has at least four vertices.*

(2) *If $\mathcal{I}nv(\gamma, t_0)$ has at least two singular points which degenerate more than $3/2$ cusp, then γ has at least four vertices.*

Proof (1) This statement is obtained from the inequality in Lemma 2 directly.

(2) Suppose $\mathcal{I}nv(\gamma, t_0)$ has at least two singular points t_1 and t_2 which degenerate more than 3/2 cusp. By Proposition 6, we have $\int_{t_0}^{t_i} \beta(u)du = 0$ and $\beta(t_i) = 0$ for $i = 1, 2$. Thus t_1 and t_2 are singular points of γ. Moreover, by Lemma 1, there exists at least one singular point for each subset (t_1, t_2) and $I \setminus [t_1, t_2]$. Therefore, γ has at least four singular points. As a consequence, γ has at least four vertices by Lemma 2. □

4 The n-th evolutes and involutes of fronts

Let $(\gamma, \nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ, β) and without inflection points. By Proposition 3, $(\mathcal{E}v(\gamma), J(\nu)) : I \to \mathbb{R}^2 \times S^1$ and $(\mathcal{I}nv(\gamma, t_0), J^{-1}(\nu)) : I \to \mathbb{R}^2 \times S^1$ are also a Legendre immersion without inflection points for any $t_0 \in I$. Therefore, we can repeat the evolute and the involute of the front. In [10], we give the form of the n-th evolute of the front, where n is a natural number. We write $\mathcal{E}v^0(\gamma)(t) = \gamma(t)$ and $\mathcal{E}v^1(\gamma)(t) = \mathcal{E}v(\gamma)(t)$ for convenience. We define $\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v(\mathcal{E}v^{n-1}(\gamma))(t)$ and

$$\beta_0(t) = \beta(t), \quad \beta_n(t) = \frac{d}{dt}\left(\frac{\beta_{n-1}(t)}{\ell(t)}\right),$$

inductively.

Theorem 3 ([10]) *$(\mathcal{E}v^n(\gamma), J^n(\nu)) : I \to \mathbb{R}^2 \times S^1$ is a Legendre immersion with the curvature (ℓ, β_n), where the n-th evolute of the front is given by*

$$\mathcal{E}v^n(\gamma)(t) = \mathcal{E}v^{n-1}(\gamma)(t) - \frac{\beta_{n-1}(t)}{\ell(t)} J^{n-1}(\nu(t))$$

and J^n is n-times operation of J.

We also write $\mathcal{I}nv^0(\gamma, t_0)(t) = \gamma(t)$ and $\mathcal{I}nv^1(\gamma, t_0)(t) = \mathcal{I}nv(\gamma, t_0)(t)$ for convenience. We define $\mathcal{I}nv^n(\gamma, t_0)(t) = \mathcal{I}nv(\mathcal{I}nv^{n-1}(\gamma, t_0), t_0)(t)$ and

$$\beta_{-1}(t) = \ell(t) \int_{t_0}^{t} \beta(u)du, \quad \beta_{-n}(t) = \ell(t) \int_{t_0}^{t} \beta_{-n+1}(u)du$$

inductively. Then we give the form of the n-th involute of the front by using induction.

Theorem 4 $(\mathcal{I}nv^n(\gamma,t_0), J^{-n}(\nu)) : I \to \mathbb{R}^2 \times S^1$ *is a Legendre immersion with the curvature* (ℓ, β_{-n})*, where the* n*-th involute of the front* γ *at* t_0 *is given by*

$$\mathcal{I}nv^n(\gamma,t_0)(t) = \mathcal{I}nv^{n-1}(\gamma,t_0)(t) + \frac{\beta_{-n}(t)}{\ell(t)} J^{-n}(\nu(t))$$

and J^{-n} *is* n*-times operation of* J^{-1}*.*

Remark 5 We can consider n-th involutes of the front at different initial points. The difference is given by a parallel curve of the involute by Proposition 5. In this paper, we only consider the n-th involute of the front at the same initial point.

By Theorems 3 and 4, we have the following sequence of the Legendre immersions (the evolutes and the involutes) without inflection points,

$$\begin{aligned} \cdots \overset{\mathcal{I}nv}{\leftarrow} (\mathcal{I}nv^2(\gamma,t_0)(t), J^{-2}(\nu)(t)) \overset{\mathcal{I}nv}{\leftarrow} (\mathcal{I}nv(\gamma,t_0)(t), J^{-1}(\nu)(t)) \overset{\mathcal{I}nv}{\leftarrow} \\ (\gamma(t),\nu(t)) \overset{\mathcal{E}v}{\to} (\mathcal{E}v(\gamma)(t), J(\nu)(t)) \overset{\mathcal{E}v}{\to} (\mathcal{E}v^2(\gamma)(t), J^2(\nu)(t)) \overset{\mathcal{E}v}{\to} \cdots \end{aligned}$$

and the corresponding sequence of the curvatures of the evolutes and the involutes,

$$\begin{aligned} \cdots \leftarrow (\ell(t), \beta_{-2}(t)) \leftarrow (\ell(t), \beta_{-1}(t)) \leftarrow \\ (\ell(t), \beta(t)) \to (\ell(t), \beta_1(t)) \to (\ell(t), \beta_2(t)) \to \cdots . \end{aligned} \tag{2}$$

5 The arc-length parameter for $\boldsymbol{\nu}$

Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ be a Legendre immersion with the curvature (ℓ,β). If $\beta(t) \neq 0$ for all $t \in I$, then γ is a regular curve in $\mathbb{R}^2$. Thus, we can choose the arc-length parameter s of γ. On the other hand, if $\ell(t) \neq 0$ for all $t \in I$ (that is, without inflection points), then ν is a regular curve in $S^1 \subset \mathbb{R}^2$. Thus, we can choose the arc-length parameter s of ν. It follows that $\nu(s)$ and also $\boldsymbol{\mu}(s)$ are unit speed. By the same method for the arc-length parameter of regular plane curves, one can prove the following:

Proposition 8 *Let* $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ *be a Legendre immersion without inflection points, and let* $t_0 \in I$*. Then* ν *is parametrically equivalent to the unit speed curve*

$$\overline{\nu} : \overline{I} \to S^1; s \mapsto \overline{\nu}(s) = \nu \circ t(s),$$

under a change of parameter $t : \overline{I} \to I$ *with* $t(0) = t_0$ *and with* $t'(s) > 0$*.*

We call the above parameter s in Proposition 8 *the arc-length parameter for* ν. If t is the arc-length parameter for ν, then we have $|\ell(t)| = 1$ for all $t \in I$. Note that we may assume $\ell(t) = 1$ for all $t \in I$, if necessary, by a change of parameter $t \mapsto -t$.

In this section, we suppose that $\ell(t) = 1$ for all $t \in I$. Then the second components of the curvatures of the evolutes and the involutes in the sequence (2) are given by

$$\cdots \leftarrow \int_{t_0}^{t}\int_{t_0}^{t} \beta(t)dtdt \leftarrow \int_{t_0}^{t} \beta(t)dt \leftarrow \beta(t) \to \frac{d}{dt}\beta(t) \to \frac{d^2}{dt^2}\beta(t) \to \cdots .$$

As a result, the evolutes and the involutes of fronts are corresponding to the differential and the integral of the curvature of the Legendre immersion.

Next, we recall the notion of the contact between Legendre immersions (cf. [9]). Let $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1; t \mapsto (\gamma(t),\nu(t))$ and $(\widetilde{\gamma},\widetilde{\nu}) : \widetilde{I} \to \mathbb{R}^2 \times S^1; u \mapsto (\widetilde{\gamma}(u),\widetilde{\nu}(u))$ be Legendre immersions respectively and let k be a natural number. We say that (γ,ν) *and* $(\widetilde{\gamma},\widetilde{\nu})$ *have at least k-th order contact at* $t = t_0, u = u_0$ if $(d^i/dt^i)(\gamma,\nu)(t_0) = (d^i/du^i)(\widetilde{\gamma},\widetilde{\nu})(u_0)$ for $i = 0,\ldots,k-1$.

In general, we may assume that (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least first order contact at any point $t = t_0, u = u_0$, up to congruence as Legendre immersions. We denote the curvatures of the Legendre immersions $(\ell(t),\beta(t))$ of $(\gamma(t),\nu(t))$ and $(\widetilde{\ell}(u),\widetilde{\beta}(u))$ of $(\widetilde{\gamma}(u),\widetilde{\nu}(u))$, respectively.

Theorem 5 ([9, Theorem 3.1]) *If (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$, then*

$$\frac{d^i}{dt^i}(\ell,\beta)(t_0) = \frac{d^i}{du^i}(\widetilde{\ell},\widetilde{\beta})(u_0), \quad i = 0,\ldots,k-1. \tag{3}$$

Conversely, if the condition (3) *holds, then (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$, up to congruence as Legendre immersions.*

As a corollary of Theorem 5, we have the relationship between the contact of Legendre immersions, evolutes and involutes.

Corollary 4 *Under the above notations, we have the following.*

(1) *If (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$, up to congruence as Legendre immersions, then $(\mathcal{E}v(\gamma), J(\nu))$ and $(\mathcal{E}v(\widetilde{\gamma}), J(\widetilde{\nu}))$ have at least k-th order contact at $t = t_0, u = u_0$, up to congruence as Legendre immersions.*

(2) *$(\mathcal{I}nv(\gamma,t_0), J^{-1}(\nu))$ and $(\mathcal{I}nv(\widetilde{\gamma},u_0), J^{-1}(\widetilde{\nu}))$ have at least $(k+1)$-th order contact at $t = t_0, u = u_0$, up to congruence as Legendre immersions if and only if (γ,ν) and $(\widetilde{\gamma},\widetilde{\nu})$ have at least k-th order contact at $t = t_0, u = u_0$, up to congruence as Legendre immersions.*

Finally, we consider when a front and its n-th evolute or involute have the same shape as the original curve under the same parametrisation. Namely, is there a Legendre immersion $(\gamma,\nu) : I \to \mathbb{R}^2 \times S^1$ such that (γ,ν) and $(\mathcal{E}v^n(\gamma), J^n(\nu)) : I \to \mathbb{R}^2 \times S^1$ (respectively, $(\mathcal{I}nv^n(\gamma,t_0), J^{-n}(\nu)) : I \to \mathbb{R}^2 \times S^1$) are congruent?

Theorem 6 *Under the above notations, we have the following.*

(1) *Legendre immersions* (γ, ν) *and* $(\mathcal{E}v^n(\gamma), J^n(\nu))$ *are congruent as Legendre immersions if and only if the curvature of the Legendre immersion* $(\gamma(t), \nu(t))$ *is given by* $\ell(t) = 1$, $\beta(t) = \sum_{k=0}^{n-1} c_k e^{\lambda_n^k t}$, *where* c_k *is a constant,* λ_n *is a primitive* n*-th root of unity,* $\lambda_n^k = \cos(2\pi k/n) + i \sin(2\pi k/n)$ *for* $k = 0, \dots, n-1$ *and* i *is the imaginary unit.*

(2) *Legendre immersions* (γ, ν) *and* $(\mathcal{I}nv^n(\gamma, t_0), J^{-n}(\nu))$ *are congruent as Legendre immersions if and only if* (γ, ν) *is given by* $\gamma(t) = (a, b)$, $\nu(t) = (\cos t, \sin t)$, *up to congruence as Legendre immersions, where* $a, b \in \mathbb{R}$.

Proof (1) By $\ell(t) = 1$ and Theorem 3, we have $\beta(t) = (d^n\beta/dt^n)(t)$. The linear ordinary differential equation can be solved, and the general solution is given by $\beta(t) = \sum_{k=0}^{n-1} c_k e^{\lambda_n^k t}$, where c_k is a constant, λ_n is a primitive n-th root of unity, $\lambda_n^k = \cos(2\pi k/n) + i \sin(2\pi k/n)$ for $k = 0, \dots, n-1$ and i is the imaginary unit. By Theorem 2, the converse holds.

(2) By $\ell(t) = 1$ and Theorem 4, we have $\beta(t) = \int_{t_0}^t \cdots \left(\int_{t_0}^t \beta(t) dt \right) \cdots dt$, n-times integrations for $\beta(t)$. This is equivalent to the conditions $\beta(t) = (d^n\beta/dt^n)(t)$ and $(d^i\beta/dt^i)(t_0) = 0$ for $i = 0, \dots, n-1$. It follows from (1) that $c_k = 0$ for $k = 0, \dots, n-1$, namely, $\beta(t) = 0$ for all $t \in I$. By Theorem 1, we obtain $\gamma(t) = (a, b), \nu(t) = (\cos t, \sin t)$, up to congruence as Legendre immersions, where $a, b \in \mathbb{R}$. By a direct calculation, we have the converse. $\Box$

We give examples for the cases $n = 1, 2$ and 3 in Theorem 6 (1).

Example 6 (1) The case of $n = 1$ in Theorem 6 (1). Since $\ell(t) = 1$ and $\beta(t) = \dot{\beta}(t)$, we have $\beta(t) = ce^t$, where $c \in \mathbb{R}$. It follows that

$$\gamma(t) = \left(\frac{c}{2} e^t (\cos t - \sin t),\ \frac{c}{2} e^t (\cos t + \sin t) \right),\ \nu(t) = (\cos t,\ \sin t),$$

up to congruence. We draw the front $\gamma(t)$ for $c = 1$ as in Figure 4, left.

(2) The case of $n = 2$ in Theorem 6 (1). Since $\ell(t) = 1$ and $\beta(t) = \ddot{\beta}(t)$, we have $\beta(t) = c_1 e^t + c_2 e^{-t}$, where $c_1, c_2 \in \mathbb{R}$. It follows that $\gamma(t) = (\gamma_1(t), \gamma_2(t)), \nu(t) = (\cos t,\ \sin t)$, where

$$\gamma_1(t) = \frac{c_1}{2} e^t (\cos t - \sin t) + \frac{c_2}{2} e^{-t} (\cos t + \sin t),$$
$$\gamma_2(t) = \frac{c_1}{2} e^t (\cos t + \sin t) + \frac{c_2}{2} e^{-t} (\sin t - \cos t),$$

up to congruence. We draw the front $\gamma(t)$ for $c_1 = 1$ and $c_2 = -1$ as in Figure 4, middle. In this case, 0 is a singular point of γ.

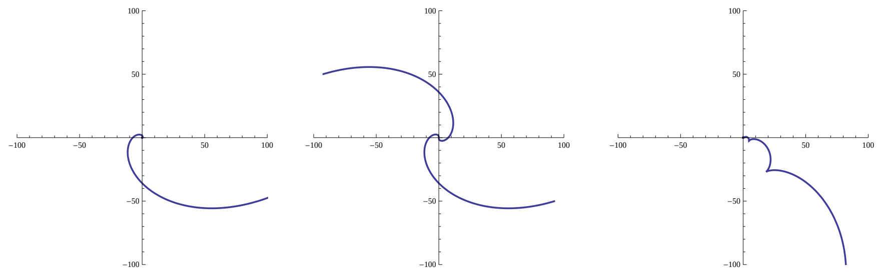
(3) The case of $n = 3$ in Theorem 6 (1). Since $\ell(t) = 1$ and $\beta(t) = \dddot{\beta}(t)$, we have

$$\beta(t) = c_1 e^t + c_2 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t + c_3 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t$$

as a smooth general solution, where $c_1, c_2, c_3 \in \mathbb{R}$. By a direct calculation, we have $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, $\nu(t) = (\cos t, \sin t)$, where

$$
\begin{aligned}
\gamma_1(t) = {} & \frac{c_1}{2} e^t (\cos t - \sin t) \\
& - c_2 e^{-t/2} \left(\frac{1}{2(2+\sqrt{3})} \left(-\frac{1}{2} \sin(\frac{\sqrt{3}}{2}+1)t - (\frac{\sqrt{3}}{2}+1) \cos(\frac{\sqrt{3}}{2}+1)t \right) \right. \\
& \left. - \frac{1}{2(2-\sqrt{3})} \left(-\frac{1}{2} \sin(\frac{\sqrt{3}}{2}-1)t - (\frac{\sqrt{3}}{2}-1) \cos(\frac{\sqrt{3}}{2}-1)t \right) \right) \\
& - c_3 e^{-t/2} \left(-\frac{1}{2(2+\sqrt{3})} \left(-\frac{1}{2} \cos(\frac{\sqrt{3}}{2}+1)t + (\frac{\sqrt{3}}{2}+1) \sin(\frac{\sqrt{3}}{2}+1)t \right) \right. \\
& \left. + \frac{1}{2(2-\sqrt{3})} \left(-\frac{1}{2} \cos(\frac{\sqrt{3}}{2}-1)t + (\frac{\sqrt{3}}{2}-1) \sin(\frac{\sqrt{3}}{2}-1)t \right) \right), \\
\gamma_2(t) = {} & \frac{c_1}{2} e^t (\cos t + \sin t) \\
& + c_2 e^{-t/2} \left(\frac{1}{2(2+\sqrt{3})} \left(-\frac{1}{2} \cos(\frac{\sqrt{3}}{2}+1)t + (\frac{\sqrt{3}}{2}+1) \sin(\frac{\sqrt{3}}{2}+1)t \right) \right. \\
& \left. + \frac{1}{2(2-\sqrt{3})} \left(-\frac{1}{2} \cos(\frac{\sqrt{3}}{2}-1)t + (\frac{\sqrt{3}}{2}-1) \sin(\frac{\sqrt{3}}{2}-1)t \right) \right) \\
& + c_3 e^{-t/2} \left(\frac{1}{2(2+\sqrt{3})} \left(-\frac{1}{2} \sin(\frac{\sqrt{3}}{2}+1)t - (\frac{\sqrt{3}}{2}+1) \cos(\frac{\sqrt{3}}{2}+1)t \right) \right. \\
& \left. + \frac{1}{2(2-\sqrt{3})} \left(-\frac{1}{2} \sin(\frac{\sqrt{3}}{2}-1)t - (\frac{\sqrt{3}}{2}-1) \cos(\frac{\sqrt{3}}{2}-1)t \right) \right),
\end{aligned}
$$

up to congruence. We draw the front $\gamma(t)$ for $c_1 = 0, c_2 = 0$ and $c_3 = 1$ as in Figure 4, right.



(1) $n = 1$, $c = 1$. (2) $n = 2$, $c_1 = 1$, $c_2 = -1$. (3) $n = 3$, $c_1 = c_2 = 0$, $c_3 = 1$.

Figure 4.

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