Dn-geometry and singularities of tangent surfaces

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By

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Abstract

The geometric model for D_n-Dynkin diagram is explicitly constructed and associated generic singularities of tangent surfaces are classified up to local diffeomorphisms. We observe, as well as the triality in D_4 case, the difference of the classification for D_3, D_4, D_5 and $D_n (n \geq 6)$, and a kind of stability of the classification in D_n for $n \to \infty$. Also we present the classifications of singularities of tangent surfaces for the cases $B_3, A_3 = D_3, G_2, C_2 = B_2$ and A_2 arising from D_4 by the processes of foldings and removings.

§ 1. Introduction

As was found by V.I. Arnol'd, the singularities of mappings are closely related to Dynkin diagrams (see [2][8]). The relations must be numerous. In this paper we are going to present one of them.

Associated to each semi-simple Lie algebra, there exists a geometric model which is a tree of fibrations of homogeneous spaces of the Lie group. We read out, from the Dynkin diagram or the root system, the associated geometric structure on the geometric model. More precisely, for each subset of vertices of Dynkin diagrams, we take the gradation on the Lie algebra. Then the gradation induces invariant distributions and cone structures on the quotients, which are called generalized flag manifolds, by the associated parabolic subgroups in the Lie group (see for instance [22][3]). Moreover the geometric structures which are homogeneous naturally induce singular objects to be

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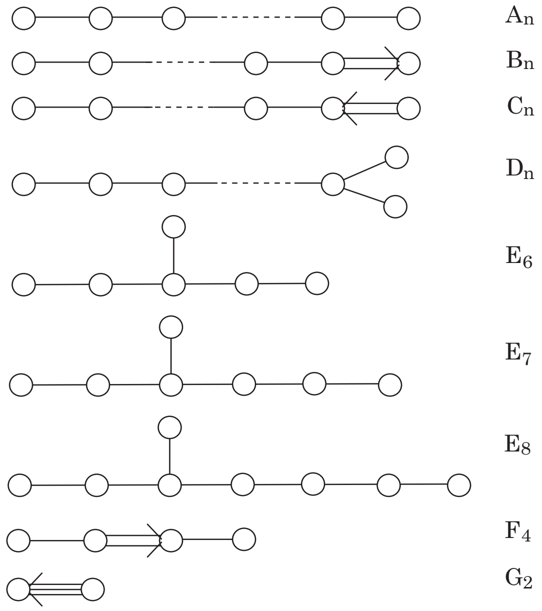
classified. The singularities of tangent surfaces are typical objects which we are going to study.

Recall the list of Dynkin diagrams of simple Lie algebras over $\mathbb{C}$,



and recall the complex Lie algebra of type A_n (resp. B_n, C_n, D_n) is the complex Lie algebra $\mathfrak{sl}(n+1,\mathbb{C})$ (resp. $\mathfrak{o}(2n+1,\mathbb{C})$, $\mathfrak{sp}(2n,\mathbb{C})$, $\mathfrak{o}(2n,\mathbb{C})$) of the classical complex Lie group $\mathrm{SL}(n+1,\mathbb{C})$ (resp. $\mathrm{O}(2n+1,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{O}(2n,\mathbb{C})$).

We have constructed, in the split real form, the geometric model and classified singularities of tangent surfaces which naturally appear in homogeneous spaces, for the case $B_2 = C_2$ in [15], and for the case G_2 in [16]. Note that the construction over $\mathbb{R}$ induces the complex construction after the complexification.

We observe that the Dynkin diagram has $\mathbb{Z}/2\mathbb{Z}$-symmetry in the cases A_n and D_n and $\mathfrak{S}_3$-symmetry only in the case D_4. The $\mathfrak{S}_3$-symmetry of Dynkin diagram (or root system) for D_4 induces the triality of D_4-geometry (see [6][7][21][18]). Cartan showed that the group of outer automorphisms of Lie algebra of type D_3 is isomorphic to $\mathfrak{S}_3$ in [6]. We would like to call the above fact and all phenomena which arise from it *triality of D_4-geometry.* In [17], we realize the geometric model explicitly for Lie algebra of type D_4 and study the triality of singularities of null tangent surfaces arising naturally from the geometric structures in D_4-geometry.

In this paper, we show the realization of the geometric models explicitly for Lie algebras of type $D_n, n \geq 3$, giving the stress on the speciality of D_4 in the class D_n and relations with other Dynkin diagrams. Then we observe the difference of the classification lists of null tangent surfaces for D_3, D_4, D_5 and $D_n (n \geq 6)$ (Theorems 6.1, 6.2, 6.3 and 6.4), and the stability of the classification lists of singularities of null tangent surfaces for D_n for $n \to \infty$. In fact, as is seen in the table of Theorem 6.4, the lists become steady without any degenerations if $n \geq 6$.

The results in case D_4 are closely related to the study of general relativity, for instance, the Kostant universe (see [9][11]). The singularities of null tangent surfaces are regarded as solution surfaces to a special kind of non-linear partial differential equation in the case D_4 (see [17]). Apart from the mathematical interest, the general classification results for D_n given in this paper will make clear the appearance of singularities in the D_4-case.

In this paper we treat a special kind of semi-Riemannian geometry ([20]). This reminds us the sub-Riemannian geometry ([19]). A sub-Riemannian structure on a manifold is a Riemannian metric on a distribution, i.e., a subbundle of the tangent bundle of the manifold. In [16], we encounter, as one of geometric structures in the G_2-geometric model, the Cartan distribution, which has the growth $(2, 3, 5)$ and, then for each point of any integral curve to the Cartan distribution, there exists the unique tangent "abnormal geodesic" to the curve at the point. Thus we have the tangent surface to the curve, whose singularities are studied in [16]. Note that also F_4-geometry is related to sub-Riemannian geometry. Also we note that B_n-geometry, for instance, $O(n+1, n)$-geometry, is related to conformal geometry. We have a plan to study them in forthcoming papers. We refer the following table:

Geometry	semi-Riemannian geometry	sub-Riemannian geometry
Geodesic	null geodesic	abnormal geodesic
Invariance	conformal invariant	distribution invariant
Tangent surface	null tangent surface	abnormal tangent surface
Simple Lie algebra	$D_n,\ B_n$	$G_2,\ F_4,\ E_6,\ E_7,\ E_8$

Note that A_n is related to projective geometry and C_n to symplectic (contact) geometry.

In §2, the D_n-geometry and the null projective space are explained, and, in §3, null tangent surfaces in the null projective space are introduced and a generic classification of singularities of null tangent surfaces is provided (Theorem 3.1). After introducing the null Grassmannians in §4, we construct the null flag manifold and the tree of fibrations for D_n-geometry in §5. We define the Engel distribution and give the detailed classification results of tangent surfaces in §6 (Theorems 6.1, 6.2, 6.3 and 6.4). For the proofs of Theorems, we describe the flag and Grassmannian coordinates and projections of Engel integral curves in §7 and relate the orders of projections with the root decompositions of Lie algebras of type D_n in §8. Then we give the proof of Theorems in §9, using the known results in [13]. We give the explicit descriptions explained in previous sections for D_3 case in §10. In §11, we show similar classifications of singularities of tangent surfaces for Dynkin diagrams arising from D_4 by the processes of foldings and removings.

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§ 2. D_n-geometry

Let $V = \mathbb{R}^{n,n} = \mathbb{R}^{2n}_n$ denote the vector space $\mathbb{R}^{2n}$ with metric of signature (n, n), $n \geq 1$. We will study the $\mathrm{O}(n, n)$ $(= \mathrm{Aut}(V))$ invariant geometry, which is called the *D_n-geometry*. Similar arguments basically work also on the complex space $\mathbb{C}^{2n}$ as well (cf. [11]). For the relation of D_n-geometry with twistor theory, see also [3].

Let us take coordinates $x_1, x_2 \ldots, x_n, x_{n+1}, x_{n+2}, \ldots, x_{2n}$ such that the inner product is written by

$$(v \mid v') = \tfrac{1}{2} \textstyle\sum_{i=1}^{2n} x_i x'_{2n+1-i}, \quad (v, v' \in V).$$

A linear subspace $W \subset V$ is called **null** if $(v|v') = 0 \quad (v, v' \in W)$. As is easily shown, if $W \subset V$ is null, then $\dim(W) \leq n$.

Consider the set of null lines in V, called the **null projective space**,

$$\begin{aligned} \mathcal{N}_1 &= \{V_1 \mid V_1 \subset V \text{ null}, \ \dim(V_1) = 1\} \\ &= \{[x] \in P(V) \mid \textstyle\sum_{i=1}^{n} x_i x_{2n+1-i} = 0\} \subset P(V), \end{aligned}$$

which is regarded as a smooth quadric hypersurface of dimension $2n-2$, in the projective space $P(V) = (V \setminus \{0\})/(\mathbb{R} \setminus \{0\})$ of dimension $2n - 1$.

Then we have $\mathcal{N}_1$ is diffeomorphic to $(S^{n-1} \times S^{n-1})/\mathbb{Z}_2$, the quotient by the diagonal action $(x, x') \mapsto (-x, -x')$, when S^{n-1} is the standard sphere.

Since the tangent space $T_{V_1}\mathcal{N}_1$ at $V_1 \in \mathcal{N}_1$ is isomorphic to $V_1^{\perp}/V_1$ up to similarity transformations, the given metric on V induces the canonical **conformal structure** on $\mathcal{N}_1$ of type $(n-1, n-1)$. In other words, the conformal structure on $\mathcal{N}_1$ is defined, for each $x = V_1 \in \mathcal{N}_1$, by the quadric tangent cone C_x of the conical Schubert variety

$$S_x := \{W_1 \in \mathcal{N}_1 \mid W_1 \subset V_1^{\perp}\} = P(V_1^{\perp}) \cap \mathcal{N}_1 \subset \mathcal{N}_1.$$

For the given (indefinite) conformal structure $\{C_x\}_{x\in\mathcal{N}_1}$ on $\mathcal{N}_1$, a tangent vector $v \in T_x\mathcal{N}_1$ is called **null** if $v \in C_x$. Moreover we call a curve $\gamma : I \to \mathcal{N}_1$ from an open interval I, a **null curve** if

$$\gamma'(t) \in C_{\gamma(t)}, \quad (t \in I),$$

that is, if the velocity vectors of γ are null.

Recall that, on a semi-Riemannian manifolds (with an indefinite metric), a regular curve is called a geodesic if the velocity vector field is parallel for the Levi-Civita connection. A geodesic is called a **null geodesic** if it is a null curve. Then the class of null geodesics is intrinsically defined by the conformal class of the metric (see [20]).

In fact null geodesics on $\mathcal{N}_1$ are null lines:

Proposition 2.1. ([9]) *The null geodesics on $\mathcal{N}_1$, for the conformal structure $C \subset T\mathcal{N}_1$, are given by null lines, namely, projective lines on $\mathcal{N}_1(\subset P(\mathbb{R}^{n,n}))$.*

§ 3. Null tangent surfaces

Given a space curve in an affine space or a projective space, we can construct a surface, which is called the **tangent surface**, ruled by tangent lines to the curve (see [13]). A tangent surface has singularities at least along the original space curve, even if the original space curve is non-singular.

Even for curves in a general space, we do declare: *Where there is a notion of "tangent lines", there is a tangent surface.* We will take null geodesics (= null lines) tangential to null curves on the null projective space $\mathcal{N}_1$ as "tangent lines".

A surface $f : U \to \mathcal{N}_1$ from a planar domain U, is called a **null surface** if

$$f_*(T_uU) \subset C_{f(u)}, (u \in U).$$

We do not assume f is an immersion. We are very interested in singularities of null surfaces which we face naturally in D_n-geometry.

Then one of main theorems in this paper is

Theorem 3.1. (Local diffeomorphism classification of null tangent surfaces.) *For a generic null curve $\gamma : I \to \mathcal{N}_1$ in the special class of null curves which are projections of an Engel integral curve (see §6), the tangent surface* $\mathrm{Tan}(\gamma)$*, that is a surface in the $(2n-2)$-dimensional conformal manifold $\mathcal{N}_1$, is a null surface with singularities. Moreover the tangent surface is locally diffeomorphic, at each point of γ, to the cuspidal edge or to the open swallowtail in D_3 case,*
to the cuspidal edge, the open swallowtail or to the open Mond surface in D_4 case,
to the cuspidal edge, the open swallowtail, the open Mond surface or to the open folded umbrella in D_n $(n \geq 5)$ case.

Here we mean the genericity in the sense of C^∞ topology.

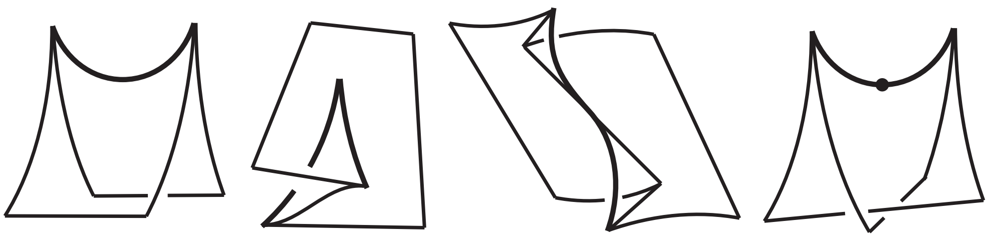
The *cuspidal edge* (resp. *open swallowtail*, *open Mond surface*, *open folded umbrella*) is defined as the local diffeomorphism class of tangent surface-germ to a curve of type $(1, 2, 3, \cdots)$ (resp. $(2, 3, 4, 5, \cdots)$, $(1, 3, 4, 5, \cdots)$, $(1, 2, 4, 5, \cdots)$) in an affine space. Here the **type** of a curve is the strictly increasing sequence of orders (degrees of initial terms, possibly infinity) of components in an appropriate system of affine coordinates. Note that, if a curve has a type $(1, 2, 3, \cdots)$ (resp. $(2, 3, 4, 5, \cdots)$, $(1, 3, 4, 5, \cdots)$, $(1, 2, 4, 5, \cdots)$) in a space of fixed dimension, the local diffeomorphism class of tangent surface-germs is uniquely determined ([13]). Their normal forms are given explicitly as follows:

$$\begin{aligned}
\mathrm{CE} \ &: (\mathbb{R}^2, 0) \to (\mathbb{R}^m, 0), \ m \geq 3, \\
&(u, t) \mapsto (u, t^2 - 2ut, 2t^3 - 3ut^2, 0, \dots, 0).
\end{aligned}$$

$$\begin{aligned}
\mathrm{OSW} &: (\mathbb{R}^2, 0) \to (\mathbb{R}^m, 0), \ m \geq 4, \\
&(u, t) \mapsto (u, t^3 - 3ut, t^4 - 2ut^2, 3t^5 - 5ut^3, 0, \dots, 0).
\end{aligned}$$

$$\begin{aligned}
\mathrm{OM} \ &: (\mathbb{R}^2, 0) \to (\mathbb{R}^m, 0), \ m \geq 4, \\
&(u, t) \mapsto (u, 2t^3 - 3ut^2, 3t^4 - 4ut^3, 4t^5 - 5ut^4, 0, \dots, 0).
\end{aligned}$$

$$\begin{aligned}
\mathrm{OFU} &: (\mathbb{R}^2, 0) \to (\mathbb{R}^m, 0), \ m \geq 4, \\
&(u, t) \mapsto (u, t^2 - 2ut, 3t^4 - 4ut^3, 4t^5 - 5ut^4, 0, \dots, 0).
\end{aligned}$$



CE OSW OM OFU

Here CE means the cuspidal edge, OSW the open swallowtail, OM the open Mond surface, and OFU the open folded umbrella.

§ 4. Null Grassmannians

In general, consider the Grassmannians of null k-subspaces:

$$\mathcal{N}_k := \{W \mid W \subset V \text{ null}, \ \dim(W) = k\}, \quad k = 1, 2, \dots, n.$$

Then we have $\dim(\mathcal{N}_k) = 2kn - \frac{k(3k+1)}{2}$. In particular $\dim(\mathcal{N}_1) = 2n - 2$ and $\dim(\mathcal{N}_n) = \frac{n(n-1)}{2}$.

Example 4.1. In D_1 case where $V = \mathbb{R}^{1,1}$, $\mathcal{N}_1$ consists of two points. In D_2 case where $V = \mathbb{R}^{2,2}$, $\mathcal{N}_1 \cong (S^1 \times S^1)/\mathbb{Z}_2$ and $\mathcal{N}_2 \cong S^1 \sqcup S^1$. In D_3 case where $V = \mathbb{R}^{3,3}$, $\mathcal{N}_1 \cong (S^2 \times S^2)/\mathbb{Z}_2$ and $\mathcal{N}_3 \cong \mathrm{SO}(3) \sqcup \mathrm{SO}(3)$.

The Grassmannian $\mathcal{N}_n$ of maximal null subspaces in $V = \mathbb{R}^{n,n}$ decomposes into two disjoint families $\mathcal{N}_n^+, \mathcal{N}_n^-$: $W, W' \in \mathcal{N}_n$ belong to the same family if and only if $\dim(W \cap W') \equiv n$ (mod.2).

For any $(n-1)$-dimensional null subspace V_{n-1}, there exist uniquely n null subspaces $V_n^\pm \in \mathcal{N}_n^\pm$ such that $V_{n-1} = V_n^+ \cap V_n^-$.

If n is even, Schubert varieties, for $y = V_n^\pm \in \mathcal{N}_n^\pm$,

$$S_y^\pm := \{W_n \in \mathcal{N}_n^\pm \mid W_n \cap V_n^\pm \neq \{0\}\} \subset \mathcal{N}_n^\pm$$

induce invariant cone fields $C_y^\pm$ on $\mathcal{N}_n^\pm$ of degree $\frac{n}{2}$, defined by a Pfaffian. Note that if n is odd, then $S_y^\pm = \mathcal{N}_n^\pm$.

We remark that, only if $n = 4$, the cone $C_y^\pm$ is of degree 2, and we have invariant conformal structures on $\mathcal{N}_n^\pm$.

§ 5. D_n-flags

Now we proceed to construct the geometric model.

Let $V_1 \subset V_2 \subset \cdots \subset V_{n-1}$ be a flag of null subspaces in $V = \mathbb{R}^{n,n}$ with $\dim(V_i) = i$. Then, as is stated in §4, there exist uniquely $V_n^+ \in \mathcal{N}_n^+$ and $V_n^- \in \mathcal{N}_n^-$ such that $V_{n-1} = V_n^+ \cap V_n^-$. Note that $V_n^+ \cup V_n^-$ is contained in $V_{n-1}^\perp := \{x \in V \mid (x|y) = 0, \text{ for any } y \in V_{n-1}\}$.

Consider the set $\mathcal{Z} = \mathcal{Z}(D_n)$ of all complete flags

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \begin{array}{c} \subset V_n^+ \subset \\ \\ \subset V_n^- \subset \end{array} V_{n-1}^\perp \subset \cdots \subset V_2^\perp \subset V_1^\perp \subset V.$$

Note that the flag is determined by $V_1, \ldots, V_{n-2}, V_n^+$ and V_n^-. Also the flag is determined by $V_1, \ldots, V_{n-2}, V_{n-1}$. In fact V_n^+ and V_n^- are uniquely determined by V_{n-1}. The flag manifold $\mathcal{Z}(D_n)$ is of dimension $n(n-1)$. Moreover we have the sequence of natural fibrations

$$\begin{array}{ccccc} & & \mathcal{Z}(D_n) & & \\ \pi_1 \swarrow & \pi_2 \swarrow & \ldots\ \pi_{n-1} \downarrow & \pi_n^+ \searrow & \pi_n^- \searrow \\ \mathcal{N}_1 & \mathcal{N}_2 & \ldots\ \mathcal{N}_{n-2} & \mathcal{N}_n^+ & \mathcal{N}_n^- \end{array}$$

spelling out from the Dynkin diagram of type D_n. Here $\pi_1 : \mathcal{Z} \to \mathcal{N}_1$ is the projection to the first component. Other projections are defined similarly.

§ 6. Engel distribution and tangent surfaces

We define the *D_n-Engel distribution* $\mathcal{E} \subset T\mathcal{Z}$ on the flag manifold $\mathcal{Z}$ as the set of tangent vectors represented by a smooth curves on $\mathcal{Z}$

$$V_1(t) \subset V_2(t) \subset \cdots \subset V_{n-2}(t) \begin{array}{l} \subset V_n^+(t) \\ \subset V_n^-(t) \end{array}$$

such that

$$V_1'(t) \subset V_2(t),\ V_2'(t) \subset V_3(t),\ \ldots,\ V_{n-2}'(t) \subset V_{n-1}(t) (= V_n^+(t) \cap V_n^-(t)).$$

Here $V_i'(t)$ means the subspace generated by $f_1'(t), \ldots, f_i'(t)$ for a frame $f_1(t), \ldots, f_i(t)$ of $V_i(t)$.

A curve $f : I \to \mathcal{Z}$ is called an **Engel integral curve** if

$$f'(t) \in \mathcal{E}_{f(t)},\ (t \in I).$$

Let $f : I \to \mathcal{Z}$ be an Engel integral curve and consider the projections $\pi_1, \ldots, \pi_{n-2}, \pi_n^{\pm}$ of f to $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_{n-2}, \mathcal{N}_n^{\pm}$.

The composition $\gamma = \pi_1 \circ f : I \to \mathcal{N}_1$ is a null curve on the conformal manifold $\mathcal{N}_1$ with well defined null tangent lines as explained in §2. In fact for each flag $V_1 \subset V_2 \subset \cdots$ in $\mathcal{Z}$, the "line" through $V_1 \in \mathcal{N}_1$, $\{W_1 \in \mathcal{N}_1 \mid W_1 \subset V_2\} = P(V_2)$ is defined. Then the tangent surface $\mathrm{Tan}(\gamma) : I \times \mathbb{R}P^1 \to \mathcal{N}_1$ is a null surface.

We remark that the tangent surface of a null curve in $\mathcal{N}_1$ is obtained also as a (closure of) two dimensional stratum of the *envelope* for the one parameter family of *null cones* (conical Schubert varieties) along the curve, which may be called the *D_n-evolute.*

Moreover, for the projection of an Engel-integral curve $f : I \to \mathcal{Z}$ to any null Grassmannian $\mathcal{N}_1, \mathcal{N}_n^+, \mathcal{N}_n^-, \mathcal{N}_2, \mathcal{N}_3, \ldots, \mathcal{N}_{n-2}$, we have a notion of tangent lines and thus we have the tangent surfaces for all cases. For example, for each flag $z \in \mathcal{Z}$,

$$z = (V_1, \ldots, V_{n-2}, V_n^+, V_n^-),$$

the "tangent line" $\ell_n^+(z)$ through $\pi_n^+(z) = V_n^+$ in $\mathcal{N}_n^+$ is defined by

$$\ell_n^+(z) := \pi_n^+((\pi_{n-2})^{-1}(\pi_{n-2}(z)) \cap (\pi_n^-)^{-1}(\pi_n^-(z))),$$

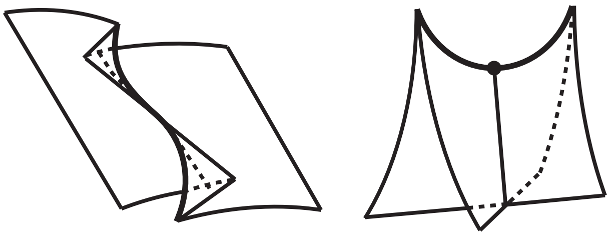
namely, by the set of null n-spaces $W \in \mathcal{N}_n^+$ satisfying $V_{n-2} \subset W$ and $\dim(W \cap V_n^-) = n-1$. Then the tangent surface $\mathrm{Tan}(\pi_n^+ f)$ of $\pi_n^+ f : I \to \mathcal{N}_n^+$ are formed by the lines $\ell_n^+(f(t))$ through $\pi_n^+ f(t), (t \in I)$. Note, for any $t \in I$, that the line $\ell_n^+(f(t))$ is tangent to the curve $\pi_n^+ f$ at $\pi_n^+ f(t) \in \mathcal{N}_n^+$.

Then we have

Theorem 6.1. (D_3). *For a generic Engel integral curve $f : I \to \mathcal{Z}(D_3)$, the singularities of tangent surfaces to the curves $\pi_1 f, \pi_3^+ f, \pi_3^- f$ on $\mathcal{N}_1, \mathcal{N}_3^+, \mathcal{N}_3^-$, respectively, at any point $t_0 \in I$ is classified, up to local diffeomorphisms, into the following four cases:*

$\mathcal{N}_1$	$\mathcal{N}_3^+$	$\mathcal{N}_3^-$
CE	CE	CE
OSW	M	M
CE	SW	FU
CE	FU	SW

The abbreviation SW (resp. M, FU) is used for the **swallowtail** (resp. **Mond surface**, **folded umbrella**). See [12][13].



M FU

Theorem 6.2. (D_4). *For a generic Engel integral curve $f : I \to \mathcal{Z}(D_4)$, the singularities of tangent surfaces to the curves $\pi_1 f, \pi_4^+ f, \pi_4^- f, \pi_2 f$ on $\mathcal{N}_1, \mathcal{N}_4^+, \mathcal{N}_4^-, \mathcal{N}_2$, respectively, at any point $t_0 \in I$ is classified, up to local diffeomorphisms, into the following five cases:*

$\mathcal{N}_1$	$\mathcal{N}_4^+$	$\mathcal{N}_4^-$	$\mathcal{N}_2$
CE	CE	CE	CE
OSW	CE	CE	CE
CE	OSW	CE	CE
CE	CE	OSW	CE
OM	OM	OM	OSW

Theorem 6.3. (D_5). *For a generic Engel integral curve $f : I \to \mathcal{Z}(D_5)$, the singularities of tangent surfaces to the curves $\pi_1 f, \pi_5^+ f, \pi_5^- f, \pi_2 f, \pi_3 f$ on $\mathcal{N}_1, \mathcal{N}_4^+, \mathcal{N}_4^-$,*

$\mathcal{N}_2, \mathcal{N}_3$, respectively, at any point $t_0 \in I$ is classified, up to local diffeomorphisms, into the following 6 cases:

$\mathcal{N}_1$	$\mathcal{N}_5^+$	$\mathcal{N}_5^-$	$\mathcal{N}_2$	$\mathcal{N}_3$
CE	CE	CE	CE	CE
OSW	CE	CE	CE	CE
CE	OSW	CE	CE	CE
CE	CE	OSW	CE	CE
OM	CE	CE	OSW	CE
OFU	OM	OM	CE	OSW

Theorem 6.4. ($D_n, n \geq 6$). *Let $n \geq 6$. For a generic Engel integral curve $f : I \to \mathcal{Z}(D_n)$, the singularities of tangent surfaces to the curves*

$$\pi_1 f, \pi_n^+ f, \pi_n^- f, \pi_2 f, \pi_3 f, \pi_4 f, \dots, \pi_{n-2} f,$$

on $\mathcal{N}_1, \mathcal{N}_4^+, \mathcal{N}_4^-, \mathcal{N}_2, \mathcal{N}_3, \mathcal{N}_3, \dots, \mathcal{N}_{n-2}$, respectively, at any point $t_0 \in I$ is classified, up to local diffeomorphisms, into the following $n+1$ cases:

$\mathcal{N}_1$	$\mathcal{N}_n^+$	$\mathcal{N}_n^-$	$\mathcal{N}_2$	$\mathcal{N}_3$	$\mathcal{N}_4$	$\cdots$	$\mathcal{N}_{n-2}$
CE	CE	CE	CE	CE	CE	$\cdots$	CE
OSW	CE	CE	CE	CE	CE	$\cdots$	CE
CE	OSW	CE	CE	CE	CE	$\cdots$	CE
CE	CE	OSW	CE	CE	CE	$\cdots$	CE
OM	CE	CE	OSW	CE	CE	$\cdots$	CE
OFU	CE	CE	CE	OSW	CE	$\cdots$	CE
CE	CE	CE	CE	CE	OSW	$\cdots$	CE
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
CE	OM	OM	CE	CE	CE	$\cdots$	OSW

§ 7. Flag and Grassmannian coordinates

Let $(V_1, V_2, \dots, V_{n-2}, V_n^+, V_n^-) \in \mathcal{Z}(D_n)$ be a flag. We take $f_1, f_2, \dots, f_{n-1} \in V = \mathbb{R}^{n,n}$ such that $f_1, f_2, \dots, f_i$ form a basis of V_i, $i = 1, 2, \dots, n-2$ and $f_1, f_2, \dots, f_{n-1}$ form a basis of $V_{n-1} = V_n^+ \cap V_n^-$ and they are written as

$$\begin{cases} f_1 &= e_1 + x_{2,1}e_2 + \cdots + x_{n,1}e_n + x_{n+1,1}e_{n+1} + x_{n+2,1}e_{n+2} + \cdots + x_{2n,1}e_{2n} \\ f_2 &= \qquad\quad e_2 + \cdots + x_{n,2}e_n + x_{n+1,2}e_{n+1} + x_{n+2,2}e_{n+2} + \cdots + x_{2n,2}e_{2n} \\ \vdots & \\ f_{n-1} &= \quad e_{n-1} + x_{n,n-1}e_n + x_{n+1,n-1}e_{n+1} + x_{n+2,n-1}e_{n+2} + \cdots + x_{2n,n-1}e_{2n} \end{cases}$$

for some $x_{i,j} \in \mathbb{R}$. Moreover we take

$$f_n \;=\; e_n + x_{n+1,n}e_{n+1} + x_{n+2,n}e_{n+2} + \cdots + x_{2n,n}e_{2n},$$

from V_n^+ so that $f_1, f_2, \ldots, f_{n-1}, f_n$ form a basis of V_n^+, and take

$$f_{n+1} \;=\; x_{n,n+1}e_n + e_{n+1} + x_{n+2,n+1}e_{n+2} + \cdots + x_{2n,n+1}e_{2n},$$

from V_n^- so that $f_1, f_2, \ldots, f_{n-1}, f_{n+1}$ form a basis of V_n^-.

Then we can choose some of $x_{i,j}$ as coordinates, so called *flag coordinates*, on $\mathcal{Z}(D_n)$. Similarly we have natural charts, so called *Grassmannian coordinates*, of $\mathcal{N}_i, (1 \leq i \leq n-2)$ and $\mathcal{N}_n^\pm$.

For example, the Grassmannian coordinates on $\mathcal{N}_n^+$ are given as follows: take a frame $g_1, g_2, \ldots, g_n$ of an n-dimensional subspace W of $V = \mathbb{R}^{n,n}$ in a neighborhood of $W_0^+ = \langle e_1, e_2, \ldots, e_n \rangle$ of the form:

$$\begin{cases} g_1 = e_1 + y_{n+1,1}e_{n+1} + \cdots + y_{2n,1}e_{2n} \\ g_2 = e_2 + y_{n+1,2}e_{n+1} + \cdots + y_{2n,2}e_{2n} \\ \vdots \qquad \ddots \\ g_{n-1} = e_{n-1} + y_{n+1,n-1}e_{n+1} + \cdots + y_{2n,n-1}e_{2n} \\ g_n = e_n + y_{n+1,n}e_{n+1} + \cdots + y_{2n,n}e_{2n} \end{cases}$$

for some $y_{i,j} \in \mathbb{R}$. Then the condition that $W \in \mathcal{N}_n^+$ is given by the condition that the $n \times n$-matrix $Y = (y_{2n+1-i,j})_{1 \leq i,j \leq n}$ is skew-symmetric. Thus we choose, as coordinates, the components in the strictly upper triangle with respect to the diagonal "upward to the right". The condition that $\dim(W \cap W_0) > 0$ is given by the condition that $\det(Y) = 0$. Then, if n is even, the Schubert variety $\mathcal{S}_{W_0}$ is given by the condition that the Pfaffian of Y is equal to zero, which gives a cone of degree $\frac{n}{2}$, as stated in §4,

For naturally chosen charts as above on $\mathcal{Z}(D_n), \mathcal{N}_i, (1 \leq i \leq n-2), \mathcal{N}_n^\pm$, the projections $\pi_i, i = 1, 2, \ldots, n-2, \pi_n^+, \pi_n^-$ are weighted homogeneous mappings respectively. Moreover the tangent lines in $\mathcal{N}_i, (1 \leq i \leq n-2), \mathcal{N}_n^\pm$ are actually expressed as lines in the Grassmannian coordinates.

§ 8. Projections of Engel integral curves

Let $\mathfrak{g} = \mathfrak{o}(n,n)$ denote the Lie algebra of Lie group $O(n,n)$ (see [10][5]). With respect to a basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$ of $\mathbb{R}^{n,n}$ with inner products

$$(e_i | e_{2n+1-j}) = \frac{1}{2}\delta_{i,j}, \quad 1 \leq i,j \leq 2n,$$

where $\delta_{i,j}$ is Kronecker delta, we have

$$\begin{aligned} \mathfrak{o}(n,n) &= \{A \in \mathfrak{gl}(2n,\mathbb{R}) \mid {}^tAK + KA = O\} \\ &= \{A = (a_{i,j}) \in \mathfrak{gl}(2n,\mathbb{R}) \mid a_{2n+1-j,2n+1-i} = -a_{i,j}, 1 \leq i,j \leq 2n\}, \end{aligned}$$

where $K = (k_{i,j})$ is the $2n \times 2n$-matrix defined by $k_{i,2n+1-j} = \frac{1}{2}\delta_{i,j}$. Let $E_{i,j}$ denote the 8×8-matrix whose (k, ℓ)-component is defined by $\delta_{i,k}\delta_{j,\ell}$. Then

$$\mathfrak{h} := \langle E_{i,i} - E_{2n+1-i,2n+1-i} \mid \varepsilon_i \in \mathbb{R}, 1 \leq i \leq n \rangle_{\mathbb{R}}$$

is a Cartan subalgebra of $\mathfrak{g}$. Let $(\varepsilon_i \mid 1 \leq i \leq n)$ denote the dual basis of $\mathfrak{h}^*$ to the basis $(E_{i,i} - E_{2n+1-i,2n+1-i} \mid 1 \leq i \leq 4)$ of $\mathfrak{h}$. Then the root system is given by $\pm\varepsilon_i \pm \varepsilon_j, 1 \leq i < j \leq n$, and $\mathfrak{g}$ is decomposed, over $\mathbb{R}$, into the direct sum of root spaces

$$\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \langle E_{i,j} - E_{2n+1-j,2n+1-i} \rangle_{\mathbb{R}},\ \mathfrak{g}_{\varepsilon_i + \varepsilon_j} = \langle E_{i,2n+1-j} - E_{j,2n+1-i} \rangle_{\mathbb{R}},$$
$$\mathfrak{g}_{-\varepsilon_i + \varepsilon_j} = \langle E_{j,i} - E_{2n+1-i,2n+1-j} \rangle_{\mathbb{R}},\ \mathfrak{g}_{-\varepsilon_i - \varepsilon_j} = \langle E_{2n+1-j,i} - E_{2n+1-i,j} \rangle_{\mathbb{R}},$$

$(1 \leq i < j \leq n)$ (cf. [4]).

The simple roots are given by

$$\alpha_1 := \varepsilon_1 - \varepsilon_2, \quad \alpha_2 := \varepsilon_2 - \varepsilon_3, \quad \ldots, \quad \alpha_{n-1} := \varepsilon_{n-1} - \varepsilon_n, \quad \alpha_n := \varepsilon_{n-1} + \varepsilon_n.$$

As an example, we illustrate the root decomposition of $\mathfrak{o}(5,5)$ (D_5), by labeling the roots just on the left-upper-half part:

0	α_1	$\alpha_1 + \alpha_2$	$\alpha_1 + \alpha_2$ $+\alpha_4$	$\alpha_1 + \alpha_2$ $\alpha_3 + \alpha_5$ $+\alpha_5$	$\alpha_1 + \alpha_2$ $+\alpha_3 + \alpha_4$ $+\alpha_5$	$\alpha_1 + \alpha_2$ $+\alpha_3 + \alpha_4$ $+\alpha_5$	$\alpha_1 + \alpha_2$ $+2\alpha_3 + \alpha_4$ $+\alpha_5$	$\alpha_1 + \alpha_2$ $+2\alpha_3 + \alpha_4$ $+\alpha_5$	
$-\alpha_1$	0	α_2	$\alpha_2 + \alpha_3$	$\alpha_2 + \alpha_3$ $+\alpha_4$	$\alpha_2 + \alpha_3$ $+\alpha_5$	$\alpha_2 + \alpha_3$ $+\alpha_4 + \alpha_5$	$\alpha_2 + 2\alpha_3$ $+\alpha_4 + \alpha_5$		
$-\alpha_1 - \alpha_2$	$-\alpha_2$	0	α_3	$\alpha_3 + \alpha_4$	$\alpha_3 + \alpha_5$	$\alpha_3 + \alpha_4$ $+\alpha_5$			
$-\alpha_1 - \alpha_2$ $-\alpha_3$	$-\alpha_2 - \alpha_3$	$-\alpha_3$	0	α_4	α_5				
$-\alpha_1 - \alpha_2$ $-\alpha_3 - \alpha_4$	$-\alpha_2 - \alpha_3$ $-\alpha_4$	$-\alpha_3 - \alpha_4$	$-\alpha_4$	0					
$-\alpha_1 - \alpha_2$ $-\alpha_3 - \alpha_5$	$-\alpha_2 - \alpha_3$ $-\alpha_5$	$-\alpha_3 - \alpha_5$	$-\alpha_5$						
$-\alpha_1 - \alpha_2$ $-\alpha_3 - \alpha_4$ $-\alpha_5$	$-\alpha_2 - \alpha_3$ $-\alpha_4 - \alpha_5$	$-\alpha_3 - \alpha_4$ $-\alpha_5$							
$-\alpha_1 - \alpha_2$ $-2\alpha_3 - \alpha_4$ $-\alpha_5$	$-\alpha_2 - 2\alpha_3$ $-\alpha_4 - \alpha_5$								
$-\alpha_1 - 2\alpha_2$ $-2\alpha_3 - \alpha_4$ $-\alpha_5$									

For D_4-case, see [17]. Also for $D_n, n = 3$ or $n \geq 6$, we have similar root decomposition of $\mathfrak{g} = \mathfrak{o}(n, n)$.

By explicit representations of Engel systems, we have the following:

Lemma 8.1. *Given (abstract) weights* $w_{2,1}, w_{3,2}, \dots, w_{n-1,n-2}, w_{n,n-1}, w_{n+1,n-1}$ *of*

$$x_{2,1}, x_{3,2}, \dots, x_{n-1,n-2}, x_{n+2,n+1} = -x_{n,n-1}, x_{n+2,n} = -x_{n+1,n-1},$$

the weights of other variables are determined by the Engel differential system, and then the weights of components of the projections $\pi_i, (1 \leq i \leq n-2), \pi_n^{\pm}$ *to* $\mathcal{N}_i, (1 \leq i \leq n-2), \mathcal{N}_n^{\pm}$ *are given by the unique expressions of the corresponding roots by simple roots.*

See [17] for the detailed calculations for D_4-case.

We can perform the calculations also for general D_n-cases. For example the orders of components of the curve $\pi_1 f$ in $\mathcal{N}_1$ for an Engel integral curve f are given by the weights

$$\begin{gathered} w_{2,1} = \mathrm{ord}(x_{2,1} f), \quad w_{3,2} = \mathrm{ord}(x_{3,2} f), \quad \dots, \quad w_{n-1,n-2} = \mathrm{ord}(x_{n-1,n-2} f), \\ w_{n,n-1} = \mathrm{ord}(x_{n,n-1} f) = \mathrm{ord}(x_{n+2,n+1} f), \\ w_{n+1,n-1} = \mathrm{ord}(x_{n+1,n-1} f) = \mathrm{ord}(x_{n+2,n} f), \end{gathered}$$

as follows:

$$\begin{cases} w_{2,1}, \\ w_{3,1} = w_{2,1} + w_{3,2}, \\ \vdots \\ w_{n-1,1} = w_{2,1} + w_{3,2} + \cdots + w_{n-2,n-3} + w_{n-1,n-2}, \\ w_{n,1} = w_{2,1} + w_{3,2} + \cdots + w_{n-2,n-3} + w_{n-1,n-2} + w_{n,n-1}, \\ w_{n+1,1} = w_{2,1} + w_{3,2} + \cdots + w_{n-2,n-3} + w_{n-1,n-2} + w_{n+1,n-1}, \\ w_{n+2,1} = w_{2,1} + w_{3,2} + \cdots + w_{n-2,n-3} + w_{n-1,n-2} + w_{n,n-1} + w_{n+1,n-1}, \\ w_{n+3,1} = w_{2,1} + w_{3,2} + \cdots + w_{n-2,n-3} + 2w_{n-1,n-2} + w_{n,n-1} + w_{n+1,n-1}, \\ \vdots \\ w_{2n-1,1} = w_{2,1} + 2w_{3,2} + \cdots + 2w_{n-2,n-3} + 2w_{n-1,n-2} + w_{n,n-1} + w_{n+1,n-1}. \end{cases}$$

For other projections we have similar calculations. Then we have

Lemma 8.2. *Let* $f : I \to \mathcal{Z}(D_n)$ *be a generic Engel-integral curve. Then, for any* $t_0 \in I$ *and for any flag chart* $(x_{i,j})$ *on* $\mathcal{Z}(D_n)$ *centered at* $f(t_0)$*, we have the following*

$(n+1)$*-cases.*

	$w_{2,1}$	$w_{n,n-1}$	$w_{n+1,n-1}$	$w_{3,2}$	$w_{4,3}$	$\dots$	$w_{n-1,n-2}$
a_0	1	1	1	1	1	$\dots$	1
a_1	2	1	1	1	1	$\dots$	1
a_{n-1}	1	2	1	1	1	$\dots$	1
a_n	1	1	2	1	1	$\dots$	1
a_2	1	1	1	2	1	$\dots$	1
a_3	1	1	1	1	2	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_{n-2}	1	1	1	1	1	$\dots$	2

Here $w_{i,j}$ is the vanishing order of the component $x_{i,j}f$ at t_0. Then the sets of orders on components for the projections $\pi_1 f, \pi_n^+ f, \pi_n^- f, \pi_2 f \dots, \pi_{n-2} f$, are given as in the following table if $n \geq 6$:

cases	$\pi_1 f$	$\pi_n^+ f$	$\pi_n^- f$	$\pi_2 f$	$\pi_3 f$	$\cdots$	$\pi_{n-2} f$
a_0	$1,2,3,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$\cdots$	$1,2,3,\dots$
a_1	$2,3,4,5,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$\cdots$	$1,2,3,\dots$
a_{n-1}	$1,2,3,\dots$	$2,3,4,5,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$\cdots$	$1,2,3,\dots$
a_n	$1,2,3,\dots$	$1,2,3,\dots$	$2,3,4,5,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$\cdots$	$1,2,3,\dots$
a_2	$1,3,4,5,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$2,3,4,5,\dots$	$1,2,3,\dots$	$\cdots$	$1,2,3,\dots$
a_3	$1,2,4,5,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$2,3,4,5,\dots$	$\cdots$	$1,2,3,\dots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
a_{n-2}	$1,2,3,\dots$	$1,3,4,5,\dots$	$1,3,4,5,\dots$	$1,2,3,\dots$	$1,2,3,\dots$	$\cdots$	$2,3,4,5,\dots$

Here $1,2,3,\dots$ (resp. $2,3,4,5,\dots,\ 1,3,4,5,\dots,\ 1,2,4,5,\dots$) means that there are components having the orders $1,2,3$ (resp. $2,3,4,5,\ 1,3,4,5,\ 1,2,4,5$) and that orders of other components are at least 3 (resp. 5).

The list of orders for D_4 is given in [17]. Also for D_5 we can calculate orders from the table of root decomposition of $\mathfrak{o}(5,5)$ as above.

Then we obtain the normal forms of the tangent surfaces, by applying the general theory on tangent surfaces [13], which are expressed using the notion of "openings".

§ 9. Tangent surfaces of curves and openings

We treat singularities of tangent surfaces in local coordinates where "tangent lines" are actually given as lines.

Let $\gamma : I \to \mathbb{R}^{N+1}$ be a C^∞ curve,

$$\gamma(t) = (x_1(t), x_2(t), \dots, x_{N+1}(t)).$$

Take $t_0 \in I$ and set $\mathrm{ord}(x_i(t) - x_i(t_0)) = a_i$, the order of the leading term with respect to $t - t_0$. We do not assume that a_i is strictly increasing, but suppose, by changing the numbering if necessary, that

$$0 < a_1 < a_2 \le \min\{a_i \mid i \ge 3\}.$$

Set $\alpha(t) = t^{a_1 - 1}$ and define

$$f_i(t, s) := x_i(t) + \frac{s}{\alpha(t)} x_i'(t), \ (1 \le i \le N+1),$$

so that

$$f(t, s) = \mathrm{Tan}(\gamma) := \gamma(t) + \frac{s}{\alpha(t)} \gamma'(t) : I \times \mathbb{R} \to \mathbb{R}^{N+1},$$

is a parametrization of the tangent surface of γ.

Consider the Wronskians

$$W_{i,j}(t) = \begin{vmatrix} x_i'(t) & x_j'(t) \\ x_i''(t) & x_j''(t) \end{vmatrix}.$$

Lemma 9.1. (cf. Lemma 4.5 of [13]) *For the exterior differential of f_i, we have, on a neighborhood of $t_0 \times \mathbb{R}$ in $I \times \mathbb{R}$,*

$$df_i = \frac{W_{i,2}}{W_{1,2}} df_1 + \frac{W_{1,i}}{W_{1,2}} df_2, \ (1 \le i \le N+1),$$

and $\dfrac{W_{i,2}}{W_{1,2}}, \dfrac{W_{1,i}}{W_{1,2}}$ are C^∞.

Proof. We have

$$df_i = \frac{x_i'}{\alpha} ds + (x_i' + s(\frac{x_i'}{\alpha})')dt.$$

In particular we have

$$(df_1 \ \ df_2) = (ds \ \ dt) \begin{pmatrix} \dfrac{x_1'}{\alpha} & \dfrac{x_2'}{\alpha} \\ x_1' + s(\dfrac{x_1'}{\alpha})' & x_2' + s(\dfrac{x_2'}{\alpha})' \end{pmatrix},$$

therefore we have

$$(ds \ \ dt) = (df_1 \ \ df_2) \frac{\alpha^2}{s W_{1,2}} \begin{pmatrix} x_2' + s(\dfrac{x_2'}{\alpha})' & -\dfrac{x_2'}{\alpha} \\ -x_1' - s(\dfrac{x_1'}{\alpha})' & \dfrac{x_1'}{\alpha} \end{pmatrix}.$$

Then we have

$$df_i = (ds \;\; dt) \begin{pmatrix} \dfrac{x_i'}{\alpha} \\ x_i' + s(\dfrac{x_i'}{\alpha})' \end{pmatrix} = (df_1 \;\; df_2) \frac{1}{W_{1,2}} \begin{pmatrix} W_{i,2} \\ W_{1,i} \end{pmatrix},$$

which shows the first equality. The order of $W_{1,2}$ is equal to $a_1 + a_2 - 3$ and the order of $W_{i,j}$ is at least $a_i + a_j - 3$. Note that $W_{i,i} = 0$. Therefore, for any i, j with $1 \leq i, j \leq n$, the quotient $W_{i,j}/W_{1,2}$, which is C^∞ outside of $t_0 \times \mathbb{R}$, extends to a C^∞ function to a neighborhood of $t_0 \times \mathbb{R}$ in $I \times \mathbb{R}$. Thus we have the result. □

In the above situation, we call f is an **opening** of $(f_1, f_2) : (I \times \mathbb{R}, t_0 \times \mathbb{R}) \to \mathbb{R}^2$ (see [13] for the details of openings).

In what follows, we take $t - t_0$ and $x - \gamma(t_0)$ as coordinates.

Lemma 9.2. *For a C^∞ curve-germ $\gamma = (x_1, \dots, x_{N+1}) : (\mathbb{R}, 0) \to (\mathbb{R}^{N+1}, 0)$, $N \geq 2$, suppose, at $t = 0$, $\mathrm{ord}(x_1) = 1, \mathrm{ord}(x_2) = 2, \mathrm{ord}(x_3) = 3$ and $\mathrm{ord}(x_i) \geq 3, (3 < i \leq N+1)$. Then the tangent surface $\mathrm{Tan}(\gamma)$ is locally diffeomorphic to the cuspidal edge.*

Proof. By Lemma 9.1, we see $\mathrm{Tan}(\gamma)$ is an opening of $\mathrm{Tan}(x_1, x_2)$ which is locally diffeomorphic to the fold map-germ $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. Moreover we have that $\mathrm{Tan}(\gamma)$ is locally diffeomorphic to the versal opening of the fold map-germ, and therefore it is locally diffeomorphic to the cuspidal edge (Proposition 6.9 and Theorem 7.1 of [13]). Note that the theory of [13] is applied to the case γ is not necessarily of finite type. For example, if the image of γ is included in a proper linear subspace, then γ is not of finite type. However the theory of [13] is applied even to such a case. □

Similarly we have, by Proposition 6.9 and Theorem 7.1 of [13]:

Lemma 9.3. *Let $\gamma = (x_1, \dots, x_{N+1}) : (\mathbb{R}, 0) \to (\mathbb{R}^{N+1}, 0)$, $N \geq 3$, be a C^∞ curve-germ.*

(1)(OSW) *If $\mathrm{ord}(x_1) = 2, \mathrm{ord}(x_2) = 3, \mathrm{ord}(x_3) = 4, \mathrm{ord}(x_4) = 5$ and $\mathrm{ord}(x_i) \geq 5, (4 < i \leq N+1)$ at 0, then the tangent surface $\mathrm{Tan}(\gamma)$ is locally diffeomorphic to the open swallowtail.*

(2)(OM) *If $\mathrm{ord}(x_1) = 1, \mathrm{ord}(x_2) = 3, \mathrm{ord}(x_3) = 4, \mathrm{ord}(x_4) = 5$ and $\mathrm{ord}(x_i) \geq 5, (4 < i \leq N+1)$ at 0, then the tangent surface $\mathrm{Tan}(\gamma)$ is locally diffeomorphic to the open Mond surface.*

(3)(OFU) *If $\mathrm{ord}(x_1) = 1, \mathrm{ord}(x_2) = 2, \mathrm{ord}(x_3) = 4, \mathrm{ord}(x_4) = 5$ and $\mathrm{ord}(x_i) \geq 5, (4 < i \leq N+1)$ at 0, then the tangent surface $\mathrm{Tan}(\gamma)$ is locally diffeomorphic to the open folded umbrella.*

Proof of Main Theorems 3.1, 6.2, 6.3, 6.4. Except for $n = 3$, Theorem 3.1 follows from Theorems 6.2, 6.3, and 6.4. Theorems 6.2, 6.3, 6.4 follow from Lemmata 8.2, 9.2, 9.3. The case $n = 3$ of Theorem 3.1 is shown in §10. □

§ 10. D_3-case

Let us examine the case $n = 3$. The system of flag coordinates is given by

$$x_{21}, x_{31}, x_{41}, x_{51}, x_{32}, x_{42}$$

and the projections $\pi_1 : \mathcal{Z}(D_3) \to \mathcal{N}_1$, $\pi_3^{\pm} : \mathcal{Z}(D_3) \to \mathcal{N}_3^{\pm}$ are given as follows:

$$\pi_1(x_{21}, x_{31}, x_{41}, x_{51}, x_{32}, x_{42}) = (x_{21}, x_{31} + x_{32}x_{21}, x_{41} + x_{42}x_{21}, x_{51} - x_{42}x_{32}x_{21}),$$

$$\pi_3^{+}(x_{21}, x_{31}, x_{41}, x_{51}, x_{32}, x_{42}) = (x_{41}, x_{42}, x_{51} + x_{42}x_{31}),$$

$$\pi_3^{-}(x_{21}, x_{31}, x_{41}, x_{51}, x_{32}, x_{42}) = (x_{31}, x_{32}, x_{51} + x_{41}x_{32}).$$

The Engel system on $\mathcal{Z}(D_3)$ is given by

$$\begin{cases} dx_{31} + x_{21}dx_{32} = 0 \\ dx_{41} + x_{21}dx_{42} = 0 \\ dx_{51} - x_{21}x_{42}dx_{32} - x_{21}x_{32}dx_{42} = 0 \end{cases}$$

The orders of components of $\pi_1 f, \pi_3^+ f$ and $\pi_3^- f$ for a generic Engel integral curve f are given by the following table (cf. Lemma 8.2):

cases	$\pi_1 f$	$\pi_3^+ f$	$\pi_3^- f$
a_0	$1, 2, 2, 3$	$1, 2, 3$	$1, 2, 3$
a_1	$2, 3, 3, 4$	$1, 3, 4$	$1, 3, 4$
a_2	$1, 2, 3, 4$	$2, 3, 4$	$1, 2, 4$
a_3	$1, 2, 3, 4$	$1, 2, 4$	$2, 3, 4$

Proof of Theorem 6.1 (*and Theorem 3.1, $n = 3$*). It is known that the singularity of tangent surfaces of a curve of type $(2, 3, 4)$ (resp. $(1, 3, 4)$, $(1, 2, 4)$) is diffeomorphic to the swallowtail (resp. Mond surface, folded umbrella) (see [12]). In the case (a_1), let h_1, h_2, h_3, h_4 be the components of $\pi_1 f$ of order $2, 3, 3, 4$ respectively. Write

$$h_2 = a_3t^3 + a_4t^4 + a_5t^5 + \cdots, \; h_3 = b_3t^3 + b_4t^4 + b_5t^5 + \cdots, \; h_4 = c_4t^4 + c_5t^5 + \cdots,$$

with $a_3 \neq 0, b_3 \neq 0, c_4 \neq 0$. Set $k_1 = h_1, k_2 = h_2$,

$$k_3 := b_3h_2 - a_3h_3 = (b_3a_4 - a_3b_4)t^4 + (b_3a_5 - a_3b_5)t^5 + \cdots,$$

and

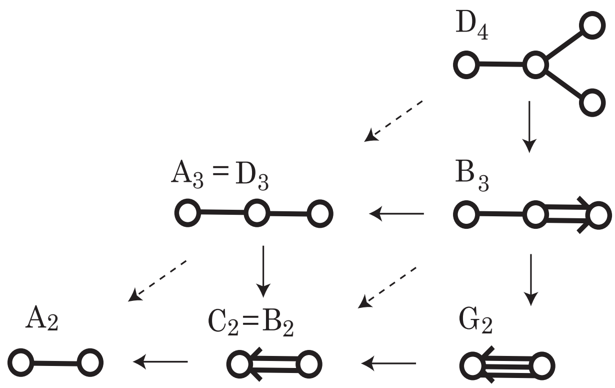
$$k_4 := c_4 k_3 - (b_3 a_4 - a_3 b_4) h_4 = \{(b_3 a_5 - a_3 b_5) c_4 - (b_3 a_4 - a_3 b_4) c_5\} t^5 + \cdots .$$

Generically we have that $b_3 a_4 - a_3 b_4 \neq 0$ and $(b_3 a_5 - a_3 b_5) c_4 - (b_3 a_4 - a_3 b_4) c_5 \neq 0$. Then the orders of k_1, k_2, k_3, k_4 are $2, 3, 4, 5$ respectively. Therefore we see that $\pi_1 f$ is of type $(2, 3, 4, 5)$ for a generic f. Then, by Lemmata 9.2, 9.3, we have the results. □

§ 11. Foldings and removings

We consider the natural problem: How are the D_n-cases related to other Dynkin diagrams ?

For example, we have the following sequence of diagrams from the D_4-diagram by "foldings" and "removings":



In fact, for each Dynkin diagram P, we can associate a *tree of fibrations* T_P such that a **folding** of Dynkin diagram $P \to Q$ corresponds to an *embedding* $T_Q \to T_P$ between trees of fibrations, and a **removing** $R \to S$ corresponds to a *local projection* $T_R \to T_S$ between trees of fibrations.

In this section we present the results for the cases obtained from the Dynkin diagram D_4 by foldings and removings. In each case we can define "Engel distribution" (standard distribution) on each flag manifold as in D_n-cases, and we can consider "a diagram of classification results" on singularities of tangent surfaces associated to generic "Engel integral curves".

By using the split octonions we constructed the geometric model for G_2-case (see [16] for details). The geometric model consists of double fibrations

$$\mathcal{Y} \xleftarrow{\Pi_{\mathcal{Y}}} \mathcal{Z} \xrightarrow{\Pi_{\mathcal{X}}} \mathcal{X},$$

with $\dim(\mathcal{Z}) = 6, \dim(\mathcal{Y}) = \dim(\mathcal{X}) = 5$. The Engel distribution in G_2-case is given by

$$\mathcal{E} = \mathrm{Ker}(\Pi_{\mathcal{Y}*}) \oplus \mathrm{Ker}(\Pi_{\mathcal{X}*}) \subset T\mathcal{Z}.$$

Then we see that $\mathcal{E}$ is of rank 2 and with the small growth vector $(2, 3, 4, 5, 6)$ and the big growth vector $(2, 3, 4, 6)$.

A curve $f : I \to (\mathcal{Z}, \mathcal{E})$ from an open interval I is called an **Engel integral curve** if $f_*(TI) \subset \mathcal{E}(\subset T\mathcal{Z})$. The tangent surface of $\Pi_{\mathcal{Y}} f$ (resp. $\Pi_{\mathcal{X}} f$) is given by $\Pi_{\mathcal{Y}}\Pi_{\mathcal{X}}^{-1}\Pi_{\mathcal{X}} f(I)$ (resp. $\Pi_{\mathcal{X}}\Pi_{\mathcal{Y}}^{-1}\Pi_{\mathcal{Y}} f(I)$).

Theorem 11.1. (G_2, [16]). *For a generic Engel integral curve $f : I \to (\mathcal{Z}, \mathcal{E})$, the pair of types of $\Pi_{\mathcal{Y}} f, \Pi_{\mathcal{X}} f$ at any point $t_0 \in I$ is given by one of the following three cases:*

$$\begin{aligned} \mathrm{I} &: ((1,2,3,4,5),(1,2,3,4,5)),\\ \mathrm{II} &: ((1,3,4,5,7),(2,3,4,5,7)),\\ \mathrm{III} &: ((2,3,5,7,8),(1,3,5,7,8)). \end{aligned}$$

The pair of diffeomorphism classes of tangent surfaces of curves $\Pi_{\mathcal{Y}} f$ and $\Pi_{\mathcal{X}} f$ at any point $t_0 \in I$ is classified, up to local diffeomorphisms, into the following three cases:

I : (cuspidal edge, cuspidal edge),
II : (open Mond surface, open swallowtail),
III : (generic open folded pleat, open Shcherbak surface).

The open Shcherbak surface is the singularity of tangent surface of a curve of type $(1, 3, 5, 7, 8)$. Note that the local diffeomorphism class of tangent surfaces of curves of type $(1, 3, 5, 7, 8)$ is uniquely determined (Proposition 7.2 of [16]).

We exhibit only classification results for the remaining cases B_3, $A_3 = D_3$, $C_2 = B_2$ and A_2.

B_3**-case**. Starting from $V = \mathbb{R}^{3,4}$, we have the following table:

⑤	⑥	⑦
CE	CE	CE
OSW	CE	CE
UFU	OSW	CE
OM	OM	OSW

Here numbers of the first line give the dimensions of Grassmannians corresponding to vertices of the Dynkin diagrams. The abbreviation UFU is used for "unfurled folded umbrella", which is the tangent surface of a curve of type $(1, 2, 4, 6, 7)$.

$A_3 = D_3$**-case**. Starting from $V = \mathbb{R}^4$, we have the following essentially same table that D_3 (cf. [12]):

③	③	④
CE	CE	CE
SW	FU	CE
M	M	OSW
FU	SW	CE

$C_2 = B_2$**-case**. Starting from $V = \mathbb{R}^4$ (symplectic), or $\mathbb{R}^{2,3}$, we have the following classification ([15]):

③	③
cuspidal edge	cuspidal edge
Mond surface	swallowtail
generic folded pleat	Shcherbak surface

A_2**-case**. ([15]). Starting from $V = \mathbb{R}^3$, we have :

②	②
fold	fold
beak-to-beak	Whitney's cusp
Whitney's cusp	beak-to-beak

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