Lefschetz invariants and Young characters for representations of the hyperoctahedral groups

<table>
<thead>
<tr>
<th>メタデータ</th>
<th>言語: eng</th>
</tr>
</thead>
<tbody>
<tr>
<td>出版者:</td>
<td></td>
</tr>
<tr>
<td>公開日: 2018-08-09</td>
<td></td>
</tr>
<tr>
<td>キーワード (Ja):</td>
<td></td>
</tr>
<tr>
<td>キーワード (En):</td>
<td></td>
</tr>
<tr>
<td>作成者: ODA, Fumihito, TAKEGAHARA, Yugen, YOSHIDA, Tomoyuki</td>
<td></td>
</tr>
<tr>
<td>メールアドレス:</td>
<td></td>
</tr>
<tr>
<td>所属:</td>
<td></td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/10258/00009673">http://hdl.handle.net/10258/00009673</a></td>
</tr>
</tbody>
</table>
Lefschetz invariants and Young characters for representations of the hyperoctahedral groups

Fumihito Oda
Department of Mathematics, Kindai University, Higashi-Osaka, 577-8502, Japan
E-mail: odaf@math.kindai.ac.jp

Yugen Takegahara*
Muroran Institute of Technology, 27-1 Mizumoto, Muroran 050-8585, Japan
E-mail: yugen@mmm.muroran-it.ac.jp

Tomoyuki Yoshida
Graduate School of Economics, Hokusei Gakuen University, 2-3-1, Ohyachi-Nishi,
Atsubetsu-ku, Sapporo 004-8631, Japan
E-mail: ytomoyuki@mub.biglobe.ne.jp

Abstract

The ring $R(B_n)$ of virtual $\mathbb{C}$-characters of the hyperoctahedral group $B_n$ has two $\mathbb{Z}$-bases consisting of permutation characters, and the ring structure associated with each basis of them defines a partial Burnside ring of which $R(B_n)$ is a homomorphic image. In particular, the concept of Young characters of $B_n$ arises from a certain set $\mathcal{U}_n$ of subgroups of $B_n$, and the $\mathbb{Z}$-basis of $R(B_n)$ consisting of Young characters, which is presented by L. Geissinger and D. Kinch [7], forces $R(B_n)$ to be isomorphic to a partial Burnside ring $\Omega(B_n, \mathcal{U}_n)$. The linear $\mathbb{C}$-characters of $B_n$ are analyzed with reduced Lefschetz invariants which characterize the unit group of $\Omega(B_n, \mathcal{U}_n)$. The parabolic Burnside ring $\mathcal{PB}(B_n)$ is a subring of $\Omega(B_n, \mathcal{U}_n)$, and the unit group of $\mathcal{PB}(B_n)$ is isomorphic to the four group. The unit group of the parabolic Burnside ring of the even-signed permutation group $D_n$ is also isomorphic to the four group.

*This work was supported by JSPS KAKENHI Grant Number JP16K05052.

2010 Mathematics Subject Classification. Primary 19A22; Secondary 20B30, 20B35, 20C15, 20C30.
Keywords. Burnside ring, Character ring, Hyper octahedral group, Lefschetz invariant, Parabolic subgroup, Sign character, Symmetric group, Young subgroup.
1 Introduction

Let $G$ be a finite group, and let $G$-set be the category of finite left $G$-sets and $G$-equivariant maps. The Burnside ring $\Omega(G)$, which is the Grothendieck ring of the category $G$-set, is the commutative unital ring consisting of all $\mathbb{Z}$-linear combinations of isomorphism classes $[X]$ of finite left $G$-sets $X$ with disjoint union for addition and cartesian product for multiplication. We denote by $R(G)$ the ring of virtual $\mathbb{C}$-characters of $G$. Set $[n] = \{1, 2, \ldots, n\}$, and let $S_n$ be the symmetric group on $[n]$. We denote by $\mathcal{Y}_n$ the set of Young subgroups of $S_n$, which is closed under intersection and conjugation. By [15, §7], $\Omega(S_n)$ possesses the partial Burnside ring $\Omega(S_n, \mathcal{Y}_n)$ relative to the Young subgroups as a subring, and $\Omega(S_n, \mathcal{Y}_n) \cong R(S_n)$. This fact means that the characters $1_{2^n}^S$ induced from the trivial characters $1_Y$ of $Y$ for $Y \in \mathcal{Y}_n$ form a $\mathbb{Z}$-basis of $R(S_n)$ (see, e.g., [2, Proposition 3]). Let $C_2$ be a cyclic group of order 2, and let $V_n$ be the direct product $C_2^\otimes n$ of $n$ copies of $C_2$. We denote by $B_n$ the hyperoctahedral group, that is, the wreath product $C_2 \wr S_n$ defined to be a semidirect product $V_n \rtimes S_n$ of $V_n$ with $S_n$. Let $\mathcal{Z}_n$ be the set of all products $K Y$ of $K \leq V_n$ and $Y \in \mathcal{Y}_n$ with $|V_n : K| \leq 2$ and $Y \leq N_{S_n}(K)$.

We establish in §3 that $R(B_n)$ is a homomorphic image of the partial Burnside ring $\Omega(B_n, \mathcal{Z}_n)$ relative to the set $\mathcal{Z}_n$ of intersections of subgroups contained in $\mathcal{Z}_n$.

For a ring $R$, we denote by $R^\times$ the unit group of $R$. By [13, Example 2], $R(S_n)^\times$ is isomorphic to the four group. There exists a unit of $\Omega(S_n, \mathcal{Y}_n)$ which enables us to describe the sign character $\text{sgn}_n : S_n \to \mathbb{C}$ as a $\mathbb{Z}$-linear combination of the characters $1_{2^n}^S$ for $Y \in \mathcal{Y}_n$ (see [2, Corollary 2] and [9, §4]); such a description is called Solomon’s formula. The ring $R(B_n)$ includes exactly four linear $\mathbb{C}$-characters, and $R(B_n)^\times$ is generated by the nontrivial linear $\mathbb{C}$-characters and $-1_{B_n}$. In §4 we identify $R(B_n)^\times$ with a subgroup of $\Omega(B_n, \mathcal{Z}_n)^\times$, and then describe the linear $\mathbb{C}$-characters of $B_n$ as $\mathbb{Z}$-linear combinations of the characters $1_H^B$ for $H \in \mathcal{Z}_n$.

There is a set $\mathcal{U}_n$ of subgroups of $B_n$ such that the characters $1_H^B$ for $H \in \mathcal{U}_n$ form a $\mathbb{Z}$-basis of $R(B_n)$ (cf. [7, Corollary II.4]). In §5 we define the partial Burnside ring $\Omega(B_n, \mathcal{U}_n)$ relative to the Young subgroups of $B_n$, which is a subring of $\Omega(B_n)$ isomorphic to $R(B_n)$. The parabolic Burnside ring $\mathcal{PB}(B_n)$ (cf. [1, §4]) is a subring of $\Omega(B_n, \mathcal{U}_n)$. By [4, (66.29) Corollary], the sign character $\varepsilon_n : B_n \to \mathbb{C}$ is described as a $\mathbb{Z}$-linear combination of the characters $1_H^B$ for parabolic subgroups $H$ of $B_n$, whence $\mathcal{PB}(B_n)$ includes a unit $\alpha_n$ corresponding to $\varepsilon_n : B_n \to \mathbb{C}$. There also is a unit $\beta_n$ of $\Omega(B_n, \mathcal{U}_n)$ corresponding to a natural extension of $\text{sgn}_n : S_n \to \mathbb{C}$ to $B_n$ such that $\alpha_n \beta_n$ corresponds to the restriction of $\text{sgn}_{2n} : S_{2n} \to \mathbb{C}$ to $B_n$. By the description of $\beta_n$ in terms of the characters $1_H^B$ for $H \in \mathcal{Z}_n \cap \mathcal{U}_n$, we have

\[
\beta_n \in \Omega(B_n, \mathcal{Z}_n)^\times \cap (\Omega(B_n, \mathcal{U}_n)^\times - \mathcal{PB}(B_n)^\times),
\]

which proves $\mathcal{PB}(B_n)^\times$ to be isomorphic to the four group.

Let $X \in G$-set. To explore the units of $\Omega(G)$, we are mainly concerned with the reduced Lefschetz invariant $\Lambda_{P(X)}$ of the $G$-poset $P(X)$ consisting of nonempty
and proper subsets of $X$. The reduced Euler-Poincaré characteristic $\bar{\chi}(P(X)^K)$ of the set of $K$-invariants $P(X)^K$ in $P(X)$ with $K \leq G$ is $(-1)^{|K\setminus X|}$, so that $\Lambda_{P(X)}$ is a unit of $\Omega(G)$ (cf. [11, §5]). As a sequel to this fact, the linear $\mathbb{C}$-characters of $B_n$ are analyzed with reduced Lefschetz invariants which characterize $\Omega(B_n, U_n)^\times$.

Let $D_n$ be the group of even-signed permutations on $[n]$, which is also a Coxeter group of type $D$. In §6 we explore the units of the parabolic Burnside ring of $D_n$.

2 Lefschetz invariant

Following [4, §80], we review the Burnside ring of $G$ and related facts. Let $F(G)$ be the free abelian group on the set of isomorphism classes of finite left $G$-sets. Given $X \in G$-set, we denote by $\overline{X}$ the isomorphism class of left $G$-sets including $X$. Let $F(G)_0$ be the subgroup of $F(G)$ generated by the elements $\overline{X_1 \cup X_2} - \overline{X_1} - \overline{X_2}$ for $X_1, X_2 \in G$-set. We define a multiplication on the generators of $F(G)$ by

$$X_1 \cdot X_2 = X_1 \times X_2,$$

where $X_1 \times X_2$ is the cartesian product of $X_1$ and $X_2$, and extend it to $F(G)$ by $\mathbb{Z}$-linearly. Then $F(G)$ is a commutative unital ring, and $F(G)_0$ is an ideal of $F(G)$. We define a commutative unital ring $\Omega(G)$ to be the quotient $F(G)/F(G)_0$, and call it the Burnside ring of $G$. For each $X \in G$-set, let $[X]$ be the coset $\overline{X} + F(G)_0$ of $F(G)_0$ in $F(G)$ represented by $\overline{X}$. Then by [4, (80.4) Lemma], $[X_1] = [X_2]$ if and only if $\overline{X_1} = \overline{X_2}$. Hence we may regard $[X]$ as the isomorphism class of left $G$-sets including $X \in G$-set. Multiplication on the generators of $\Omega(G)$ is given by

$$[X_1] \cdot [X_2] = [X_1 \times X_2].$$

Let $C(G)$ be a full set of non-conjugate subgroups of $G$. Given $H \leq G$, we denote by $G/H$ the set of left cosets $gH$, $g \in G$, of $H$ in $G$, and make $G/H$ into a left $G$-set by defining $d(gH) = dgH$ for all $d, g \in G$. For $H, K \leq G$, $G/H \simeq G/K$ if and only if $H$ is a conjugate of $K$ (cf. [4, (80.5) Proposition]). The elements $[G/H]$ for $H \in C(G)$ form a free $\mathbb{Z}$-basis of $\Omega(G)$. We have

$$[G/H] \cdot [G/U] = \sum_{Hg \in H\setminus G/U} [G/(H \cap gU)]$$

(1)

for all $H, U \leq G$, where $gU = gUg^{-1}$ (cf. [4, §80 Exercise 2]). The identity of $\Omega(G)$ is $[G/G]$. For shortness’ sake, we usually write 1 = $[G/G]$.

Let $H \leq G$. For each $X \in G$-set, we denote by $\text{inv}_H(X)$ or $X^H$ the set of $H$-invariants in $X$. There exists a ring homomorphism $\phi_H : \Omega(G) \to \mathbb{Z}$ given by

$$[G/U] \mapsto |\text{inv}_H(G/U)|$$

for all $U \in C(G)$. For each $X \in G$-set, it is obvious that

$$\phi_H([X]) = |X^H|.$$
We set $\Omega(G) = \prod_{H \in C(G)} \mathbb{Z}$, and define a map $\phi : \Omega(G) \to \bar{\Omega}(G)$ by

$$x \mapsto (\phi_H(x))_{H \in C(G)}$$

for all $x \in \Omega(G)$. By [4, (80.12) Proposition], this map is a ring monomorphism. We call $\bar{\Omega}(G)$ the ghost ring of $\Omega(G)$, and call $\phi : \Omega(G) \to \bar{\Omega}(G)$ the Burnside homomorphism or the mark homomorphism. Obviously, $\bar{\Omega}(G)^\times = \prod_{H \in C(G)} \mathbb{Z}^\times$. Hence $\bar{\Omega}(G)^\times$ is an elementary abelian 2-group, and so is $\Omega(G)^\times$.

We turn to the concept of (reduced) Lefschetz invariants for finite $G$-sets. A finite (left) $G$-set $P$ equipped with order relation $\leq$ is called a finite $G$-poset if $\leq$ is invariant under the $G$-action. Let $P$ be a finite $G$-poset. For each nonnegative integer $n$, we denote by $Sd_n(P)$ the set of chains $p_0 < p_1 < \cdots < p_n$ of elements of $P$ of cardinality $n + 1$, and make $Sd_n(P)$ into a $G$-set by defining

$$g(p_0 < p_1 < \cdots < p_n) = gp_0 < gp_1 < \cdots < gp_n$$

for all $g \in G$ and $p_0 < p_1 < \cdots < p_n \in Sd_n(P)$. The Lefschetz invariant $\Lambda_P$ of $P$ and the reduced Lefschetz invariant $\bar{\Lambda}_P$ of $P$ are two elements of $\Omega(G)$ given by

$$\Lambda_P = \sum_{i=0}^{\infty} (-1)^i [Sd_i(P)] \quad \text{and} \quad \bar{\Lambda}_P = \Lambda_P - 1,$$

respectively, which are introduced by Thévenaz (cf. [3, 11]).

Given $X \in G$-set, we denote by $P(X)$ the $G$-poset consisting of nonempty and proper subsets of $X$, and explore $\bar{\Lambda}_{P(X)}$ from the point of view of combinatorics.

**Definition 2.1** Let $X \in G$-set. Given $X_0 \in G$-set, we define a finite left $G$-set $\text{Map}(X, X_0)$ to be the set of maps from $X$ to $X_0$ with the action given by

$$(gf)(x) = gf(g^{-1}x)$$

for all $g \in G$, $f \in \text{Map}(X, X_0)$, and $x \in X$ (cf. [5, $\S$2]). Given a nonnegative integer $i$ and $X_0, X_1, \ldots, X_i \in G$-set, we denote by $\text{Map}(X, X_0, X_1, \ldots, X_i)$ the set of all $f \in \text{Map}(X, X_0 \cup X_1 \cup \cdots \cup X_i)$ such that $\text{Im} f \cap X_j \neq \emptyset$ for any $j = 1, 2, \ldots, i$, and make it into a left $G$-set by defining

$$(gf)(x) = gf(g^{-1}x)$$

for all $g \in G$, $f \in \text{Map}(X, X_0, X_1, \ldots, X_i)$, and $x \in X$.

**Lemma 2.2** Let $X \in G$-set. Set $n = |X|$ and $X_1 = \cdots = X_n = G/G$. Then

$$\bar{\Lambda}_{P(X)} = \sum_{i=1}^{n} (-1)^i [\text{Map}(X, \emptyset, X_1, \ldots, X_i)].$$
Proof. Obviously, $[\text{Map}(X, \emptyset, X_1)] = [\text{Map}(X, G/G)] = 1$. We assume that $2 \leq i \leq n$, and define a bijection $\Delta : \text{Map}(X, \emptyset, X_1, \ldots, X_i) \rightarrow S_{d_{i-2}}(P(X))$ by

$$f \mapsto p_0 < p_1 < \cdots < p_{i-2},$$

where

$$p_k = \{x \in X \mid f(x) \in X_j \text{ for some } j \in \{1, 2, \ldots, k+1\}\}$$

for each integer $k$ with $0 \leq k \leq i-2$. Let $g \in G$, and let $f \in \text{Map}(X, \emptyset, X_1, \ldots, X_i)$. We have $(gf)(gx) = f(x)$ for any $x \in X$. Hence, if $\Delta(f) = p_0 < p_1 < \cdots < p_{i-2}$, then $\Delta(gf) = gp_0 < gp_1 < \cdots < gp_{i-2}$. Consequently, we have

$$[\text{Map}(X, \emptyset, X_1)] = 1 \quad \text{and} \quad [\text{Map}(X, \emptyset, X_1, \ldots, X_i)] = [S_{d_{i-2}}(P(X))]
$$

for all integer $i$ with $2 \leq i \leq n$, which implies that

$$\tilde{\Lambda}_{P(X)} = -1 + \sum_{i=0}^{\infty} (-1)^i [S_{d_i}(P(X))] = \sum_{i=1}^{n} (-1)^i [\text{Map}(X, \emptyset, X_1, \ldots, X_i)].$$

This completes the proof. $\square$

By Eq.(1), the set $\Omega(G)^+$ consisting of all elements $\sum_{U \in C(G)} \ell_U[G/U]$, $\ell_U \geq 0$, of $\Omega(G)$ is an additive semigroup closed under multiplication. We fix $X \in G\text{-set}$, and define a multiplicative map $\text{Map}(X, -) : \Omega(G)^+ \rightarrow \Omega(G)$ by

$$[Y] \mapsto [\text{Map}(X, Y)]$$

for all $Y \in G\text{-set}$. There exists a unique polynomial map (multiplicative map) $(-)^{|X|} : \Omega(G) \rightarrow \Omega(G)$, $y \mapsto y^{|X|}$ extending $\text{Map}(X, -)$ (see [5, §2] and [14, §3]). If $X = X_1 \cup X_2$, then $y^{|X|} = y^{|X_1|} \cdot y^{|X_2|}$ for any $y \in \Omega(G)$.

By [14, Lemma 3.6], $\phi((-1)^{|X|}) = ((-1)^{|K \setminus X|})_{K \in C(G)}$, where $K \setminus X$ is the set of $K$-orbits in $X$, and thus $(-1)^{|X|} \in \Omega(G)\times$. The following proposition is equivalent to [9, Proposition 4.1] and [11, Proposition 5.1].

**Proposition 2.3** For any $X \in G\text{-set}$, $\tilde{\Lambda}_{P(X)} = (-1)^{|X|} \in \Omega(G)\times$.

We derive Proposition 2.3 from the combinatorial identity

$$(-1)^n = \sum_{i=1}^{n} (-1)^i S(n, i) i!, \quad (2)$$

where $S(n, i)$ is the Stirling number of the second kind (cf. [10, (24d)]). While Eq.(2) is equivalent to [9, Lemma 4.2], the former is nicer than the later for our argument based on entry 3 of the Twelvefold Way (cf. [10, p. 33]).
Proof of Proposition 2.3. Set \( n = |X| \) and \( X_1 = \cdots = X_n = G/G \). By Lemma 2.2,

\[
\tilde{\Lambda}_{P(X)} = \sum_{i=1}^{n} (-1)^i |\text{Map}(X, \emptyset, X_1, \ldots, X_i)|.
\]

Let \( K \in C(G) \), and set \( m_K = |K \setminus X| \). Then for each integer \( i \) with \( 1 \leq i \leq n \),

\[
|\text{Map}(X, \emptyset, X_1, \ldots, X_i)^K| = S(m_K, i)!,
\]

because \( S(m_K, i) \) is the number of partitions of an \( m_K \)-set into \( i \) nonempty subsets. Combining the preceding facts with Eq. (2), we have

\[
\phi(\tilde{\Lambda}_{P(X)}) = \left( \sum_{i=1}^{m_K} (-1)^i S(m_K, i)! \right)_{K \in C(G)} = ((-1)^m_K)_{K \in C(G)},
\]

completing the proof. \( \square \)

Remark 2.4 For each \( X \in G\text{-set} \), the elements \( y^{[X]} \) for \( y \in \Omega(G) \), which may be called exponentials, were introduced by A. Dress (cf. [5, \S 2]), including \((-1)^{[X]} \) (cf. [5, \S 3]), and the fact that \( \phi(\tilde{\Lambda}_{P(X)}) = ((-1)^{[K \setminus X]})_{K \in C(G)} \) was generalized in terms of the exponentials (see [12, \S 6] and [14, \S 3]).

3 The character ring of \( B_n \)

Set \( C_2 = \mathbb{Z}^\times \), and let \( V_n \) be the direct product \( C_2^{(n)} \) of \( n \) copies of \( C_2 \). The wreath product \( B_n := C_2 \wr S_n \) of \( C_2 \) with \( S_n \) is defined to be the semidirect product

\[
V_n \rtimes S_n = \{(x_1, \ldots, x_n) \sigma \mid (x_1, \ldots, x_n) \in V_n \text{ and } \sigma \in S_n\}
\]

in which each permutation on \([n]\) acts as an inner automorphism on \( V_n \):

\[
\sigma(x_1, \ldots, x_n) \sigma^{-1} = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).
\]

If \( L \leq V_n \) or if \( F \leq S_n \), then we regard \( L \) or \( F \) as a subgroup of \( B_n \). Given \( K \leq V_n \) and \( F \leq N_{S_n}(K) := N_{B_n}(K) \cap S_n \), \( KF \) is the semidirect product \( K \rtimes F \).

Given \( J \subset [n] \), we denote by \( S_J \) the symmetric group on \( J \), and view it as a subgroup of \( S_n \). For a cycle type \( \lambda = (1^{m_1}, \ldots, n^{m_n}) \) of a permutation on \([n]\), let \( S_\lambda \) denote a Young subgroup of \( S_n \) isomorphic to \( S_1^{(m_1)} \times \cdots \times S_n^{(m_n)} \), where each \( S_i^{(m_i)} \) is the direct product of \( m_i \) copies of \( S_i \).

Let \( J \subset [n] \). There exists a linear \( \mathbb{C} \)-character \( \vartheta_J \) of \( V_n \) given by

\[
\vartheta_J((x_1, \ldots, x_n)) = \vartheta(x_1) \cdots \vartheta(x_n) \quad \text{with} \quad \vartheta(x_j) = \begin{cases} x_j & \text{if } j \in J, \\ 1 & \text{otherwise} \end{cases}
\]
for all \((x_1, \ldots, x_n) \in V_n\). Set \(\mathcal{J} = [n] - J\). The inertia group \(I_{B_n}(\vartheta_J)\) of \(\vartheta_J\), which is defined to be \(\{a \in B_n \mid \vartheta_J(aba^{-1}) = \vartheta_J(b)\text{ for all } b \in V_n\}\), is

\[
V_n(S_J S_{\mathcal{J}}) = \{(x_1, \ldots, x_n) \sigma \in B_n \mid (x_1, \ldots, x_n) \in V_n \text{ and } \sigma \in S_J S_{\mathcal{J}}\}
\]

(cf. [8, Lemma 25.5]). There exists an extension \(\widehat{\vartheta}_J\) of \(\vartheta_J\) to \(I_{B_n}(\vartheta_J)\) given by

\[
\widehat{\vartheta}_J((x_1, \ldots, x_n) \sigma) = \vartheta_J((x_1, \ldots, x_n))
\]

for all \((x_1, \ldots, x_n) \in V_n\) and \(\sigma \in S_J S_{\mathcal{J}}\). Obviously, \(I_{B_n}(\vartheta_J)/V_n \cong S_J S_{\mathcal{J}}\). For a \(\mathbb{C}\)-character \(\psi\) of \(S_J S_{\mathcal{J}}\), we denote by \(\overline{\psi}\) the \(\mathbb{C}\)-character of \(I_{B_n}(\vartheta_J)\) given by

\[
\overline{\psi}(g \sigma) = \psi(\sigma)
\]

for all \(g \in V_n\) and \(\sigma \in S_J S_{\mathcal{J}}\). Set \(K_J = \ker \vartheta_J\). Then \(S_J S_{\mathcal{J}} \leq I_{B_n}(\vartheta_J) \leq N_{B_n}(K_J)\).

For each integer \(i\) with \(0 \leq i \leq n\), we indicate with \([i] \subset [n]\) that \([i]\) is the subset \(\{1, 2, \ldots, i\}\) of \([n]\), where \([0]\) is the empty set.

Let \([i] \subset [n]\). We write \(\vartheta_i = \vartheta_{[i]}\), \(K_i = \ker \vartheta_i\), \(S_i = S_{[i]}\), and \(S_{\mathcal{J}} = S_{[\mathcal{J}]}\) for shortness’ sake. Let \(\text{Irr}(S_i S_{\mathcal{J}})\) be the set of irreducible \(\mathbb{C}\)-characters of \(S_i S_{\mathcal{J}}\).

The following proposition is well-known (cf. [7, §II]).

**Proposition 3.1** The irreducible \(\mathbb{C}\)-characters of \(B_n\) consist of the \(\mathbb{C}\)-characters \((\vartheta_i \overline{\psi})^B_n\) induced from the product \(\vartheta_i \overline{\psi}\) of \(\vartheta_i\) and \(\overline{\psi}\) for \([i] \subset [n]\) and \(\overline{\psi} \in \text{Irr}(S_i S_{\mathcal{J}})\).

Let \(J \subset [n]\), and let \(\mathcal{P}(J)\) be the set of cycle types of permutations on \(J\). We write \(\mathcal{P}(n) = \mathcal{P}([n])\). Recall that for each \(\lambda \in \mathcal{P}(J)(= \mathcal{P}([J]))\), \(S_{\lambda, J}\) denotes a Young subgroup of \(S_{[J]}\). We set \(\mathcal{P}(J, \mathcal{J}) = \mathcal{P}(J) \times \mathcal{P}(\mathcal{J})\). Given \((\lambda_J, \lambda_{\mathcal{J}}) \in \mathcal{P}(J, \mathcal{J})\), let \(S_{\lambda_J, \lambda_{\mathcal{J}}}\) denote the product \(HK\) of a subgroup \(H\) of \(S_J\) and a subgroup \(K\) of \(S_{\mathcal{J}}\) such that \(H = \text{conj}(S_{\lambda_J})\) in \(S_J\) and \(K = \text{conj}(S_{\lambda_{\mathcal{J}}})\) in \(S_{\mathcal{J}}\).

For each \(X \in G\)-set, let \(\pi_X\) be the permutation character of \(G\) which assigns each \(g \in G\) the number of fixed elements of \(X\) by \(g\), that is, \(\pi_X(g) = |X^{(g)}|\). For each \(H \leq G\), \(\pi_{G/H}\) is the character \(1_{H}^G\) induced from the trivial character \(1_{H}\) of \(H\).

**Theorem 3.2** The characters \(1_{K_i S_{\lambda_J, \lambda_{\mathcal{J}}}}^{B_n}\) induced from the trivial characters \(1_{K_i S_{\lambda_J, \lambda_{\mathcal{J}}}}\) of \(K_i S_{\lambda_J, \lambda_{\mathcal{J}}}\) for \([i] \subset [n]\) and \((\lambda_J, \lambda_{\mathcal{J}}) \in \mathcal{P}([i], [\mathcal{J}])\) form a \(\mathbb{Z}\)-basis of \(R(B_n)\). In particular, the number of irreducible \(\mathbb{C}\)-characters of \(B_n\) is \(\sum_{i=0}^{n} |\mathcal{P}([i], [\mathcal{J}])|\).

**Proof.** The second assertion is well-known, and is also an immediate consequence of the first one. Let \(J \subset [n]\), and let \((\lambda_J, \lambda_{\mathcal{J}}) \in \mathcal{P}(J, \mathcal{J})\). If \(g \in V_n\) and \(\sigma \in S_J S_{\mathcal{J}}\), then

\[
g \sigma (h \tau K_J S_{\lambda_J, \lambda_{\mathcal{J}}}) = h \tau K_J S_{\lambda_J, \lambda_{\mathcal{J}}} \iff \tau^{-1} h^{-1} (g \sigma) h \tau \in K_J S_{\lambda_J, \lambda_{\mathcal{J}}}
\]

\[
\iff \tau^{-1} (h^{-1} g) \tau^{-1} \sigma \tau \in K_J S_{\lambda_J, \lambda_{\mathcal{J}}}
\]

\[
\iff g \sigma h \in h \tau K_J \text{ and } \sigma \tau \in \tau S_{\lambda_J, \lambda_{\mathcal{J}}}
\]

\[
\iff ghK_J = hK_J \text{ and } \sigma \tau S_{\lambda_J, \lambda_{\mathcal{J}}} = \tau S_{\lambda_J, \lambda_{\mathcal{J}}}
\]
for all \( h \in V_n \) and \( \tau \in S_j S_\tau \), because \( \sigma \in N_{S_n}(K_j) \) and \(|V_n : K_j| \leq 2\), and thus

\[
1^{B_n(\varrho_j)}_{K_j S_{\lambda_j \tau}}(g \sigma) = \pi_{B_n(\varrho_j)/(K_j S_{\lambda_j \tau})}(g \sigma) \\
= \pi_{V_n/K_j}(g) \cdot \pi_{(S_j S_\tau)/S_{\lambda_j \tau}}(\sigma) \\
= 1^{V_n}_{K_j}(g) \cdot 1^{S_j S_\tau}_{S_{\lambda_j \tau}}(\sigma).
\]

In particular, \( 1^{I_{V_n S_\lambda}}_{V_n} = 1^{S_\lambda}_{S_\lambda} \). Moreover, if \( J \neq \emptyset \), then \( \vartheta_J = 1^{V_n}_{K_j} - 1^{V_n}_J \) and

\[
(1^{B_n(\varrho_j)}_{K_j S_{\lambda_j \tau}} - 1^{S_j S_\tau}_{S_{\lambda_j \tau}})(g \sigma) = (1^{V_n}_{K_j}(g) - 1^{V_n}_J(g)) \cdot 1^{S_j S_\tau}_{S_{\lambda_j \tau}}(\sigma) = (\vartheta_J 1^{S_j S_\tau}_{S_{\lambda_j \tau}})(g \sigma)
\]

for all \( g \in V_n \) and \( \sigma \in S_j S_\tau \), and consequently,

\[
1^{B_n}_{K_j S_{\lambda_j \tau}} = 1^{S_j S_\tau}_{S_{\lambda_j \tau}} + (\vartheta_J 1^{S_j S_\tau}_{S_{\lambda_j \tau}})^{B_n} \cdot 1^{S_n}_{S_{\lambda_j \tau}} + (\vartheta_J 1^{S_j S_\tau}_{S_{\lambda_j \tau}})^{B_n}.
\]

Let \([i] \subset [n]\). By the above fact with \( J = [i] \) and Proposition 3.1, it suffices to verify that the characters \( 1^{S_j S_\tau}_{S_{\lambda_j \tau}} \) for \((\lambda_i, \lambda_t) \in \mathcal{P}([i], [t])\) form a \( \mathbb{Z}\)-basis of \( R(S_i S_\tau) \). We identify \( S_i S_\tau \) and the subgroups \( S_{\lambda_i \lambda_t} \) of \( S_i S_\tau \) for \((\lambda_i, \lambda_t) \in \mathcal{P}([i], [t])\) with \( S_i \times S_{n-i} \) and the subgroups \( S_\mu \times S_\nu \) of \( S_i \times S_{n-i} \) for \( \mu \in \mathcal{P}(i) \) and \( \nu \in \mathcal{P}(n-i) \), respectively. By \([2, \text{ Proposition 3}] \) and \([4, \text{ Exercise 6}]\), the characters \( 1^{S_\mu \times S_{n-i}}_{S_\mu} \) for \( \mu \in \mathcal{P}(i) \) and \( \nu \in \mathcal{P}(n-i) \) form a \( \mathbb{Z}\)-basis of \( R(S_i \times S_{n-i}) \). This, combined with \([4, (10.19) \text{ Corollary}]\), shows that the characters \( 1^{S_\mu \times S_{n-i}}_{S_\mu} \) for \( \mu \in \mathcal{P}(i) \) and \( \nu \in \mathcal{P}(n-i) \) form a \( \mathbb{Z}\)-basis of \( R(S_i \times S_{n-i}) \), as desired. This completes the proof. \( \square \)

We quote part of \([15, \text{ §3}]\) and review the concept of generalized Burnside rings.

**Definition 3.3** For a set \( \mathcal{D} \) of subgroups of \( G \), we define a \( \mathbb{Z}\)-lattice \( \Omega(G, \mathcal{D}) \) to be an additive group consisting of all \( \mathbb{Z}\)-linear combinations of the elements \([G/H]\) of \( \Omega(G) \) for \( H \in \mathcal{D} \), and define \( \overline{\mathcal{D}} := \{ gH \mid g \in G \text{ and } H \in \mathcal{D} \} \).

The following theorem is a concise version of \([15, 3.11 \text{ Theorem}]\).

**Theorem 3.4** Let \( \mathcal{D} \) be a set of subgroups of \( G \) including \( G \), and suppose that

\[
\bigcap_{U \leq H \in \overline{\mathcal{D}}} H \in \mathcal{D}
\]

for all \( U \in \overline{\mathcal{D}} \) and \( g \in N_G(U) \). Then \( \Omega(G, \overline{\mathcal{D}}) \) has a unique ring structure such that the group homomorphism \( \Omega(G, \overline{\mathcal{D}}) \to \prod_{H \in C(G) \cap \overline{\mathcal{D}}} \mathbb{Z} \) given by

\[
x \mapsto (\phi_H(x))_{H \in C(G) \cap \overline{\mathcal{D}}}
\]

for all \( x \in \Omega(G, \overline{\mathcal{D}}) \) is a ring homomorphism, and the identity of \( \Omega(G, \overline{\mathcal{D}}) \) is 1. If \( \overline{\mathcal{D}} \) is closed under intersection, then \( \Omega(G, \overline{\mathcal{D}}) \) is a subring of \( \Omega(G) \).
We set $X_n = \{ KJS_{\lambda_J \lambda_T} \mid J \subset [n] \text{ and } (\lambda_J, \lambda_T) \in \mathcal{P}(J, J) \}$. Let $\mathcal{V}_n$ be the set of Young subgroups of $S_n$, and let $\mathcal{Z}_n$ be the set consisting of all products $KY$ of $K \leq V_n$ and $Y \in \mathcal{V}_n$ with $|V_n : K| \leq 2$ and $Y \leq N_{S_n}(K)$. We define

$$\tilde{Z}_n := \left\{ \bigcap_{H \in S} H \left| S \in \text{Sub}(\mathcal{Z}_n) \right. \right\},$$

where $\text{Sub}(\mathcal{Z}_n)$ is the set of nonempty subsets of $\mathcal{Z}_n$.

**Lemma 3.5** The following statements hold.

(a) The set $\overline{X}_n$ coincides with $\mathcal{Z}_n$. In particular, $\mathcal{Z}_n$ is closed under conjugation.

(b) The set $\tilde{\mathcal{Z}}_n$ is closed under intersection and conjugation.

**Proof.** Suppose that $J \subset [n]$ and $(\lambda_J, \lambda_T) \in \mathcal{P}(J, J)$. Let $\sigma \in S_n$, and let $g \in V_n$. Then we have $\sigma(KJS_{\lambda_J \lambda_T}) = K_{\sigma(J)}S_{\lambda_J \lambda_T}$, $S_{\lambda_J \lambda_T} \in \mathcal{V}_n$, and $S_{\lambda_J \lambda_T} \leq N_{S_n}(K_{\sigma(J)})$, where $\sigma(J) = \{ \sigma(j) \mid j \in J \}$. Since $\vartheta_{\sigma(J)}(g \varphi g) = 1$ for any $\tau \in SJS_T$, it follows that

$$\sigma(KJS_{\lambda_J \lambda_T}) = \{ gh \varphi g \mid h \in K_J \text{ and } \tau \in S_{\lambda_J \lambda_T} \} = KJS_{\lambda_J \lambda_T}.$$

In particular, $\overline{X}_n \subset \mathcal{Z}_n$. Suppose that $K \leq V_n$ and $Y \in \mathcal{V}_n$ with $|V_n : K| \leq 2$ and $Y \leq N_{S_n}(K)$. There exists a subset $J$ of $[n]$ such that $K = K_J$. For each $\sigma \in Y$, we have $K_J = \sigma(K_J) = K_{\sigma(J)}$, whence $\sigma(J) = J$ and $Y = S_{\lambda_J \lambda_T}$ for some $\tau \in SJS_T$ and $(\lambda_J, \lambda_T) \in \mathcal{P}(J, J)$. This means that $KY$ is a conjugate of $KJS_{\lambda_J \lambda_T}$. Consequently, $\overline{X}_n \supset \mathcal{Z}_n$, and the statement (a) holds. Obviously, $\tilde{\mathcal{Z}}_n$ is closed under intersection. Hence the statement (b) follows from (a). This completes the proof.

By Lemma 3.5, $\tilde{\mathcal{Z}}_n$ satisfies the hypothesis of Theorem 3.4 with $\mathcal{D} = \overline{\mathcal{D}} = \tilde{\mathcal{Z}}_n$, so that $\Omega(B_n, \tilde{\mathcal{Z}}_n)$ is a subring of $\Omega(B_n)$ which is called a partial Burnside ring.

We now define a ring homomorphism $\text{char}_G : \Omega(G) \rightarrow R(G)$ by

$$[X] \mapsto \pi_X$$

for all $X \in G$-set (cf. [14, §6]), and usually write char $= \text{char}_G$ by omitting subscript $G$. Given $x \in \Omega(G)$ and $g \in G$, $\text{char}(x)(g) = \phi_{(g)}(x)$.

We are successful in finding a natural relationship between $\Omega(B_n, \tilde{\mathcal{Z}}_n)$ and $R(B_n)$.

**Theorem 3.6** The ring homomorphism $\text{char} : \Omega(B_n) \rightarrow R(B_n)$ induces an epimorphism from the partial Burnside ring $\Omega(B_n, \tilde{\mathcal{Z}}_n)$ to $R(B_n)$.

**Proof.** The theorem is a consequence of Theorem 3.2. □
4 Units of the character ring of $B_n$

The set $[n]$ is viewed as a left $S_n$-set. According to [9, Eq.(3)],

$$\widetilde{\Lambda}_P([n]) = \sum_{\lambda=(1^{m_1}, \ldots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \cdots + m_n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} [S_n/S_\lambda],$$  \hfill (3)

so that the sign character $\text{sgn}_n : S_n \to \mathbb{C}$ is described as

$$\text{sgn}_n = \sum_{\lambda=(1^{m_1}, \ldots, n^{m_n}) \in \mathcal{P}(n)} (-1)^{m_1 + \cdots + m_n + n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} \text{sgn}_n S_\lambda$$  \hfill (4)

(see [2, Corollary 2] and [9, Theorem 4.4]). Note that the numbers

$$\frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!}$$

for nonnegative integers $m_1, \ldots, m_n$ are multinomial coefficients (cf. [10, 1.2]).

Let $\kappa_n : B_n \to \mathbb{C}$ be a linear $\mathbb{C}$-character of $B_n$ given by

$$(x_1, \ldots, x_n) \sigma \mapsto \prod_{i=1}^n x_i$$

for all $(x_1, \ldots, x_n) \in V_n$ and $\sigma \in S_n$. There also exists an extension $\rho_n : B_n \to \mathbb{C}$ of the sign character $\text{sgn}_n : S_n \to \mathbb{C}$ to $B_n$ given by

$$(x_1, \ldots, x_n) \sigma \mapsto \text{sgn}_n(\sigma)$$

for all $(x_1, \ldots, x_n) \in V_n$ and $\sigma \in S_n$. Let $\varepsilon_n : B_n \to \mathbb{C}$ be the product $\kappa_n \rho_n$ of $\kappa_n$ and $\rho_n$, which coincides with the sign character of $B_n$.

We view the set $\mathbb{Z}^\times = \{1, -1\}$ as a left $B_n$-set with the action given by

$$(x_1, \ldots, x_n) \sigma \cdot x = x \cdot \prod_{i=1}^n x_i$$

for all $(x_1, \ldots, x_n) \in V_n$, $\sigma \in S_n$, and $x \in \mathbb{Z}^\times$. The set $[n]$ is naturally viewed as a left $B_n$-set on which $V_n$ acts trivially. Let $[n]^\circ$ denote the $B_n$-set $\mathbb{Z}^\times \cup [n]$.

**Lemma 4.1** There are exactly three nontrivial linear $\mathbb{C}$-characters $\kappa_n : B_n \to \mathbb{C}$, $\rho_n : B_n \to \mathbb{C}$, and $\varepsilon_n : B_n \to \mathbb{C}$ defined as above in $R(B_n)$, and $\kappa_n(y) = (-1)^{|(y)\mathbb{Z}^\times|}$, $\rho_n(y) = (-1)^{|(y)\mathbb{Z}^\times|+n}$, and $\varepsilon_n(y) = (-1)^{|(y)[n]^\circ|+n}$ for each $y \in B_n$.

**Proof.** By Proposition 3.1, there are exactly three nontrivial linear $\mathbb{C}$-characters of $B_n$. Let $(x_1, \ldots, x_n) \in V_n$, and let $\sigma \in S_n$. Set $y = (x_1, \ldots, x_n) \sigma \in B_n$, and
assume that \( \sigma \) is a product of pairwise disjoint \( n_j \)-cycles \( \sigma_j \) for \( j = 1, 2, \ldots, r \) with \( \sum_j n_j = n \). Obviously, \( \kappa_n(y) = (-1)^{|y|/2} \). We have \( |y|/n| = r \) and

\[
|y|/n| = \begin{cases} r + 1 & \text{if } \prod_{i=1}^n x_i = -1, \\ r + 2 & \text{if } \prod_{i=1}^n x_i = 1. \end{cases}
\]

Moreover, if \( \ell = \sum \{ j \mid n_j \text{ is odd} \} \), \( \rho_n(y) = \text{sgn}(\sigma) = (-1)^{\ell-n} = (-1)^{r+n} \) and \( \varepsilon_n(y) = (-1)^{r+n} \prod_{i=1}^n x_i \), because \( \ell \equiv n \pmod{2} \). This completes the proof. \( \square \)

**Lemma 4.2** \( R(B_n)^x = \langle \kappa_n, \eta_n, -1_{B_n} \rangle \).

**Proof.** The lemma is a consequence of [6, Theorem 5.5.6] (see also Theorem 3.2), [13, Corollary 1.2 and Lemma 2.1], and Lemma 4.1. \( \square \)

We are now in position to establish the following proposition.

**Proposition 4.3** The nontrivial linear \( \mathbb{C} \)-characters of \( B_n \) are characterized by the reduced Lefschetz invariants. Indeed, \( \kappa_n = \text{char}(\tilde{\Lambda}_{P(\mathbb{Z}^x)}) \), \( \rho_n = (-1)^n\text{char}(\tilde{\Lambda}_{P([n])}) \), and \( \varepsilon_n = (-1)^n\text{char}(\tilde{\Lambda}_{P([n]^{\circ})}) \). The reduced Lefschetz invariants \( \tilde{\Lambda}_{P(\mathbb{Z}^x)} \) and \( \tilde{\Lambda}_{P([n])} \), together with \(-1\), generate an elementary abelian subgroup of \( \Omega(B_n, \mathbb{Z}_n)^x \) isomorphic to \( R(B_n)^x \), and \( \tilde{\Lambda}_{P([n]^{\circ})} = \tilde{\Lambda}_{P([n])} \cdot \tilde{\Lambda}_{P(\mathbb{Z}^x)} \). Moreover,

\[
\tilde{\Lambda}_{P(\mathbb{Z}^x)} = [B_n/(K_nS_n)] - [B_n/B_n], \\
\tilde{\Lambda}_{P([n])} = \sum_{\lambda=1}^{m_1+\cdots+m_n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} [B_n/(V_nS_{\lambda})], \\
\tilde{\Lambda}_{P([n]^{\circ})} = \sum_{\lambda=1}^{m_1+\cdots+m_n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} [B_n/(K_nS_{\lambda})] \\
- \sum_{\lambda=1}^{m_1+\cdots+m_n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} [B_n/(V_nS_{\lambda})].
\]

**Proof.** The first assertion follows from Proposition 2.3 and Lemma 4.1. We prove the last two assertions. By Lemma 2.2 with \( X = \mathbb{Z}^x \) and \( X_1 = X_2 = B_n/B_n \),

\[
\tilde{\Lambda}_{P(\mathbb{Z}^x)} = -[\text{Map}(\mathbb{Z}^x, \emptyset, X_1)] + [\text{Map}(\mathbb{Z}^x, \emptyset, X_1, X_2)] = -[B_n/B_n] + [B_n/(K_nS_n)].
\]

We obtain the description of \( \tilde{\Lambda}_{P([n])} \) in a similar fashion to the proof of [9, Eq.(3)]. By Proposition 2.3, \( \tilde{\Lambda}_{P([n]^{\circ})} = \tilde{\Lambda}_{P([n])} \cdot \tilde{\Lambda}_{P(\mathbb{Z}^x)} \), which yields the description of \( \tilde{\Lambda}_{P([n]^{\circ})} \), and the reduced Lefschetz invariants \( \tilde{\Lambda}_{P(\mathbb{Z}^x)}, \tilde{\Lambda}_{P([n])}, \) and \( \tilde{\Lambda}_{P([n]^{\circ})} \) are contained
in $\Omega(B_n, \tilde{Z}_n)^\times$. Hence it follows from Lemma 4.2 that $\tilde{\lambda}_{P(\mathbb{Z})}$, $\tilde{\lambda}_{P([n])}$, and $-1$ generate an elementary abelian subgroup of $\Omega(B_n, \tilde{Z}_n)^\times$ isomorphic to $R(B_n)^\times$. This completes the proof. $\square$

The following descriptions of nontrivial linear $\mathbb{C}$-characters of $B_n$ are obtained; see Eq.(5) in §5 for Solomon’s formula of the sign character $\varepsilon_n : B_n \to \mathbb{C}$.

**Corollary 4.4**

$$\kappa_n = 1_{K_n S_n} B_n - 1_{B_n},$$

$$\rho_n = \sum_{\lambda = (m_1, \ldots, n \in \mathbb{N})} (-1)^{m_1 + \cdots + m_n + n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} 1_{V_n S_{\lambda}},$$

$$\varepsilon_n = \sum_{\lambda = (m_1, \ldots, n \in \mathbb{N})} (-1)^{m_1 + \cdots + m_n + n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} 1_{V_n S_{\lambda}}.$$

**Proof.** The corollary is an immediate consequence of Proposition 4.3. (The formulae of $\kappa_n$ and $\rho_n$ can also be obtained by a calculation and Eq.(4), respectively.) $\square$

**5 The Young subgroups of the hyperoctahedral groups**

Given $J \subset [n]$, we define a subgroup $L_J$ of $V_n$ by

$$L_J = \{(x_1, \ldots, x_n) \in V_n \mid x_k = 1 \text{ for all } k \in \overline{J}\}.$$

Let $U_n$ denote the set of products $L_J S_{\lambda J}$ of $L_J$ and $S_{\lambda J}$ for $J \subset [n]$ and $(\lambda_J, \lambda_J) \in \mathcal{P}(J, \overline{J})$, and let $\mathcal{E}_n$ denote the set of products $L_J (S_{\lambda J} S_J)$ of $L_J$ and $S_{\lambda J} S_J$ for $J \subset [n]$ and $(\lambda_J, \lambda_J) \in \mathcal{P}(J, \overline{J})$. Obviously, $\mathcal{E}_n \subset U_n$.

We call the subgroups $L_J S_{\lambda J}$ of $B_n$ and the characters $1_{L_J S_{\lambda J}}$ for $J \subset [n]$ and $(\lambda_J, \lambda_J) \in \mathcal{P}(J, \overline{J})$ the Young subgroups and the Young characters, respectively.

The sets $U_n$ and $\mathcal{E}_n$ are closed under conjugation, however. Recall that $\overline{D} = \{ yH \mid y \in B_n \text{ and } H \in \mathcal{D} \}$ where $\mathcal{D}$ is $U_n$ or $\mathcal{E}_n$. Given $[i] \subset [n]$ and $\lambda \in \mathcal{P}(i)$, we write $L_i = L_{[i]}$ and $S_{i S_{\lambda N-i}} = L_{[i]} (S_{\lambda N-i})$. The set $\mathcal{T}_n$ consists of the conjugates of the parabolic subgroups $S_{\lambda N-i}$ for $[i] \subset [n]$ and $\lambda \in \mathcal{P}(i)$, and is closed under intersection (cf. [6, Exercise 2.2]). To explore $\mathcal{U}_n$, we make $\mathbb{Z}^\times \times [n]$ into a left $B_n$-set by defining

$$(x_1, x_2, \ldots, x_n) \cdot (x, i) = (x_{\sigma(i)} x, \sigma(i))$$

for all $(x_1, x_2, \ldots, x_n) \in V_n$, $\sigma \in S_n$, and $(x, i) \in \mathbb{Z}^\times \times [n]$. 
Lemma 5.1 The set $\mathcal{U}_n$ is closed under intersection.

Proof. Suppose that $J_1, J_2 \subset [n], (\lambda_{J_1}, \lambda_{J_2}) \in \mathcal{P}(J_1, J_1), (\lambda_{J_2}, \lambda_{J_2}) \in \mathcal{P}(J_2, J_2)$, $g \in V_n$, and $\sigma \in S_n$. Then $\sigma^*(L_{\sigma(J_1)}S_{\lambda_{J_1}, \lambda_{J_2}}) \cap L_{\sigma(J_2)}S_{\lambda_{J_2}, \lambda_{J_2}}$ is considered to be the intersection of the stabilizers of disjoint subsets

$$N_1^+, \ldots, N_k^+, N_1^-, \ldots, N_k^-, N_{k+1}, \ldots, N_r$$

obtained by a certain partition of $\mathbb{Z}^\times \times [n]$ into nonempty subsets such that

$$N_i^+ = \{q, 1, q \mid q \in Q_i\} \quad \text{and} \quad N_i^- = \{q, 1, q \mid q \in Q_i\}$$

with $Q_i \subset [n]$ and $g_i \in L_{Q_i}$ for $i = 1, 2, \ldots, k$ and

$$N_i = \{(1, q), (-1, q) \mid q \in Q_i\}$$

with $Q_i \subset [n]$ for $i = k + 1, \ldots, r$. Set $g' = g_1 \cdots g_k$ and $J = Q_{k+1} \cup \cdots Q_r$. Then

$$\sigma^*(L_{J, S_{\lambda_{J_1}, \lambda_{J_2}}}) \cap L_{J, S_{\lambda_{J_2}, \lambda_{J_2}}} = \sigma^*(L_{\sigma(J_1), S_{\lambda_{J_1}, \lambda_{J_2}}}) \cap L_{\sigma(J_2), S_{\lambda_{J_2}, \lambda_{J_2}}}$$

$$= \sigma'(L_{J, S_{\lambda_{J_2}, \lambda_{J_2}}})$$

$$= \sigma'(L_{J, S_{\lambda_{J_2}, \lambda_{J_2}}})$$

for some $\tau \in S_{J, \lambda_{J_2}}$ and $(\lambda_{J_1}, \lambda_{J_2}) \in \mathcal{P}(J, J)$. Consequently, $\mathcal{U}_n$ is closed under intersection. This completes the proof. $\square$

By Lemma 5.1 and [6, Exercise 2.2], $\Omega(B_n, \mathcal{U}_n)$ and $\Omega(B_n, \mathcal{E}_n)$ are subrings of $\Omega(B_n)$ (cf. Theorem 3.4) called partial Burnside rings. The partial Burnside ring $\Omega(B_n, \mathcal{E}_n)$ is known as the parabolic Burnside ring. As for the partial Burnside ring $\Omega(B_n, \mathcal{U}_n)$ relative to the Young subgroups of $B_n$, we quote [7, Corollary II.4]:

Theorem 5.2 The characters $1^B_n B_{n} \subset \mathcal{U}_n$ induced from the trivial characters $1_{L_{\tau, S_{\lambda_{J_2}, \lambda_{J_2}}}}$ for $[i] \subset [n]$ and $(\lambda_{J_1}, \lambda_{J_2}) \in \mathcal{P}(i, i)$ form a $\mathbb{Z}$-basis of $R(B_n)$.

Corollary 5.3 The ring homomorphism $\operatorname{char} : \Omega(B_n) \rightarrow R(B_n)$ induces a ring isomorphism $\operatorname{char} : \Omega(B_n, \mathcal{U}_n) \rightarrow R(B_n)$. In particular, $\Omega(B_n, \mathcal{U}_n)^\times \simeq R(B_n)^\times$.

Proof. The corollary is a consequence of Theorem 5.2, because $\mathcal{U}_n$ is a set of conjugates of the subgroups $L_{\tau, S_{\lambda_{J_2}, \lambda_{J_2}}}$ for $[i] \subset [n]$ and $(\lambda_{J_1}, \lambda_{J_2}) \in \mathcal{P}(i, i)$. $\square$

The rest of this section is devoted to quite a new view of the units of $\Omega(B_n, \mathcal{U}_n)$.

Proposition 5.4 $|\Omega(B_n, \mathcal{E}_n)^\times| = 4$. 

Proof. By [4, (66.29) Corollary] and Corollary 5.3, there is a unique unit \( \alpha_n \) of \( \Omega(B_n, \mathcal{E}_n) \) such that \( \text{char}(\alpha_n) = \varepsilon_n \). Obviously, \(-1 \in \Omega(B_n, \mathcal{E}_n)^\times \). Hence we have \(|\Omega(B_n, \mathcal{E}_n)^\times| \geq 4 \). By Proposition 4.3 and Theorem 5.2, \( \widetilde{\Lambda}_{P([n])} \in \Omega(B_n, \mathcal{U}_n)^\times \) and \( \widetilde{\Lambda}_{P([n])} \notin \Omega(B_n, \mathcal{E}_n)^\times \). Thus \(|\Omega(B_n, \mathcal{U}_n)^\times : \Omega(B_n, \mathcal{E}_n)^\times| \geq 2 \). By Lemma 4.1 and Corollary 5.3, we have \(|\Omega(B_n, \mathcal{U}_n)^\times| = |R(B_n)^\times| = 8 \), whence \(|\Omega(B_n, \mathcal{E}_n)^\times| = 4 \). This completes the proof. \( \square \)

We present a technical lemma by which [4, (66.29) Corollary] deduces Eq. (4) and a description of \( \varepsilon_n : B_n \to \mathbb{C} \) (see also [6, Propositions 2.3.8 and 2.3.10]):

\[
\varepsilon_n = \sum_{i=0}^{n} \sum_{\lambda=(1^{m_1}, \ldots, 1^{m_i}) \in \mathcal{P}(i)} (-1)^{m_1 + \cdots + m_i + n} \frac{(m_1 + \cdots + m_i)!}{m_1! \cdots m_i!} 1_{S_\lambda B_n-1}^{B_n}. \tag{5}
\]

Lemma 5.5 Let \((S_n, X)\) be the Coxeter system of type \( A_{n-1} \). Given \( \lambda \in \mathcal{P}(n) \), let \( W(\lambda) \) be the set of parabolic subgroups \( W_I \) of \( S_n \) for \( I \subset X \) which are conjugates of \( S_\lambda \). Suppose that \( I \subset X \) and \( W_I \in W(\lambda) \) with \( \lambda = (1^{m_1}, \ldots, n^{m_n}) \in \mathcal{P}(n) \). Then \(|I| \equiv m_1 + \cdots + m_n + n \pmod{2} \), so that \((-1)^{|I|} = (-1)^{m_1 + \cdots + m_n + n} \).

Proof. We use induction with respect to the partially order \( \leq \) on \( \mathcal{P}(n) \) given by

\[ \mu \leq \nu :\iff S_\mu \text{ is a conjugate of a subgroup of } S_\nu. \]

If \( \lambda = (1^n) \), then \( I = \emptyset \), and hence \(|I| \equiv 2n \pmod{2} \). Assume that \((1^n) < \lambda \). Then \( m_k \neq 0 \) and \( m_{k+1} = \cdots = m_n = 0 \) for some \( k \in [n] \). We set

\[ \mu = \begin{cases} (1^{m_1+2}, 2^{m_2-1}) & \text{if } k = 2, \\ (1^{m_1+1}, 2^{m_2}, \ldots, (k-1)^{m_{k-1}+1}, k^{m_k-1}, 0, \ldots, 0) & \text{if } k > 2. \end{cases} \]

Suppose that \( I' \subset X \) and \( W_{I'} \in W(\mu) \). Then \( \mu < \lambda \) and \(|I'| = |I| - 1 \). By the inductive assumption, \(|I'| \equiv m_1 + \cdots + m_n + 1 + n \pmod{2} \). Since \(|I| = |I'| + 1 \), it follows that \(|I| \equiv m_1 + \cdots + m_n + n \pmod{2} \). This completes the proof. \( \square \)

What about a unique unit \( \gamma_n \) of \( \Omega(B_n, \mathcal{U}_n) \) satisfying \( \text{char}(\gamma_n) = \kappa_n \)? We are interested in the reduced Lefschetz invariant \( \widetilde{\Lambda}_{P(\mathbb{Z} \times [n])} \).

Lemma 5.6 \( \kappa_n = \text{char}(\widetilde{\Lambda}_{P(\mathbb{Z} \times [n])}) \).

Proof. By Proposition 2.3, \( \text{char}(\widetilde{\Lambda}_{P(\mathbb{Z} \times [n])})(y) = (-1)^{|\langle y \rangle \setminus (\mathbb{Z} \times [n])|} \) for all \( y \in B_n \).

Let \( \sigma \in S_n \), and assume that \( \sigma \) is the product of pairwise disjoint \( u_j \)-cycles \( \sigma_j \) for \( j = 1, 2, \ldots, r \) with \( \sum_j u_j = n \). Let \( (x_1, \ldots, x_n) \in V_n \), and set \( y = (x_1, \ldots, x_n)\sigma \). For each \( j \in \{1, 2, \ldots, r\} \), let \( I_j \) be the minimal subset of \([n]\) with \( \sigma_j \in S_{I_j} \), and set

\[ y_j = (x_1^{(j)}, x_2^{(j)}, \ldots, x_n^{(j)})\sigma_j \quad \text{with} \quad x_i^{(j)} = \begin{cases} x_i & \text{if } i \in I_j, \\ 1 & \text{otherwise}. \end{cases} \]
Obviously, \( y = \prod_{j=1}^{r} y_j \). We now set \( s = \{ j \in \{ 1, 2, \ldots, r \} \mid \prod_{i=1}^{n} x_i^{(j)} = 1 \} \), so that \( |\langle y \rangle \setminus (\mathbb{Z}^\times \times [n])| = r + s \). Hence it turns out that
\[
\kappa_n(y) = \prod_{i=1}^{n} x_i = \prod_{j=1}^{r} \prod_{i=1}^{n} x_i^{(j)} = (-1)^{r-s} = (-1)^{|\langle y \rangle \setminus (\mathbb{Z}^\times \times [n])|}.
\]
Consequently, we obtain \( \kappa_n = \text{char}(\bar{A}_P(\mathbb{Z}^\times \times [n])) \), completing the proof. \( \square \)

The following lemma, which is a basic fact for the left \( B_n \)-set \( \mathbb{Z}^\times \times [n] \), is crucial.

**Lemma 5.7** Let \( \{ M_1, \ldots, M_i \} \), \( i \) a positive integer, be a partition of \( \mathbb{Z}^\times \times [n] \) into nonempty subsets, and view them as elements of the \( B_n \)-poset \( P(\mathbb{Z}^\times \times [n]) \). If each \( M_j \) for \( j = 1, 2, \ldots, i \) does not include both \( (1, q) \) and \( (-1, q) \) for any \( q \in [n] \), then there exists an element \( \lambda \) of \( P(n) \) such that the intersection of stabilizers of \( M_j \) in \( B_n \) for \( j = 1, 2, \ldots, i \) is a conjugate of \( S_\lambda \).

**Proof.** There is a partition \( \{ N_1, \ldots, N_k \} \), \( k \) a positive integer, of \( [n] \) into nonempty subsets such that each \( M_j \) for \( j = 1, 2, \ldots, i \) consists of either \( (1, q) \) or \( (-1, q) \), but not both, for each \( q \in N_{\ell_1} \cup \cdots \cup N_{\ell_r} \), with \( \{ N_{\ell_1}, \ldots, N_{\ell_r} \} \subset \{ N_1, \ldots, N_k \} \). Let \( \bar{P}(n) \) be the set of all cycle types to which such partitions \( \{ N_1, \ldots, N_k \} \) of \( [n] \) into nonempty subsets correspond, and take the maximal element \( \mu \) of \( \bar{P}(n) \) with respect to the partially order \( \leq \) on \( P(n) \) given in the proof of Lemma 5.5. Let \( \{ N_1, \ldots, N_k \} \) be a partition of \( [n] \) into nonempty subsets corresponding to \( \mu \) which satisfy the above condition. We set \( J = N_{\ell} \), where \( \ell \) is an arbitrary integer with \( 1 \leq \ell \leq k \).

There exists a unique subset \( Q \) of \( J \) such that
\[
J^+ := \{(1, q) \mid q \in Q \}\cup\{(-1, q) \mid q \in J - Q \} \subset M_{j_1}
\]
and
\[
J^- := \{(1, q) \mid q \in J - Q \}\cup\{(-1, q) \mid q \in Q \} \subset M_{j_2}
\]
for some integers \( j_1 \) and \( j_2 \) with \( 1 \leq j_1 \neq j_2 \leq i \). Let \( g = (x_1, \ldots, x_n) \in L_Q \), and suppose that \( x_q = -1 \) for all \( q \in Q \). Then the stabilizer of \( J^+ \) in \( B_n \) is \( \langle L_Q S_{j_1} S_{j_2} \rangle \), and so is that of \( J^- \) in \( B_n \). Observe now that the intersection of stabilizers of \( M_j \) for \( j = 1, 2, \ldots, i \) in \( B_n \) coincides with the intersection of such subgroups of \( B_n \). Hence the assertion is a consequence of Lemma 5.1. This completes the proof. \( \square \)

Identifying \( (-1, q) \) with \( n + q \in [2n] \) for all \( q \in [n] \), we may consider \( S_{2n} \) to be the symmetric group on \( \mathbb{Z}^\times \times [n] \). In particular, \( B_n \) is viewed as a subgroup of \( S_{2n} \).

**Lemma 5.8** Let \( \lambda \in \bar{P}(2n) \). Then \( B_n \cap ^\sigma S_\lambda \in \mathcal{U}_n \) for all \( \sigma \in S_{2n} \), and
\[
[\text{res}_{B_n}^{S_{2n}}(S_{2n}/S_\lambda)] = \sum_{\sigma \in B_n \setminus S_{2n}/S_\lambda} [B_n/(B_n \cap ^\sigma S_\lambda)] \in \Omega(B_n, \mathcal{U}_n),
\]
where \( \text{res}_{B_n}^{S_{2n}} \) indicates restriction of operators from \( S_{2n} \) to \( B_n \) and \( B_n \setminus S_{2n}/S_\lambda \) is a complete set of representatives of double cosets \( B_n^\sigma S_\lambda, \sigma \in S_{2n} \), in \( S_{2n} \).
Proof. Let \( \sigma \in S_{2n} \). By Lemma 5.7, \( B_n \cap \sigma S_\lambda = \sigma^\tau(L_J S_{\mu \tau}) \) for some \( J \subset [n] \), \( g \in L_\tau \), \( \tau \in S_3 S_T \), and \( (\mu, J, \tau) \in P(J, \tau) \). Hence \( B_n \cap \sigma S_\lambda \in \mathcal{U}_n \). The second assertion follows from [4, (80.27) Subgroup Theorem]. This completes the proof. \( \square \)

There is a formula of the reduced Lefschetz invariant \( \tilde{\Lambda}_{P(Z \times [n])} \) (cf. Eq.(6)) which is implicit in the proof of a conclusion from the proceeding facts:

**Theorem 5.9** Define three elements \( \alpha_n, \beta_n, \) and \( \gamma_n \) of \( \Omega(B_n, \mathcal{U}_n) \) by

\[
\alpha_n = \sum_{i=0}^{n} \sum_{\lambda=(1^{m_1}, \ldots, 1^{m_i}) \in P(i)} (-1)^{m_1 + \cdots + m_i + n} \frac{(m_1 + \cdots + m_i)!}{m_1! \cdots m_i!} [B_n/(S_\lambda B_{n-i})],
\]

\[
\beta_n = (-1)^n \tilde{\Lambda}_{P([n])}, \quad \text{and} \quad \gamma_n = \tilde{\Lambda}_{P(Z \times [n])}.
\]

Then \( \varepsilon_n = \text{char}(\alpha_n) \), \( \rho_n = \text{char}(\beta_n) \), \( \kappa_n = \text{char}(\gamma_n) \), and \( \alpha_n = (-1)^n \tilde{\Lambda}_{P([n])}(Z \times [n]) \). Moreover, \( \Omega(B_n, \mathcal{E}_n)^\times = \langle \alpha_n, -1 \rangle \), \( \Omega(B_n, \mathcal{U}_n)^\times = \langle \beta_n, \gamma_n, -1 \rangle \), and \( \alpha_n = \beta_n \gamma_n \).

**Proof.** By Eq.(5), \( \varepsilon_n = \text{char}(\alpha_n) \). Obviously, \( \alpha_n \in \Omega(B_n, \mathcal{E}_n) \). Since \( \alpha_n \neq 1, -1 \), it follows from Proposition 5.4 that \( \Omega(B_n, \mathcal{E}_n)^\times \) is generated by \( \alpha_n \) and \( -1 \). By Proposition 4.3 and Lemma 5.6, we have \( \rho_n = \text{char}(\beta_n) \), \( \beta_n \in \Omega(B_n, \mathcal{U}_n) \), and \( \kappa_n = \text{char}(\gamma_n) \). The reduced Lefschetz invariant \( \tilde{\Lambda}_{P([2n])} \) of the left \( S_{2n} \)-set \( [2n] \) is an element of \( \Omega(S_{2n}, \mathcal{E}_{2n}) \) (cf. [9, §4]); for its description, see Eq.(3). We may identify \( \tilde{\Lambda}_{P(Z \times [n])} \) with \( \text{res}_{B_n} \tilde{\Lambda}_{P([2n])} \) which is the element of \( \Omega(B_n) \) obtained by restriction of operators on \( S_{2n} \)-sets appearing in the components of \( \tilde{\Lambda}_{P([2n])} \) from \( S_{2n} \) to \( B_n \). By Lemma 5.8, \( \text{res}_{B_n} \tilde{\Lambda}_{P([2n])} \in \Omega(B_n, \mathcal{U}_n) \), and thus \( \tilde{\Lambda}_{P(Z \times [n])} \in \Omega(B_n, \mathcal{U}_n) \). Moreover, it follows from Lemma 4.2 and Corollary 5.3 that \( \Omega(B_n, \mathcal{U}_n)^\times \) is generated by \( \beta_n, \gamma_n, \) and \( -1 \). Also, \( \alpha_n = \beta_n \gamma_n \), because \( \varepsilon_n = \rho_n \kappa_n \). By Proposition 2.3, it turns out that \( \alpha_n = (-1)^n \tilde{\Lambda}_{P([n])}(Z \times [n]) \). This completes the proof. \( \square \)

Since \( \tilde{\Lambda}_{P(Z \times [n])} = \text{res}_{B_n} \tilde{\Lambda}_{P([2n])} \), it follows from Eq.(3) and Lemma 5.8 that

\[
\tilde{\Lambda}_{P(Z \times [n])} = \sum_{\lambda=(1^{m_1}, \ldots, 2n^{m_{2n}}) \in P(2n)} (-1)^{m_1 + \cdots + m_{2n}} \frac{(m_1 + \cdots + m_{2n})!}{m_1! \cdots m_{2n}!} [B_n/(B_n \cap \sigma S_\lambda)]. \tag{6}
\]

We close this section with a character theoretical explanation of the formula of \( \kappa_n \) obtained by Eq.(6). For each \( \mathbb{C} \)-character \( \chi \) of \( G \), let \( \chi|_H \) with \( H \leq G \) denote the \( \mathbb{C} \)-character obtained by restriction of \( \chi \) from \( G \) to \( H \).

**Lemma 5.10** Let \( M : G \to GL_n(\mathbb{C}) \) be a \( \mathbb{C} \)-representation of \( G \) affording a real valued character \( \chi \) of \( G \). Then for any \( g \in G \),

\[
\det M(g) = (-1)^{n-\langle \chi|_g, 1_g \rangle},
\]

where \( \langle \chi|_g, 1_g \rangle \) is the inner product of \( \chi|_g \) and \( 1_g \).
Lemma 5.6, shows that the linear
which affords the permutation character
6 The parabolic Burnside rings of even-signed permutation groups
We set
and call it the even-signed permutation group on
for all \( i \in S_n \). Recall that \( B_n \) is viewed as a subgroup of \( S_{2n} \). By Lemma 5.10,
for all \( \sigma \in S_{2n} \) (see also [9, Lemma 3.3]). This, combined with Proposition 2.3 and
Lemma 5.6, shows that the linear \( \mathbb{C} \)-character \( \det M_{2n} : S_{2n} \to \mathbb{C} \) coincides with
\( \kappa_n : B_n \to \mathbb{C} \). Consequently, we have \( \kappa_n = \text{sgn}_{2n}|_{B_n} \). Hence it follows from Eq.(4)
and Lemma 5.8 (see also [4, (10.13) Subgroup Theorem]) that
\[
\kappa_n = \sum_{\lambda=(1^{m_1}, \ldots, (2n)^{m_{2n}}) \in \mathcal{P}(2n)} \sum_{\sigma \in B_n \setminus S_{2n}/S_\lambda} (-1)^{m_1 + \cdots + m_{2n}} \frac{(m_1 + \cdots + m_{2n})!}{m_1! \cdots m_{2n}!} \delta_{B_n/\mathcal{U}_n, \lambda} \]
and \( B_n \cap S_\lambda \in \mathcal{U}_n \) for all \( \lambda \in \mathcal{P}(2n) \) and \( \sigma \in S_{2n} \).

6 The parabolic Burnside rings of even-signed permutation groups
We set \( D_n = \ker \kappa_n \) and call it the even-signed permutation group on \( [n] \). Obvi-
ously, \( D_n = K_n S_n \), where \( K_n = \ker \vartheta_n \). Suppose that \( [i] \subset [n] \) and \( \lambda \in \mathcal{P}(i) \). We set \( S_\lambda D_{n-i} = (K_n \cap L_{\lambda}) S_\lambda S_1 \) and set \( t = (0, 0, \ldots, 1) \in V_n \). Observe that
\[
[\text{res}_{D_n}^B(B_n/(S_\lambda B_{n-i}))] = \begin{cases} [D_n/(S_\lambda D_{n-i})] & \text{if } 0 \leq i \leq n-1, \\
[D_n/S_\lambda] + [D_n/S^\lambda] & \text{if } i = n \end{cases}
\]
by [4, (80.27) Subgroup Theorem], which are contained in the parabolic Burnside ring \( \mathcal{P}(D_n) \) (cf. [6, 2.3.11]). We define a map \( \text{res}_{D_n}^B : \mathcal{P}(B_n) \to \mathcal{P}(D_n) \) by
\[
[B_n/(S_\lambda B_{n-i})] \mapsto [\text{res}_{D_n}^B(B_n/(S_\lambda B_{n-i}))]
\]
for all \( [i] \subset [n] \) and \( \lambda \in \mathcal{P}(i) \). Set \( \alpha'_n = \text{res}_{D_n}^B(\alpha_n) \) (see Theorem 5.9). Then
\[
\alpha'_n = \sum_{i=0}^{n-1} \sum_{\lambda=(1^{m_1}, \ldots, (i^{m_i}) \in \mathcal{P}(i)} (-1)^{m_1 + \cdots + m_i + n} \frac{(m_1 + \cdots + m_i)!}{m_1! \cdots m_i!} [D_n/(S_\lambda D_{n-i})] + \sum_{\lambda=(1^{m_1}, \ldots, (2n)^{m_{2n}}) \in \mathcal{P}(n)} (-1)^{m_1 + \cdots + m_{2n} + n} \frac{(m_1 + \cdots + m_n)!}{m_1! \cdots m_n!} ([D_n/S_\lambda] + [D_n/S^\lambda]).
\]
Proposition 6.1 \( \mathcal{PB}(D_n)^\times = \langle \alpha'_n, -1 \rangle \).

Proof. By the proof of [1, Theorem 4.5], there is an injection from \( \mathcal{PB}(D_n)^\times \) to \( R(D_n)^\times \) inherited from the ring homomorphism \( \text{char} : \Omega(D_n) \rightarrow R(D_n) \). The sign character \( \varepsilon_n|_{D_n} : D_n \rightarrow \mathbb{C} \) is the only nontrivial \( \mathbb{C} \)-character of \( D_n \) and \( \mathbb{Q} \) is a splitting field for \( D_n \) (cf. [6, \S 5.6]). This, combined with [13, Corollary 1.2 and Lemma 2.1], shows that \( R(D_n)^\times \) is isomorphic to the four group. Moreover, by [4, (10.13) Subgroup Theorem] and Eq.(5), we have \( \varepsilon_n|_{D_n} = \text{char}(\alpha'_n) \). Consequently, \( \mathcal{PB}(D_n)^\times \) is generated by \( \alpha'_n \) and \(-1\). This completes the proof. \( \square \)

Remark 6.2 Let \((W,S)\) be a Coxeter system of type \( E_6, E_7, \) or \( E_8 \). Then every character of \( W \) is rational-valued (cf. [6, 5.3.6]). Moreover, there are exactly two linear \( \mathbb{C} \)-characters of \( W \) (cf. [6, pp. 413–416]). Hence \( R(W)^\times \) is isomorphic to the four group and \( \mathcal{PB}(W)^\times \) is isomorphic to a subgroup of \( R(W)^\times \) (see the proof of Proposition 6.1). Thus it follows from [4, (66.29) Corollary] that \( \mathcal{PB}(W)^\times \) is of order 4 and is generated by \( \sum_{J \subseteq S} (-1)^{|J|} [W/W_J] \) and \(-1\), where \( W_J = \langle s \mid s \in J \rangle \).

REFERENCES


