

On the reciprocity formula for generalized Dedekind sums

メタデータ	<p>言語: eng</p> <p>出版者: Department of Mathematics, Faculty of Science, Hokkaido University</p> <p>公開日: 2019-05-20</p> <p>キーワード (Ja):</p> <p>キーワード (En):</p> <p>作成者: 竹ヶ原, 裕元</p> <p>メールアドレス:</p> <p>所属:</p>
URL	http://hdl.handle.net/10258/00009883

On the reciprocity formula for generalized Dedekind sums.

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(Received February 23, 1988, Revised May 16, 1988)

T. Yoshida recently defined a function on a finite abelian group A ;

$$\Delta_A[\lambda] := \sum_{j=1}^{|A|-1} \left(\frac{j}{|A|} - \frac{1}{2} \right) \lambda^j,$$

where λ is a linear character of A ([2]). For a positive integer k and an integer h , the Dedekind sum $s(h, k)$ is defined by

$$s(h, k) = \sum_{j=1}^{k-1} \left(\left(\frac{j}{k} \right) \right) \left(\left(\frac{hj}{k} \right) \right),$$

$$((x)) = x - [x] - \frac{1}{2} + \frac{1}{2} \delta(x), \quad x \in \mathbf{R},$$

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ is an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

Yoshida has shown that if the order of λ is k , then

$$\langle \Delta_A[\lambda], \Delta_A[\lambda^h] \rangle_A = s(h, k),$$

where h is an integer and \langle, \rangle_A means a usual scalar product on the space of complex valued functions on A . Furthermore, by using the character theory, he proved the following well known formula ;

$$s(h, k) + s(k, h) = \frac{k^2 + h^2 + 1 - 3kh}{12kh},$$

where k and h are positive integers such that $(h, k) = 1$. Now a generalized Dedekind sum is defined by

$$s(h, k, r) = \sum_{j=1}^{k-1} \left(\left(\frac{j}{k} \right) \right) \left(\left(\frac{hj+r}{k} \right) \right),$$

where k is a positive integer, h is an integer and r is a real number ([1]). Clearly $s(h, k, r) = s(h, k, -r)$. If r is an integer and k is the order of λ , then we have

$$\langle \Delta_A[\lambda], \Delta_A[\lambda^h] \cdot \lambda^r \rangle_A = s(h, k, r).$$

Here λ is as above. This formula will be proved in Corollary 4.

The purpose of this paper is to show the reciprocity formula for generalized Dedekind sums due to Knuth from a different angle approach.

THEOREM 1([1]). *Let k and h be positive integers such that $(k, h)=1$, and let r be a real number such that $-h < r < k$. For $z \in \mathbf{Z}$, put*

$$e(r, z) = \begin{cases} 1 & \text{if } r=0 \text{ or} \\ & r \neq 0, r \not\equiv 0 \pmod{z}, \\ 0 & \text{if } r \neq 0, r \equiv 0 \pmod{z}. \end{cases}$$

Then we have

$$\begin{aligned} s(h, k, r) + s(k, h, r) = & \frac{k^2 + h^2 + 1}{12kh} + \frac{[r]([r] + 1 - \delta(r))}{2kh} \\ & + \frac{1}{2} \left(\left[\frac{r}{k} \right] - \left[\frac{r}{h} \right] \right) - \frac{1}{4} (e(r, h) - e(r, k) + 1). \end{aligned}$$

If we assume that $k > h > 0$ and $k > r \geq 0$, then this formula is the same that as in [1].

Throughout this paper, let k and h be positive integers such that $(k, h)=1$. Let k' and h' be integers such that $kk' + hh' = 1$. We start with the following lemma.

LEMMA 2. *Let c, d and r be integers such that $0 \leq c < k$, $0 \leq d < h$ and $0 \leq r < k$.*

$$(1) \quad s(1, k, r) = \frac{(k-1)(k-2)}{12k} + \frac{r^2}{2k} - \frac{r}{2} + \frac{1}{4} \left(1 - \delta\left(\frac{r}{k}\right) \right).$$

$$(2) \quad s(hh', kh, hh'c + kk'd) = s(h, k, d-c) - \frac{1}{2} \left(\left(\frac{d-c}{k} \right) \right) \left(1 - \delta\left(\frac{d}{h}\right) \right).$$

PROOF.

$$\begin{aligned} (1) \quad s(1, k, r) - s(1, k) &= \sum_{j=1}^{k-1} \left(\left(\frac{j}{k} \right) \right) \left\{ \left(\left(\frac{j+r}{k} \right) \right) - \left(\left(\frac{j}{k} \right) \right) \right\} \\ &= \sum_{j=1}^{k-r-1} \left(\left(\frac{j}{k} \right) \right) \cdot \frac{r}{k} + \left(\left(\frac{k-r}{k} \right) \right) \left(\frac{r-k}{k} + \frac{1}{2} \right) + \sum_{j=k-r+1}^{k-1} \left(\left(\frac{j}{k} \right) \right) \left(\frac{r}{k} - 1 \right). \end{aligned}$$

It is well known that, for a real number x and a positive integer n ,

$$((-x)) = -((x)), \quad ((x+n)) = ((x))$$

and
$$((n \cdot x)) = \sum_{i=0}^{n-1} \left(\left(\frac{i}{n} + x \right) \right).$$

Therefore the above sum is equal to $\sum_{j=1}^{r-1} \left(\left(\frac{j}{k} \right) \right) + \frac{1}{2} \left(\left(\frac{r}{k} \right) \right)$, and we have

$$\sum_{j=1}^{r-1} \left(\left(\frac{j}{k} \right) \right) + \frac{1}{2} \left(\left(\frac{r}{k} \right) \right) = \frac{r^2}{2k} - \frac{r}{2} + \frac{1}{4} \left(1 - \delta \left(\frac{r}{k} \right) \right).$$

Since $s(1, k) = \frac{(k-1)(k-2)}{12k}$, we obtain the required equality.

(2) Since $kk' + hh' = 1$, we have

$$\begin{aligned} s(hh', kh, hh'c + kk'd) &= \sum_{j=0}^{kh-1} \left(\left(\frac{j}{kh} \right) \right) \left(\left(\frac{hh'(j+c-d)+d}{kh} \right) \right) \\ &= \sum_{\beta=0}^{k-1} \sum_{\alpha=0}^{h-1} \left(\left(\frac{\alpha k + \beta}{kh} \right) \right) \\ &\quad \left(\left(\frac{hh'(\alpha k + \beta + c - d)}{kh} + \frac{d}{kh} \right) \right) \\ &= \sum_{\beta=0}^{k-1} \left\{ \sum_{\alpha=0}^{h-1} \left(\left(\frac{\alpha}{h} + \frac{\beta}{kh} \right) \right) \right\} \\ &\quad \left(\left(\frac{hh'(\beta + c - d)}{kh} + \frac{d}{kh} \right) \right). \end{aligned}$$

Here we get $\sum_{\alpha=0}^{h-1} \left(\left(\frac{\alpha}{h} + \frac{\beta}{kh} \right) \right) = \left(\left(\frac{\beta}{k} \right) \right)$. Hence we have

$$\begin{aligned} s(hh', kh, hh'c + kk'd) &= \sum_{\beta=0}^{k-1} \left(\left(\frac{\beta}{k} \right) \right) \left(\left(\frac{h'(\beta + c - d)}{k} + \frac{d}{kh} \right) \right) \\ &= \sum_{j=0}^{k-1} \left(\left(\frac{hj + d - c}{k} \right) \right) \left(\left(\frac{j}{k} + \frac{d}{kh} \right) \right). \end{aligned}$$

Now we can see

$$\left(\left(\frac{j}{k} + \frac{d}{kh} \right) \right) = \begin{cases} \left(\left(\frac{j}{k} \right) \right) + \frac{d}{kh} & \text{if } j \neq 0, \\ \frac{d}{kh} - \frac{1}{2} \left(1 - \delta \left(\frac{d}{h} \right) \right) & \text{if } j = 0. \end{cases}$$

Therefore we get the required formula, and the lemma was proved.

NOTE. The assertion (1) of Lemma 2 is a special case of the Knuth's formula which is in the proof of lemma 3 of the paper [1].

Next we state the properties of the function $\Delta_A[\lambda]$ without proofs.

LEMMA 3([2]). Let A be a finite abelian group and B a subgroup of A . Let λ be a linear character of A .

(1) If $\lambda^k = 1_A$, then we have $\Delta_A[\lambda] = \sum_{j=1}^{k-1} \left(\frac{j}{k} - \frac{1}{2} \right) \lambda^j$.

(2) $\Delta_A[\lambda]_{|B} = \Delta_B[\lambda_{|B}]$, here $_{|B}$ denotes the restriction to B .

(3) Let $a \in A$. Then $\Delta_A[\lambda](a^{-1}) = -\Delta_A[\lambda](a)$, and

$$\Delta_A[\lambda] = \begin{cases} 0 & \text{if } \lambda(a) = 1, \\ \frac{1}{2} \cdot \frac{\lambda(a) + 1}{\lambda(a) - 1} & \text{if } \lambda(a) \neq 1. \end{cases}$$

COROLLARY 4. Under the notation of Lemma 3, if the order of λ is k , then

$$\langle \Delta_A[\lambda], \Delta_A[\lambda^h] \cdot \lambda^r \rangle_A = s(h, k, r),$$

where h and r are integers.

PROOF. By (1) of Lemma 3, we have $\Delta_A[\lambda] = \sum_{j=1}^{k-1} \left(\frac{j}{k} - \frac{1}{2} \right) \lambda^j$.

Therefore we have

$$\begin{aligned} \langle \Delta_A[\lambda], \Delta_A[\lambda^h] \cdot \lambda^r \rangle_A &= \left\langle \sum_{j=1}^{k-1} \left(\frac{j}{k} - \frac{1}{2} \right) \lambda^j, \sum_{i=1}^{k-1} \left(\frac{i}{k} - \frac{1}{2} \right) \lambda^{hi+r} \right\rangle_A \\ &= \sum_{\substack{i=1 \\ hi+r \not\equiv 0 \pmod{k}}}^{k-1} \left(\frac{hi+r}{k} - \left[\frac{hi+r}{k} \right] - \frac{1}{2} \right) \left(\frac{i}{k} - \frac{1}{2} \right). \end{aligned}$$

This proves the corollary.

Hereafter, let M and N be cyclic groups of order k and h , and let λ and μ be generators of the linear characters of M and N , respectively. Put $A = M \times N$. In the paper [2] Yoshida defined the function θ on A as

$$\begin{aligned} \theta &= \Delta_A[\lambda \times 1_N] \cdot \Delta_A[1_M \times \mu] \\ &\quad - \Delta_A[\lambda \times \mu] (\Delta_A[\lambda \times 1_N] + \Delta_A[1_M \times \mu]), \end{aligned}$$

and proved the preceding reciprocity formula for Dedekind sums by considering the product $\langle \theta, 1_A \rangle_A$.

PROPOSITION 5. Let r be an integer such that $-h < r < k$. Then we have

$$\begin{aligned} s(h, k, r) + s(k, h, r) &= \frac{k^2 + h^2 + 1}{12kh} + \frac{r^2}{2kh} \\ &\quad + \frac{1}{2} \left(\left[\frac{r}{k} \right] - \left[\frac{r}{h} \right] \right) - \frac{1}{4} (e(r, h) - e(r, k) + 1). \end{aligned}$$

PROOF. Under the above notation, let c and d be integers such that $0 \leq c < k$ and $0 \leq d < h$. We consider the product $\langle \theta, \lambda^c \times \mu^d \rangle_A$ to show the equality. Let m and n be generators of M and N respectively. Then we have the equality ;

$$\theta(m^i \times n^j) = -\frac{1}{4} \text{ for } 0 < i < k \text{ and } 0 < j < h,$$

which is proved by (3) of Lemma 3 (see [2]). Put $\omega_k = \lambda(m)$ and $\omega_h = \mu(n)$. Since $\theta(1_A) = 0$, we have

$$\begin{aligned} \langle \theta, \lambda^c \times \mu^d \rangle_A &= (k \langle \theta_M \cdot \lambda^c \rangle_M + h \langle \theta_N \cdot \mu^d \rangle_N \\ &\quad + \sum_{i=1}^{k-1} \sum_{j=1}^{h-1} \theta(m^i \times n^j) \cdot \lambda(m^{ci}) \cdot \mu(n^{dj})) / kh \\ &= (k \langle \Delta_M[\lambda], \Delta_M[\lambda] \cdot \lambda^c \rangle_M + h \langle \Delta_N[\mu], \Delta_N[\mu] \cdot \mu^d \rangle_N \\ &\quad - \frac{1}{4} \cdot \sum_{i=1}^{k-1} \sum_{j=1}^{h-1} \omega_k^{ci} \cdot \omega_h^{dj}) / kh \end{aligned}$$

by (2) and (3) of Lemma 3. Here we get

$$\sum_{i=1}^{k-1} \sum_{j=1}^{h-1} \omega_k^{ci} \cdot \omega_h^{dj} = \left(k \delta\left(\frac{c}{k}\right) - 1 \right) \left(h \delta\left(\frac{d}{h}\right) - 1 \right).$$

Hence $\langle \theta, \lambda^c \times \mu^d \rangle_A$ is equal to

$$\frac{s(1, k, r)}{h} + \frac{s(1, h, r)}{k} - \frac{1}{4kh} \left\{ \left(k \delta\left(\frac{c}{k}\right) - 1 \right) \left(h \delta\left(\frac{d}{h}\right) - 1 \right) \right\},$$

by Corollary 4. Therefore, by (1) of Lemma 2, we have

$$(*) \quad \langle \theta, \lambda^c \times \mu^d \rangle_A = \frac{k^2 + h^2 + 1}{12kh} + \frac{c^2 + d^2}{2kh} - \frac{1}{2} \left(\frac{c}{h} + \frac{d}{k} \right) - \frac{1}{4} \delta\left(\frac{cd}{kh}\right).$$

On the other hand,

$$\begin{aligned} \langle \theta, \lambda^c \times \mu^d \rangle_A &= s(hh', kh, hh'c + kk'd) + s(kk', kh, hh'c + kk'd) \\ &\quad + \langle \Delta_A[\lambda \times 1_N] \cdot \Delta_A[1_M \times \mu], \lambda^c \times \mu^d \rangle_A \\ &= s(h, k, d - c) + s(k, h, c - d) \\ &\quad - \frac{1}{2} \left(\left(\frac{d - c}{k} \right) \right) \left(1 - \delta\left(\frac{d}{h}\right) \right) - \frac{1}{2} \left(\left(\frac{c - d}{h} \right) \right) \left(1 - \delta\left(\frac{c}{k}\right) \right) \\ &\quad + \left(\frac{c}{k} - \frac{1}{2} + \frac{1}{2} \delta\left(\frac{c}{k}\right) \right) \left(\frac{d}{h} - \frac{1}{2} + \frac{1}{2} \delta\left(\frac{d}{h}\right) \right) \end{aligned}$$

by (2) Lemma 2 and Corollary 4. Here we get

$$\begin{aligned} \left(\left(\frac{d - c}{k} \right) \right) \left(1 - \delta\left(\frac{d}{h}\right) \right) &= \left(\left(\frac{d - c}{k} \right) \right) + \left(\left(\frac{c}{k} \right) \right) \delta\left(\frac{d}{h}\right) \\ &= \frac{d - c}{k} - \left[\frac{d - c}{k} \right] - \frac{1}{2} + \frac{1}{2} \delta\left(\frac{d - c}{k}\right) \\ &\quad + \left(\left(\frac{c}{k} \right) \right) \delta\left(\frac{d}{h}\right). \end{aligned}$$

Therefore we have

$$\begin{aligned}\langle \theta, \lambda^c \times \mu^d \rangle_A &= s(h, k, d-c) + s(k, h, c-d) \\ &\quad + \frac{cd}{kh} - \frac{1}{2} \left(\frac{d}{k} + \frac{c}{h} - \left[\frac{d-c}{k} \right] - \left[\frac{c-d}{h} \right] \right) \\ &\quad - \frac{1}{4} \left(\delta \left(\frac{d-c}{k} \right) + \delta \left(\frac{c-d}{h} \right) - 3 + \delta \left(\frac{cd}{kh} \right) \right).\end{aligned}$$

Now by (*), we obtain the equality ;

$$\begin{aligned}s(h, k, d-c) + s(k, h, c-d) &= \frac{k^2 + h^2 + 1}{12kh} + \frac{(c-d)^2}{2kh} - \frac{1}{2} \left(\left[\frac{d-c}{k} \right] + \left[\frac{c-d}{h} \right] \right) \\ &\quad + \frac{1}{4} \left(\delta \left(\frac{d-c}{k} \right) + \delta \left(\frac{c-d}{h} \right) - 3 \right).\end{aligned}$$

Put $r = c - d$. Then $-h < r < k$, and we have

$$\left[\frac{d-c}{k} \right] = \left[\frac{-r}{k} \right] = - \left[\frac{r}{k} \right] - 1 + \delta \left(\frac{r}{k} \right).$$

Furthermore $\delta \left(\frac{r}{h} \right) - \delta \left(\frac{r}{k} \right) = e(r, k) - e(r, h)$, this proves the proposition.

PROOF OF THEOREM 1. Let r be a real number such that $-h < r < k$. Put $r = l + x$ where l is an integer such that $-h < l < k$ and x is a real number such that $-1 < x < 1$. Furthermore we assume that both l and x are positive or negative unless $l = 0$. Then we have the following lemma. See also the proofs of Lemmas 1 and 3 in [1].

LEMMA 6. *We have*

$$\begin{aligned}s(h, k, r) + s(k, h, r) &= s(h, k, l) + s(k, h, l) \\ &\quad + \left\{ \frac{|l|}{2kh} - \frac{1}{4} \left| \delta \left(\frac{l}{k} \right) - \delta \left(\frac{l}{h} \right) \right| \right\} (1 - \delta(r)).\end{aligned}$$

PROOF. First we have

$$\begin{aligned}s(h, k, r) &= \sum_{j=1}^{k-1} \left(\left(\frac{j}{k} \right) \right) \left(\left(\frac{hj + l + x}{k} \right) \right) \\ &= \sum_{j=0}^{k-1} \left(\left(\frac{h'(j-l)}{k} \right) \right) \left(\left(\frac{j+x}{k} \right) \right) \\ &= s(h, k, l) + \frac{x}{2|x|} \left(\left(\frac{h'l}{k} \right) \right) (1 - \delta(r)).\end{aligned}$$

For $s(k, h, r)$ we have the similar form. Furthermore we have

$$\left(\left(\frac{h'l}{k} \right) \right) + \left(\left(\frac{k'l}{h} \right) \right) = \frac{l}{kh} + \frac{1}{2} \left(\delta \left(\frac{l}{k} \right) - \delta \left(\frac{l}{h} \right) \right).$$

Therefore we get the required formula from our assumption, and the lemma was proved.

By Proposition 5 and Lemma 6, we have

$$\begin{aligned} s(h, k, r) + s(k, h, r) &= \frac{k^2 + h^2 + 1}{12kh} \\ &+ \frac{[r]([r] + 1 - \delta(r))}{2kh} + \frac{1}{2} \left(\left[\frac{r}{k} \right] - \left[\frac{r}{h} \right] \right) + \frac{1}{2} (1 - e(l, k))(1 - \delta(r)) \\ &- \frac{1}{4} \left\{ e(l, h) - e(l, k) + 1 + \left| \delta\left(\frac{l}{k}\right) - \delta\left(\frac{l}{h}\right) \right| (1 - \delta(r)) \right\}. \end{aligned}$$

Finally we can see

$$\begin{aligned} e(l, h) - e(l, k) + \left| \delta\left(\frac{l}{k}\right) - \delta\left(\frac{l}{h}\right) \right| (1 - \delta(r)) \\ = e(r, h) - e(r, k) + 2(1 - e(l, k))(1 - \delta(r)), \end{aligned}$$

and this prove the theorem.

References

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